§15.5

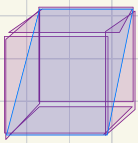
#1 Rewrite & evaluate.

$$\int_0^1\int_0^{1-x}\int_{x+z}^{1} 1\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{y}\int_0^{y-x} 1\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{y} z\Big|_0^{y-x}\,dx\,dy = \int_0^1\int_0^{y} y-x\,dx\,dy = \int_0^1 yx-\tfrac{1}{2}x^2\Big|_0^{y}\,dy$$

$$= \int_0^1 y^2-\tfrac{1}{2}y^2\,dy = \int_0^1 \tfrac{1}{2}y^2\,dy = \tfrac{1}{6}y^3\Big|_0^1 = \tfrac{1}{6}-0 = \boxed{1/6}$$



Sanity check.
less than 1/2 of a
unit cube ✓

Exercises 15.5

Triple Integrals in Different Iteration Orders

1. Evaluate the integral in Example 2 taking $F(x, y, z) = 1$ to find the volume of the tetrahedron in the order $dz\,dx\,dy$.
2. **Volume of rectangular solid** Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$. Evaluate one of the integrals.
3. **Volume of tetrahedron** Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.
4. **Volume of solid** Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.

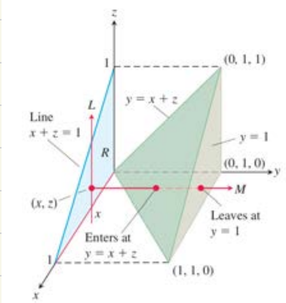


FIGURE 15.32 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Examples 2 and 3).

$$\int_0^1\int_0^{1-x}\int_{x+z}^{1} F(x, y, z)\,dy\,dz\,dx.$$

#2 $R = [0,1]\times[0,2]\times[0,3]$ Find vol of R six ways.

① $\int_0^1\int_0^2\int_0^3 1\,dz\,dy\,dx$

④ $\int_0^2\int_0^1\int_0^3 1\,dz\,dx\,dy$

② $\int_0^1\int_0^3\int_0^2 1\,dy\,dz\,dx$

⑤ $\int_0^2\int_0^3\int_0^1 1\,dz\,dy\,dx$

③ $\int_0^3\int_0^1\int_0^2 1\,dy\,dx\,dz$

⑥ $\int_0^3\int_0^2\int_0^1 1\,dx\,dy\,dz$.

Evaluate one of them: ⑥ $= \int_0^3\int_0^2 x\Big|_0^1\,dy\,dz = \int_0^3\int_0^2 1\,dy\,dz = \int_0^3 y\Big|_0^2\,dz$

$= \int_0^3 2\,dz = 2z\Big|_0^3 = 6-0 = 6$

§15.5

#1 Rewrite & evaluate.

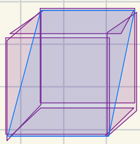
$$\int_0^1\int_0^{1-x}\int_{x+z}^{1} 1\ dy\,dz\,dx$$

$$= \int_0^1\int_0^{1-y}\int_0^{y-x} 1\ dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-y} z\Big|_0^{y-x}\,dx\,dy = \int_0^1\int_0^{1-y} y-x\ dx\,dy = \int_0^1 yx-\tfrac{1}{2}x^2\Big|_0^{1-y}\,dy$$

$$= \int_0^1 y(1-y)-\tfrac{1}{2}(1-y)^2\,dy = \int_0^1 y-y^2-\tfrac{1}{2}(1-2y+y^2)\ dy$$

$$= \int_0^1 \tfrac{1}{2}-\tfrac{3}{2}y+\tfrac{3}{2}y^2\,dy = \tfrac{1}{2}y-\tfrac{3}{4}y^2+\tfrac{3}{6}y^3\Big|_0^1 = \left(\tfrac{1}{2}-\tfrac{3}{4}+\tfrac{1}{2}\right)-0 = \boxed{1/4}$$



Sanity check. less than 1/2 of a unit cube ✓

Exercises 15.5

Triple Integrals in Different Iteration Orders

1. Evaluate the integral in Example 2 taking $F(x, y, z) = 1$ to find the volume of the tetrahedron in the order $dz\,dx\,dy$.
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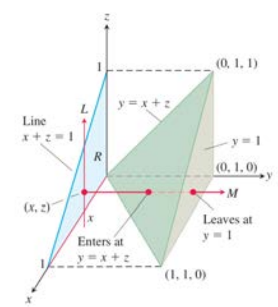

FIGURE 15.32 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Examples 2 and 3).

$$\int_0^1\int_0^{1-x}\int_{x+z}^{1} F(x, y, z)\,dy\,dz\,dx.$$

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③ $\int_0^3\int_0^1\int_0^2 1\ dy\,dx\,dz$

④ $\int_0^2\int_0^1\int_0^3 1\ dz\,dx\,dy$

⑤ $\int_0^2\int_0^3\int_0^1 1\ dz\,dy\,dx$

⑥ $\int_0^3\int_0^2\int_0^1 1\ dx\,dy\,dz$.

Evaluate one of them: ⑥ $= \int_0^3\int_0^2 x\Big|_0^1\,dy\,dz = \int_0^3\int_0^2 1\,dy\,dz = \int_0^3 y\Big|_0^2\,dz$

$$= \int_0^3 2\,dz = 2z\Big|_0^3 = 6-0 = \boxed{6}$$

Evaluate

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$

8. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

9. $\int_1^e\int_1^{e^2}\int_1^{e^3} \frac{1}{xyz}\,dx\,dy\,dz$

10. $\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx$

11. $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\,dx\,dy\,dz$

12. $\int_{-1}^1\int_0^1\int_0^2 (x+y+z)\,dy\,dx\,dz$

13. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

14. $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy$

15. $\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx$

16. $\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

17. $\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)\,du\,dv\,dw$ (uvw-space)

18. $\int_0^1\int_1^{\sqrt{e}}\int_1^e se^s\ln r\,\frac{(\ln t)^2}{t}\,dt\,dr\,ds$ (rst-space)

#8 $\displaystyle\int_0^{\sqrt2}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} 1\;dz\,dx\,dy$

$$= \int_0^{\sqrt2}\int_0^{3y} z\Big|_{x^2+3y^2}^{8-x^2-y^2} dx\,dy$$

$$= \int_0^{\sqrt2}\int_0^{3y} (8-x^2-y^2)-(x^2+3y^2)\;dx\,dy$$

$$= \int_0^{\sqrt2}\int_0^{3y} 8-2x^2-4y^2\;dx\,dy = \int_0^{\sqrt2} 8x-\tfrac{2}{3}x^3-4y^2x\Big|_0^{3y}$$

$$= \int_0^{\sqrt2} 24y-\cancel{\tfrac{2}{3}}(3y)^3-12y^3\;dy \quad \left(\cancel{27y^3}\ 9\right) = \int_0^{\sqrt2} 24y-30y^3\,dy = 12y^2-\frac{30}{4}y^4\Big|_0^{\sqrt2} = 24-\frac{15}{\cancel{2}}*\cancel{4}^{\,2} = \boxed{-6}$$

Note: If $y=\sqrt2$ and $x=3<3\sqrt2$

then $8-x^2-y^2 = 8-9-1=-2$

and $x^2+3y^2 = 9+6=15$

So the z-limits are BACKWARDS?!

(Sanity restored)

#13 $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} 1\;dz\,dy\,dx = \int_0^3\int_0^{\sqrt{9-x^2}} z\Big|_0^{\sqrt{9-x^2}} dy\,dx$

$$= \int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}\;dy\,dx = \int_0^3 \sqrt{9-x^2}\,y\Big|_0^{\sqrt{9-x^2}} dx = \int_0^3 9-x^2\,dx$$

$$= 9x-\tfrac13x^3\Big|_0^3 = \left(27-\tfrac13(3)^3\right)-(0-0) = 27-9 = \boxed{18}$$

#14 $\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} 1\;dz\,dx\,dy = \int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\Big|_0^{2x+y} dx\,dy = \int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2x+y\;dx\,dy$

$$= \int_0^2 x^2+xy\Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2 \left(\cancel{(4-y^2)}+y\sqrt{4-y^2}\right)-\left(\cancel{(4-y^2)}-y\sqrt{4-y^2}\right)dy$$

$$= \int_0^2 2y\sqrt{4-y^2}\;dy = \int_4^0 -u^{1/2}\,du = -\tfrac23u^{3/2}\Big|_4^0 = 0-\left(-\tfrac23 4^{3/2}\right) = \tfrac23*8$$

$$= \boxed{16/3}$$

u-sub

$$\boxed{\begin{aligned} u&=4-y^2\\ du&=-2y\,dy\\ -\tfrac12du&=y\,dy\end{aligned}}$$

$y=0 \Rightarrow u=4$

$y=2 \Rightarrow u=0$

Evaluate

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$

8. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$ 9. $\int_1^e\int_1^{e^2}\int_1^{e^3} \frac{1}{xyz}\,dx\,dy\,dz$

10. $\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx$ 11. $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\,dx\,dy\,dz$

12. $\int_{-1}^1\int_0^1\int_0^2 (x+y+z)\,dy\,dx\,dz$

13. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$ 14. $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy$

15. $\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx$ 16. $\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

17. $\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)\,du\,dv\,dw$ (*uvw*-space)

18. $\int_0^1\int_1^{\sqrt{e}}\int_1^e se^s\ln r\,\frac{(\ln t)^2}{t}\,dt\,dr\,ds$ (*rst*-space)

#16 $$\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx = \int_0^1\int_0^{1-x^2} xz\Big|_3^{4-x^2-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x^2} x\left((4-x^2-y)-(3)\right) dy\,dx = \int_0^1\int_0^{1-x^2} x-x^3-xy\ dy\,dx$$

$$= \int_0^1 (x-x^3)y-\tfrac{1}{2}xy^2\Big|_0^{1-x^2} dx = \int_0^1 x(1-x^2)^2-\tfrac{1}{2}x(1-x^2)^2\,dx$$

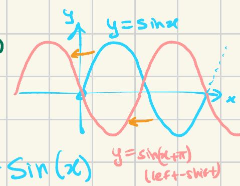
$$= \int_0^1 x(1-x^2)^2\left(1-\tfrac{1}{2}\right)dx = \int_0^1 \tfrac{1}{2}x(1-x^2)^2\,dx = \int_1^0 -\tfrac{1}{4}u^2\,du = +\tfrac{1}{12}u^3\Big|_0^1 = \tfrac{1}{12}-0$$

u-sub BOX
$u = 1-x^2$
$du = -2x\,dx$
$-\tfrac{1}{2}du = x\,dx$

$x=0 \Rightarrow u=1$
$x=1 \Rightarrow u=0$

$$= \boxed{\tfrac{1}{12}}$$

#17 $$\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)\,du\,dv\,dw = \int_0^\pi\int_0^\pi \sin(u+v+w)\Big|_0^\pi dv\,dw$$

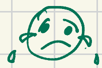


Note:
$\sin(x+\pi) = -\sin(x)$
& sim $\cos(x+\pi) = -\cos(x)$

$$= \int_0^\pi\int_0^\pi \sin(v+w+\pi)-\sin(v+w)\,dv\,dw$$

$$= \int_0^\pi\int_0^\pi -2\sin(v+w)\,dv\,dw = \int_0^\pi 2\cos(v+w)\Big|_0^\pi dw = \int_0^\pi 2(\cos(\pi+w)-\cos(w))\,dw$$

$$= \int_0^\pi -4\cos w\,dw = -4\sin w\Big|_0^\pi = -4\sin\pi-(-4\sin 0) = -0+0 = \boxed{0}$$



19. $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv$ (*tvx*-space)

20. $\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr$ (*pqr*-space)

#20 $$\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr = \int_0^7\int_0^2 \frac{qp}{r+1}\Big|_0^{\sqrt{4-q^2}} dq\,dr$$

$$= \int_0^7\int_0^2 \frac{1}{r+1}q\sqrt{4-q^2}\,dq\,dr = \int_0^7 \frac{1}{r+1}\cdot\frac{-1}{2}\cdot\frac{2}{3}(4-q^2)\Big|_0^2 dr = \int_0^7 \frac{1}{r+1}\cdot\frac{-1}{3}(0-4)\,dr$$

$u=4-q^2 \quad du=-2q\,dq$

$$= \tfrac{4}{3}\ln(1+r)\Big|_0^7 = \tfrac{4}{3}(\ln 8-\ln 1) = \boxed{\tfrac{4}{3}\ln 8}$$

Find volume

#23 D: $z=y^2$, and $z=0, x=0, x=1, y=-1, y=1$

So $R=[0,1]\times[-1,1] \ni (x,y)$ and $z \in [0,y^2]$

$$\text{Vol} = \iiint_D 1\, dV = \iint_R \int_0^{y^2} 1\, dz\, dA$$

$$= \int_0^1 \int_{-1}^1 \int_0^{y^2} 1\, dz\, dy\, dx = \int_0^1 \int_{-1}^1 z \Big|_0^{y^2} dy\, dx$$

$$= \int_0^1 \int_{-1}^1 y^2\, dy\, dx = \int_0^1 \tfrac{1}{3}y^3 \Big|_{-1}^1 dx = \int_0^1 \tfrac{1}{3} - \tfrac{-1}{3}\, dx$$

$$= \int_0^1 \tfrac{2}{3}\, dx = \tfrac{2}{3}x \Big|_0^1 = \boxed{\tfrac{2}{3}}$$

#25 D: first octant, coord planes, $y+z=2$ and $x=4-y^2$

R: $x\in[0,4]$ $y\in[0,2]$ since $x=4-y^2$, $x\geq 0$, $y \geq 0$ $\Rightarrow$ $x\in[0,4]$, $y\in[0,2]$

and $z\in[0,2-y]$

$$\text{Vol} = \iint_R \int_0^{2-y} 1\, dz\, dA = \int_0^4 \int_0^2 \int_0^{2-y} 1\, dz\, dy\, dx = \int_0^4 \int_0^2 z \Big|_0^{2-y} dy\, dx$$

$$= \int_0^4 \int_0^2 2-y\, dy\, dx = \int_0^4 2y - \tfrac{1}{2}y^2 \Big|_0^2 dx = \int_0^4 (4-2)-(0-0)\, dx$$

$$= \int_0^4 2\, dx = 2x \Big|_0^4 = \boxed{8}$$

#26 D: wedge from $x^2+y^2=1$ over $z=0$ and under $z=-y$

R: $x^2+y^2=1$, $x\in[-1,1]$ and $y\in[-1,0]$. $z\in[0,-y]$

$$V = \iint_R \int_0^{-y} 1\, dz\, dA = \int_{-1}^1 \int_{-1}^0 \int_0^{-y} 1\, dz\, dy\, dx = \int_{-1}^1 \int_{-1}^0 z \Big|_0^{-y} dy\, dx$$

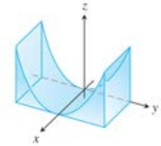
$$= \int_{-1}^1 \int_{-1}^0 -y\, dy\, dx = \int_{-1}^1 -\tfrac{1}{2}y^2 \Big|_{-1}^0 dx = \int_{-1}^1 \left(0 - \tfrac{-1}{2}\right) dx = \int_{-1}^1 \tfrac{1}{2}\, dx$$

$$= \tfrac{1}{2}x \Big|_{-1}^1 = \tfrac{1}{2} - \tfrac{-1}{2} = \boxed{1}$$

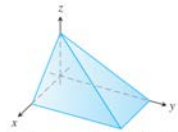
Finding Volumes Using Triple Integrals

Find the volumes of the regions in Exercises 23–36.

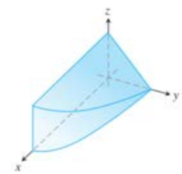
23. The region between the cylinder $z=y^2$ and the xy-plane that is bounded by the planes $x=0$, $x=1$, $y=-1$, $y=1$



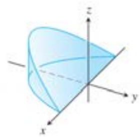
24. The region in the first octant bounded by the coordinate planes and the planes $x+z=1$, $y+2z=2$



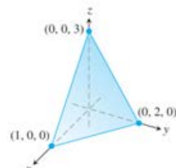
25. The region in the first octant bounded by the coordinate planes, the plane $y+z=2$, and the cylinder $x=4-y^2$



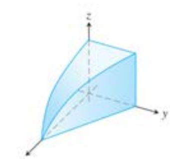
26. The wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y$ and $z=0$



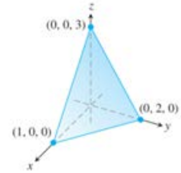
27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$



28. The region in the first octant bounded by the coordinate planes, the plane $y=1-x$, and the surface $z=\cos(\pi x/2)$, $0 \leq x \leq 1$



27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through (1, 0, 0), (0, 2, 0), and (0, 0, 3)



#27 D: tetrahedron w/ vertices $(1,0,0), (0,2,0), (0,0,3)$.

So plane is $6x+3y+2z=6$ → @ $z=0$: $6x+3y=6$ ⇒ $y=2-2x$

R: $x\in[0,1]$ and $y\in[0,2-2x]$ and $z\in[0,3-3x-\frac{3}{2}y]$

$$V=\iint_R\int_0^{3-3x-\frac{3}{2}y} 1\,dz\,dA=\int_0^1\int_0^{2-2x}\int_0^{3-3x-\frac{3}{2}y}1\,dz\,dy\,dx=\int_0^1\int_0^{2-2x} z\Big|_0^{3-3x-\frac{3}{2}y}dy\,dx$$

$$=\int_0^1\int_0^{2-2x}3-3x-\tfrac{3}{2}y\,dy\,dx=\int_0^1(3-3x)y-\tfrac{3}{4}y^2\Big|_0^{2-2x}dx=\int_0^1(3-3x)(2-2x)-\tfrac{3}{4}(2-2x)^2\,dx=\int_0^1(2-2x)\left[3-3x-\tfrac{3}{4}(2-2x)\right]dx$$

$$=\int_0^1(2-2x)\left[(3-3x)-\tfrac{1}{2}(3-3x)\right]dx=\int_0^1\cancel{2}\cdot3(1-x)^2\left[\cancel{1}-\tfrac{1}{2}\right]dx=\int_0^1(1-x)^2\,dx\overset{*}{=}-\tfrac{1}{3}(1-x)^3\Big|_0^1=-\tfrac{1}{3}[0-1]=\boxed{\tfrac{1}{3}}$$

* u-sub box: $u=1-x$, $du=-dx$

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

38. $F(x, y, z) = x + y - z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$

39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

#38 Find avg value of $F(x,y,z)=x+y-z$ over $D=[0,1]\times[0,1]\times[0,2]$.

$\mathrm{Vol}(D)=1*1*2=2$,

$$M=\int_0^1\int_0^1\int_0^2 x+y-z\,dz\,dx\,dy=\int_0^1\int_0^1(x+y)z-\tfrac{1}{2}z^2\Big|_0^2dx\,dy$$

$$=\int_0^1\int_0^1 2x+2y-2\,dx\,dy=\int_0^1x^2+(2y-2)x\Big|_0^1=\int_0^1 1+2y-2\,dy=\int_0^1 2y-1\,dy=y^2-y\Big|_0^1=1-1=0$$

So avg value is $\mathrm{Avg}_D(F)=\frac{M}{\mathrm{Vol}\,D}=\frac{0}{2}=\boxed{0}$

#40 Avg value of $F(x,y,z)=xyz$ over $D=[0,2]\times[0,2]\times[0,2]$ $\qquad \mathrm{Vol}(D)=2^3=8$

$$M=\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz=\int_0^2\int_0^2\tfrac{1}{2}x^2yz\Big|_0^2dy\,dz=\int_0^2\int_0^2 2yz\,dy\,dz=\int_0^2 y^2z\Big|_0^2dz$$

$$=\int_0^2 4z\,dz=2z^2\Big|_0^2=2(4-0)=8$$

So $\mathrm{Avg}_D(F)=\frac{M}{\mathrm{Vol}\,D}=\frac{8}{8}=\boxed{1}$

§15.7

Evaluate

#2 $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} r\,dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^3 rz\Big|_{r^2/3}^{\sqrt{18-r^2}} dr\,d\theta = \int_0^{2\pi}\int_0^3 r\sqrt{18-r^2}-\frac{r^3}{3}\,dr\,d\theta$$

u-sub Box
$u=18-r^2$
$du=-2r\,dr$
$-\frac{1}{2}du=dr$

$\int\sqrt{u}\,du=\frac{2}{3}u^{3/2}+C$

$$=\int_0^{2\pi} -\frac{1}{2}\cdot\frac{2}{3}(18-r^2)^{3/2}-\frac{1}{12}r^4\Big|_0^3 d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(27)-\frac{1}{12}(81)\right]-\left[-\frac{1}{3}(18)^{3/2}\right]d\theta$$

$$=\int_0^{2\pi}\left(-9-\frac{27}{4}\right)+\frac{1}{3}(3\sqrt{2})^3\,d\theta = \int_0^{2\pi}18\sqrt{2}-\frac{63}{4}\,d\theta = \left(18\sqrt{2}-\frac{63}{4}\right)\theta\Big|_0^{2\pi}$$

$36+27=63$

$$=\left(18\sqrt{2}-\frac{63}{4}\right)*2\pi = \boxed{36\sqrt{2}-\frac{63}{2}}$$

#10 $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta+1)r\,d\theta\,dz\,dr = \int_0^2\int_{r-2}^{\sqrt{4-r^2}} r(-r\cos\theta+\theta)\Big|_0^{2\pi}dz\,dr$

$$=\int_0^2\int_{r-2}^{\sqrt{4-r^2}} r\left[(-\cancel{r}(1)+2\pi)-(-\cancel{r}(1)+0)\right]dz\,dr = \int_0^2\int_{r-2}^{\sqrt{4-r^2}}2\pi r\,dz\,dr$$

$$=\int_0^2 2\pi rz\Big|_{r-2}^{\sqrt{4-r^2}}dr = \int_0^2 2\pi r\left[\sqrt{4-r^2}-(r-2)\right]dr$$

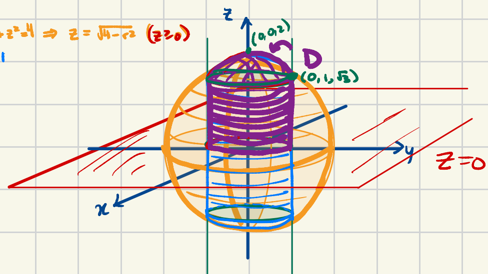
u-sub Box
$u=4-r^2$
$du=-2r\,dr$
$-\frac{1}{2}du=r\,dr$

$\int\sqrt{u}\,du = \frac{2}{3}u^{3/2}+C$

$$=\int_0^2 2\pi r\sqrt{4-r^2}-2\pi r^2+4\pi r\,dr = -\pi\cdot\frac{2}{3}(4-r^2)^{3/2}-\frac{2}{3}\pi r^3+2\pi r^2\Big|_0^2 = \left[0-\frac{16\pi}{3}+8\pi\right]-\left[-\pi\cdot\frac{2}{3}4^{3/2}-0+0\right] = \frac{24\pi-16\pi}{3}+\frac{2}{3}\pi\cdot 8 = \boxed{8\pi}$$

#11 Set up $\iiint_D 1\,dV$ in cylindrical coords.

Sphere $x^2+y^2+z^2=4 \Leftrightarrow r^2+z^2=4 \Rightarrow z=\sqrt{4-r^2}$ $(z\ge 0)$
Cylndr $x^2+y^2=1 \Leftrightarrow r=1$
intersection: $1+z^2=4$
$\Rightarrow z=\sqrt{3}$



(a) $\theta\in[0,2\pi]$ $r\in[0,1]$ $z\in[0,\sqrt{4-r^2}]$

$$\boxed{V=\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta}$$

(b) $\theta\in[0,2\pi]$ $z\in[0,2]$ & If $0\le z\le\sqrt{3}$ then $r\in[0,1]$
If $\sqrt{3}\le z\le 2$ then $r\in[0,\sqrt{4-z^2}]$

$$\boxed{V=\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\,dr\,dz\,d\theta+\int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta}$$

(c) Same as (a) by Fubini's Thm

$$\boxed{V=\int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\,d\theta\,dz\,dr}$$

Exercises 15.7

Evaluating Integrals in Cylindrical Coordinates

Evaluate the cylindrical coordinate integrals in Exercises 1–6.

1. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta$
2. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\,r\,dr\,d\theta$
3. $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\,r\,dr\,d\theta$
4. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta$
5. $\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta$
6. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2} (r^2\sin^2\theta+z^2)\,dz\,r\,dr\,d\theta$

Changing the Order of Integration in Cylindrical Coordinates

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in Exercises 7–10.

7. $\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\,dr\,dz\,d\theta$
8. $\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\,dr\,d\theta\,dz$
9. $\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}(r^2\cos^2\theta+z^2)\,r\,d\theta\,dr\,dz$
10. $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta+1)\,r\,d\theta\,dz\,dr$
11. Let D be the region bounded below by the plane $z=0$, above by the sphere $x^2+y^2+z^2=4$, and on the sides by the cylinder $x^2+y^2=1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
    a. $dz\,dr\,d\theta$ b. $dr\,dz\,d\theta$ c. $d\theta\,dz\,dr$
12. Let D be the region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the paraboloid $z=2-x^2-y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
    a. $dz\,dr\,d\theta$ b. $dr\,dz\,d\theta$ c. $d\theta\,dz\,dr$

11. Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$

12. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$

#12 Set up integrals for $V = \iiint_D 1\,dV$

Cone $z = \sqrt{x^2+y^2} \Leftrightarrow z = r$

paraboloid $z = 2 - x^2 - y^2 \Leftrightarrow z = 2 - r^2$

intersection $(z = z)$

$r = 2 - r^2$

$\Rightarrow r^2 + r - 2 = 0$

$\Rightarrow (r+2)(r-1) = 0$

$\Rightarrow r = -2$ (discard) & $r = 1$

(a) $\theta \in [0, 2\pi]$ $r \in [0, 1]$ $z \in [r, 2 - r^2]$

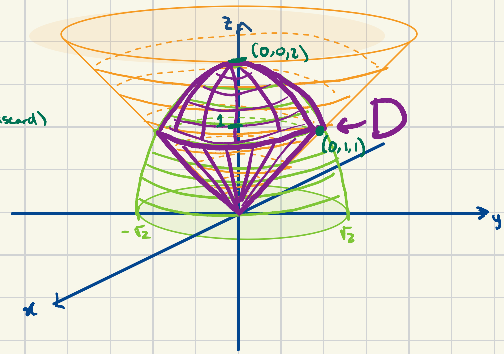
$$\boxed{V = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta}$$

(b) $\theta \in [0, 2\pi]$ $z \in [0, 2]$ if $z \in [0,1]$, $r \in [0, z]$; if $z \in [1, 2]$, $r \in [0, \sqrt{2-z}]$

$$\boxed{V = \int_0^{2\pi}\int_0^1\int_0^z r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\,dr\,dz\,d\theta}$$

(c) Same as (a) by Fubini's theorem

$$\boxed{V = \int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\,d\theta\,dz\,dr}$$



14. Convert the integral

$$\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^x (x^2 + y^2)\,dz\,dx\,dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

#14 Convert to cylind. coord & eval.

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

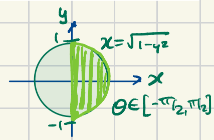
$$M = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^x \overbrace{(x^2+y^2)}^{r^2}\,\underbrace{dz\,dx\,dy}_{dV = r\,dz\,dr\,d\theta}$$

@ $x = \sqrt{1-y^2}$

$\Rightarrow x^2 + y^2 = 1 \Rightarrow r = 1$

@ $x = 0 \Rightarrow r\cos\theta = 0$, can set $r = 0$.

@ $z = x \Rightarrow z = r\cos\theta$



R: $y \in [-1, 1]$, $x \in [0, \sqrt{1-y^2}]$

* $x \geq 0 \Leftrightarrow \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

* $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $r \in [0, 1]$, $z \in [0, r\cos\theta]$

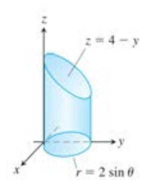
$$M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_0^{r\cos\theta} r^2 * r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^3 z\Big|_0^{r\cos\theta}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1 r^3 r\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{5} r^5\cos\theta\Big|_0^1\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{5}\cos\theta\,d\theta = \frac{1}{5}\sin\theta\Big|_{-\pi/2}^{\pi/2} = \frac{1}{5}(1 - (-1))$$

$$= \boxed{2/5}$$

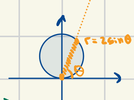
In Exercises 15–20, set up the iterated integral for evaluating $\iiint_D f(r, \theta, z)\,dz\,r\,dr\,d\theta$ over the given region D.

15. D is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$.



#15 Set up integral

R: off-set circle w/ radius 1 & center @ $(0, 1)$



$x^2 + (y-1)^2 = 1 \Leftrightarrow r = 2\sin\theta$ $\quad y \geq 0 \Leftrightarrow \theta \in [0, \pi]$

R: $\theta \in [0, \pi]$, $r \in [0, 2\sin\theta]$, then $z = 4 - y$

so $z = 4 - r\sin\theta$

$z \in [0, 4 - r\sin\theta]$

$$\boxed{V = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4 - r\sin\theta} r\,dz\,dr\,d\theta}$$

Set up $\iiint_D 1\,dV$ in cylindrical coords

#16 R: off-set circle w/ center $(\frac{3}{2},0)$ & radius $r=\frac{3}{2}$

$(x-1)^2+y^2=\frac{9}{4} \Leftrightarrow r=3\cos\theta$

Top $z=5-x \Rightarrow z=5-r\cos\theta$

$\theta\in[-\pi/2,\pi/2]$, $r\in[0,3\cos\theta]$, $z\in[0,5-r\cos\theta]$

$(r\cos\theta-\frac{3}{2})^2+r^2\sin^2\theta=\frac{9}{4}$

$\Rightarrow r^2\cos^2\theta-3r\cos\theta+\cancel{\frac{9}{4}}+r^2\sin^2\theta=\cancel{\frac{9}{4}}$

$\Rightarrow r^2=3r\cos\theta \Rightarrow r=3\cos\theta$ ✓

$x\geq 0 \Rightarrow \theta\in[-\pi/2,\pi/2]$

$$\boxed{V=\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} r\,dz\,dr\,d\theta}$$

#18 R: $\theta\in[-\pi/2,\pi/2]$ since $x\geq 0$, $z=3-y \Rightarrow z=3-r\sin\theta$

$r\in[\cos\theta, 2\cos\theta]$, $z\in[0,3-r\sin\theta]$

So $$\boxed{V=\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta}$$

Evaluate

#21 $$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=\int_0^{\pi}\int_0^{\pi}\frac{1}{3}\rho^3\sin\varphi\Big|_0^{2\sin\varphi}d\varphi\,d\theta$$

$$=\int_0^{\pi}\int_0^{\pi}\frac{1}{3}8\sin^4\varphi\,d\varphi\,d\theta$$

$$=\int_0^{\pi}\int_0^{\pi}\frac{8}{3}\left(\frac{3}{8}-\frac{1}{2}\cos2\varphi+\frac{1}{8}\cos4\varphi\right)d\varphi\,d\theta$$

$$=\int_0^{\pi}\int_0^{\pi}1-\frac{4}{3}\cos2\varphi+\frac{1}{3}\cos4\varphi\,d\varphi\,d\theta$$

$$=\int_0^{\pi}\varphi-\frac{2}{3}\sin2\varphi+\frac{1}{12}\sin4\varphi\Big|_0^{\pi}d\theta$$

$$=\int_0^{\pi}\left(\pi-\frac{2}{3}\cdot0+\frac{1}{12}\cdot0\right)-(0-0+0)\,d\theta=\pi\theta\Big|_0^{\pi}=\boxed{\pi^2}$$

$\sin^2\varphi=\frac{1-\cos2\varphi}{2}$

$\cos^2\varphi=\frac{1+\cos2\varphi}{2}$

So $(\sin^2\varphi)^2=\left(\frac{1-\cos2\varphi}{2}\right)^2$

$=\frac{1}{4}(1-2\cos2\varphi+\cos^22\varphi)$

$=\frac{1}{4}\left(1-2\cos2\varphi+\frac{1+\cos4\varphi}{2}\right)$

$=\frac{1}{4}\left(\frac{3}{2}-2\cos2\varphi+\frac{1}{2}\cos4\varphi\right)$

$=\frac{3}{8}-\frac{1}{2}\cos2\varphi+\frac{1}{8}\cos4\varphi$

#22 $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}\rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=\int_0^{2\pi}\int_0^{\pi/4}\cos\varphi\sin\varphi*\frac{1}{4}\rho^4\Big|_0^2\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/4}4\cos\varphi\sin\varphi\,d\varphi\,d\theta=\int_0^{2\pi}4*\frac{1}{2}\sin^2\varphi\Big|_0^{\pi/4}d\theta=\int_0^{2\pi}2\left(\sin^2\pi/4-\sin^2 0\right)d\theta$$

$(1/\sqrt{2})^2$ 0

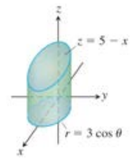
$$=\int_0^{2\pi}2*\frac{1}{2}\,d\theta=\theta\Big|_0^{2\pi}=2\pi-0=\boxed{2\pi}$$

u-sub Box

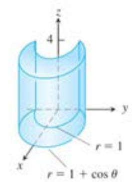
$$\boxed{\begin{aligned}u&=\sin\varphi\\ du&=\cos\varphi\,d\varphi\end{aligned}}$$

$\int u\,du=\frac{1}{2}u^2+C$

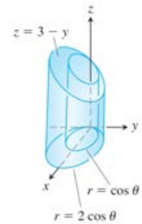
16. D is the right circular cylinder whose base is the circle $r=3\cos\theta$ and whose top lies in the plane $z=5-x$.



17. D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$ and whose top lies in the plane $z=4$.



18. D is the solid right cylinder whose base is the region between the circles $r=\cos\theta$ and $r=2\cos\theta$ and whose top lies in the plane $z=3-y$.



Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

22. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

23. $\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

24. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1}5\rho^3\sin^3\phi\,d\rho\,d\phi\,d\theta$

25. $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Evaluate

#25 $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\varphi}^{2} 3\rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \rho^3\sin\varphi \Big|_{\sec\varphi}^{2} d\varphi\, d\theta$

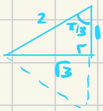
$\displaystyle = \int_0^{2\pi}\int_0^{\pi/3} 8\sin\varphi - \sec^3\varphi\sin\varphi \, d\varphi\, d\theta$

u-sub Box
$\boxed{u=\cos\varphi \quad du=-\sin\varphi\, d\varphi}$

$\displaystyle \int \sec^3\varphi\sin\varphi\, d\varphi = \int \frac{\sin\varphi}{\cos^3\varphi}\, d\varphi = \int -\frac{1}{u^3}\, du = \frac{1}{2}u^{-2}+C$

$\displaystyle = \int_0^{2\pi} -8\cos\varphi - \frac{1}{2}\cos^{-2}\varphi \Big|_0^{\pi/3} d\theta$

$\displaystyle = \int_0^{2\pi}\left[-8\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{-2}\right] - \left[-8-\tfrac{1}{2}\right] d\theta$



$\cos\pi/3 = 1/2$
$\cos 0 = 1$

$\displaystyle = \int_0^{2\pi}[-4-2]-[-7.5]\, d\theta = \int_0^{2\pi} 1.5\, d\theta$

$\displaystyle = \frac{3}{2}\theta\Big|_0^{2\pi} = \frac{3}{2}*2\pi = \boxed{3\pi}$

Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

22. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

23. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

24. $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3\sin^3\phi\, d\rho\, d\phi\, d\theta$

25. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

Changing the Order of Integration in Spherical Coordinates

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders give the same value and are occasionally easier to evaluate. Evaluate the integrals in Exercises 27–30.

27. $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3\sin 2\phi\, d\phi\, d\theta\, d\rho$

28. $\displaystyle\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2\sin\phi\, d\theta\, d\rho\, d\phi$

29. $\displaystyle\int_0^{1}\int_0^{\pi}\int_0^{\pi/4} 12\rho\sin^3\phi\, d\phi\, d\theta\, d\rho$

30. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4\sin^3\phi\, d\rho\, d\theta\, d\phi$

#28 $\displaystyle\int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi}\int_0^{2\pi} \rho^2\sin\varphi\, d\theta\, d\rho\, d\varphi = \int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi} \rho^2\sin\varphi\,\theta\Big|_0^{2\pi} d\rho\, d\varphi$

$\csc\varphi = \frac{1}{\sin\varphi}$

$\displaystyle = \int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi} 2\pi\rho^2\sin\varphi\, d\rho\, d\varphi = \int_{\pi/6}^{\pi/3} \frac{2}{3}\pi\rho^3\sin\varphi\Big|_{\csc\varphi}^{2\csc\varphi} d\varphi = \int_{\pi/6}^{\pi/3} \frac{2}{3}\pi\sin\varphi\left(8\csc^3\varphi - \csc^3\varphi\right) d\varphi$

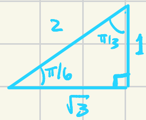
$\displaystyle = \int_{\pi/6}^{\pi/3} \frac{2}{3}\pi\sin\varphi \cdot \frac{7}{\sin^3\varphi}\, d\varphi = \int_{\pi/6}^{\pi/3} \frac{14}{3}\pi\csc^2\varphi\, d\varphi = -\frac{14}{3}\pi\cot\varphi\Big|_{\pi/6}^{\pi/3}$

$\displaystyle\int \csc^2 x\, dx = -\cot x + C$

since $\displaystyle (\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{\sin x(-\sin x) - \cos x\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} = \csc^2 x$ ✓

$-\sin^2 x - \cos^2 x = -1$

$\displaystyle = -\frac{14}{3}\pi\left(\underbrace{1/\sqrt{3}-\sqrt{3}}_{-2\sqrt{3}/3}\right) = \boxed{\frac{28\sqrt{3}\pi}{9}}$



$\cot = \frac{\text{adj}}{\text{opp}}$

$\cot\pi/3 = 1/\sqrt{3}$

$\cot\pi/6 = \sqrt{3}$

Set up and evaluate

#33 $\theta\in[0,2\pi]$, $\varphi\in[0,\pi/2]$ b/c $z\geq 0$, $\rho\in[\cos\varphi, 2]$

(a) Set up

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\varphi}^{2} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$

Note: Off-set sphere w/ radius $1/2$ and center $(0,0,1/2)$

$\boxed{x^2+y^2+\left(z-\tfrac{1}{2}\right)^2 = \tfrac{1}{4}}$

$\Rightarrow \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta + \left(\rho\cos\varphi - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$

$\Rightarrow \rho^2\sin^2\varphi + \rho^2\cos^2\varphi - \rho\cos\varphi + \cancel{\tfrac{1}{4}} = \cancel{\tfrac{1}{4}}$

$\Rightarrow \rho^2 - \rho\cos\varphi = 0 \Rightarrow \boxed{\rho = \cos\varphi}$

(b) evaluate

u-sub Box
$\boxed{u=\cos\varphi \quad du=-\sin\varphi\, d\varphi}$

$\displaystyle\int u^3\, du = \frac{1}{4}u^4 + C$

$\displaystyle V = \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{3}\sin\varphi\,\rho^3\Big|_{\cos\varphi}^{2} d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{\frac{8}{3}\sin\varphi}_{①} - \underbrace{\frac{1}{3}\cos^3\varphi\sin\varphi}_{②}\, d\varphi\, d\theta$

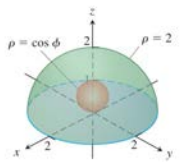
$\displaystyle ① = \int_0^{2\pi}\int_0^{\pi/2} \frac{8}{3}\sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi} -\frac{8}{3}\cos\varphi\Big|_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{8}{3}\left(\cos\frac{\pi}{2} - \cos 0\right) d\theta = \frac{8}{3}\theta\Big|_0^{2\pi} = \frac{16\pi}{3}$

$\displaystyle ② = \int_0^{2\pi} -\frac{1}{12}\cos^4\varphi\Big|_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{1}{12}\left(\cos^4\frac{\pi}{2} - \cos^4 0\right) d\theta = \int_0^{2\pi} \frac{1}{12}\, d\theta = \frac{\theta}{12}\Big|_0^{2\pi} = \frac{2\pi}{12} = \pi/6$

Finding Iterated Integrals in Spherical Coordinates

In Exercises 33–38, **(a)** find the spherical coordinate limits for the integral that calculates the volume of the given solid and then **(b)** evaluate the integral.

33. The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geq 0$



Spherical to Rectangular

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

But... but...
1/2-sphere w/ $\rho=2$
$\text{Vol} = \frac{1}{2}\cdot\frac{4}{3}\pi\rho^3 = \frac{2}{3}\pi\cdot 8 = \frac{16\pi}{3}$
minus sphere w/ $\rho = 1/2$
$\text{Vol} = \frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = \pi/6$
So $\boxed{31\pi/6}$:)

So $① - ② = \dfrac{16\pi}{3} - \dfrac{\pi}{6} = \dfrac{32\pi - \pi}{6}$

$= \boxed{31\pi/6}$

uhhh...

Set up & evaluate

#34 $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi/2]$ b/c $z \geq 0$, $\rho \in [1, 1+\cos\varphi]$

(a) Set up

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}$$

(b) evaluate

$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi/2} \tfrac{1}{3}\rho^3 \sin\varphi \Big|_1^{1+\cos\varphi} d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \tfrac{1}{3}(1+\cos\varphi)^3 \sin\varphi - \tfrac{1}{3}\sin\varphi \, d\varphi \, d\theta$$

U-sub Box: $u = 1+\cos\varphi$, $du = -\sin\varphi \, d\varphi$

$$= \int_0^{2\pi} \tfrac{-1}{12}(1+\cos\varphi)^4 + \tfrac{1}{3}\cos\varphi \Big|_0^{\pi/2} d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{-1}{12}\left(1+\underbrace{\cos\tfrac{\pi}{2}}_{0}\right)^4 + \tfrac{1}{3}\underbrace{\cos\tfrac{\pi}{2}}_{0}\right] - \left[\tfrac{-1}{12}\left(1+\underbrace{\cos 0}_{1}\right)^4 + \tfrac{1}{3}\underbrace{\cos 0}_{1}\right] d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{-1}{12}\right] - \underbrace{\left[\tfrac{-1}{12}(2)^4 + \tfrac{1}{3}\right]}_{-\frac{4}{3}+\frac{1}{3}} d\theta = \int_0^{2\pi} \tfrac{11}{12} \, d\theta = \tfrac{11}{12}\theta \Big|_0^{2\pi} = \tfrac{11}{12} * 2\pi = \boxed{\tfrac{11\pi}{6}}$$

Find the volume

#43 Cylindrical $\theta \in [0, 2\pi]$, $r \in [0, 1]$, $z \in [r^4 - 1, 4 - 4r^2]$

$$V = \int_0^{2\pi}\int_0^1\int_{r^4-1}^{4-4r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^1 rz \Big|_{r^4-1}^{4-4r^2} dr \, d\theta$$

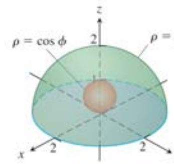
$$= \int_0^{2\pi}\int_0^1 r\left[(4-4r^2)-(r^4-1)\right] dr \, d\theta = \int_0^{2\pi}\int_0^1 -r^5 - 4r^3 + 5r \, dr \, d\theta$$

$$= \int_0^{2\pi} -\tfrac{1}{6}r^6 - r^4 + \tfrac{5}{2}r^2 \Big|_0^1 d\theta = \int_0^{2\pi} \tfrac{-1}{6} - 1 + \tfrac{5}{2} \, d\theta = \int_0^{2\pi} -\tfrac{7}{6} + \tfrac{15}{6} \, d\theta = \tfrac{4}{3}\theta \Big|_0^{2\pi} = \tfrac{4}{3}(2\pi - 0) = \boxed{8\pi/3}$$

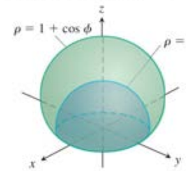
Finding Iterated Integrals in Spherical Coordinates

In Exercises 33–38, **(a)** find the spherical coordinate limits for the integral that calculates the volume of the given solid and then **(b)** evaluate the integral.

33. The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, z \geq 0$



34. The solid bounded below by the hemisphere $\rho = 1, z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos\phi$



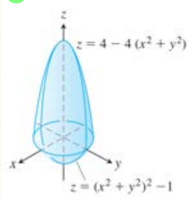
35. The solid enclosed by the cardioid of revolution $\rho = 1 - \cos\phi$

36. The upper portion cut from the solid in Exercise 35 by the xy-plane

Volumes

Find the volumes of the solids in Exercises 43–48.

43.



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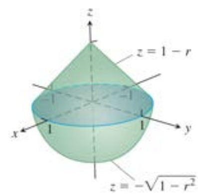

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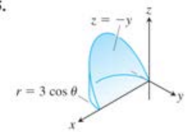
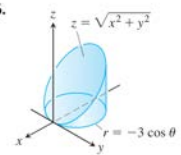


46.



(a) Solve & Find Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = |DT|$

(b) Find image of triangular region

§15.8

Exercises 15.8

Jacobians and Transformed Regions in the Plane

1. a. Solve the system

$$u = x - y, \qquad v = 2x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the xy-plane. Sketch the transformed region in the uv-plane.

2. a. Solve the system

$$u = x + 2y, \qquad v = x - y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x + 2y$, $v = x - y$ of the triangular region in the xy-plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv-plane.

3. a. Solve the system

$$u = 3x + 2y, \qquad v = x + 4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy-plane bounded

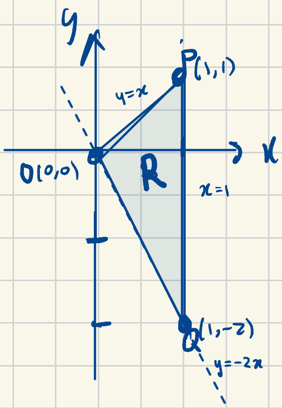
#1 $\begin{cases} u = x - y \\ v = 2x - y \end{cases}$ $\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ so $A^{-1} = \frac{1}{-1+2}\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$T^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

(a) So $\boxed{\begin{cases} x = -u + v \\ y = -2u + v \end{cases}}$ and $|DT| = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = -1 + 2 = \boxed{1}$

(b)



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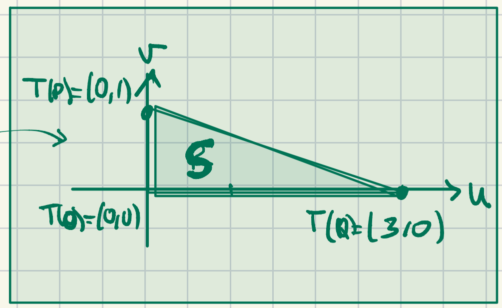


$T(0,0) = (0,0)$

$T(1,1) = (0,1)$

$T(1,-2) = (3,0)$

Area $S = \frac{1}{2} * 1 * 3 = 3/2$

Area $T(R) = |\det A|$ Area $R = 1 * \frac{3}{2}$ ✓

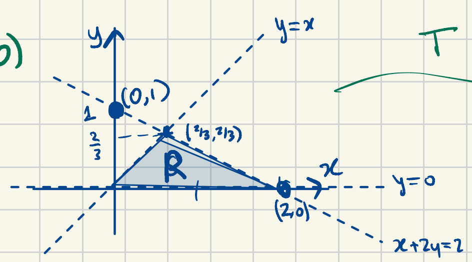
Area $S = \frac{1}{2}bh = \frac{1}{2} 3 \times 1 = 3/2$

#2 (a) $\begin{cases} u = x + 2y \\ v = x - y \end{cases}$ $\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ so $A^{-1} = \frac{1}{-1-2}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$

$\det A^{-1} = \frac{-1}{9} - \frac{2}{9} = -3/9 = -1/3$ ✓

So $\boxed{\begin{cases} x = \frac{1}{3}u + \frac{2}{3}v \\ y = \frac{1}{3}u - \frac{1}{3}v \end{cases}}$ and $|DT| = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = \boxed{-3}$ $\quad T^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 1/3u + 2/3v \\ 1/3u - 1/3v \end{bmatrix}$

(b)



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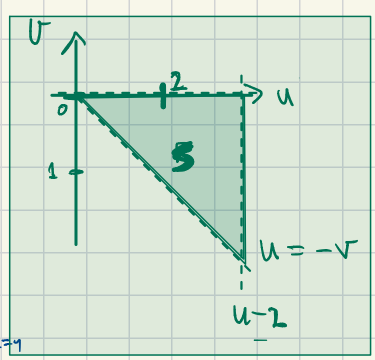


@ $x=y \Rightarrow \cancel{\frac{1}{3}u} + \frac{2}{3}v = \cancel{\frac{1}{3}u} - \frac{1}{3}v \Rightarrow v = 0$

@ $y = 0 \Rightarrow \frac{1}{3}u + \frac{1}{3}v = 0 \Rightarrow u = -v$

@ $x+2y=2 \Rightarrow \frac{1}{3}u + \cancel{\frac{2}{3}v} + \frac{2}{3}u - \cancel{\frac{2}{3}v} = 2 \Rightarrow u = 2$

$S = T(R)$

Area $S = \frac{1}{2} * 4 = 2$

Area $T(R) = |\det A| *$ Area $R = 3 * \frac{2}{3} = 2$

Area $S = \frac{1}{2}$ base $*$ height $= \frac{1}{2} * 2 * \frac{2}{3} = 2/3$

$x + 2y = 2$ @ $x = y$ $\Rightarrow 3y = 2 \Rightarrow y = 2/3$

6. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\,dx\,dy$$

for the region R in the first quadrant bounded by the lines $y=-2x+4$, $y=-2x+7$, $y=x-2$, and $y=x+1$.

#6 R: bounded by

$$\begin{cases} y=-2x+4 \\ y=-2x+7 \\ y=x-2 \\ y=x+1 \end{cases} \Rightarrow \text{@ } x=-u+v \text{ and } y=-2u+v \quad \begin{cases} -2u+v=-2(-u+v)+4 \\ -2u+v=-2(-u+v)+7 \\ -2u+\cancel{v}=-u+\cancel{v}-2 \\ -2u+\cancel{v}=-u+\cancel{v}+1 \end{cases} \Rightarrow \begin{cases} v=\frac{4+4u}{3} \\ v=\frac{7+4u}{3} \\ u=2 \\ u=-1 \end{cases}$$

So $u\in[-1,2]$ and $v\in\left[\frac{4+4u}{3},\frac{7+4u}{3}\right]$

$$V=\int_{-1}^{2}\int_{\frac{4+4u}{3}}^{\frac{7+4u}{3}} \overbrace{2(-u+v)^2-(-u+v)(-2u+v)-(-2u+v)^2}^{f(T(x,y))} * \overbrace{1}^{|\det DT|}\,dv\,du = \int_{-1}^{2}\int_{\frac{4+4u}{3}}^{\frac{7+4u}{3}} 2(\cancel{u^2}-2\cancel{uv}+\cancel{v^2})-(2\cancel{u^2}-3uv+\cancel{v^2})-(4u^2-4\cancel{uv}+\cancel{v^2})\,du\,dv$$

$$=\int_{-1}^{2}\int_{\frac{4+4u}{3}}^{\frac{7+4u}{3}} 3uv-4u^2\,dv\,du=\int_{-1}^{2}\frac{3}{2}uv^2-4u^2v\Big|_{\frac{4+4u}{3}}^{\frac{7+4u}{3}} = \int_{-1}^{2}\left[\frac{3}{2}u\left(\frac{7+4u}{3}\right)^2-4u^2\left(\frac{7+4u}{3}\right)\right]-\left[\frac{3}{2}u\left(\frac{4+4u}{3}\right)^2-4u^2\left(\frac{4+4u}{3}\right)\right]du$$

← * $a^2-b^2=(a+b)(a-b)$

$$=\int_{-1}^{2}\frac{3}{2}u\left[\left(\frac{7+4u}{3}\right)^2-\left(\frac{4+4u}{3}\right)^2\right]-4u^2\left[\frac{7+\cancel{4u}}{3}-\frac{4+\cancel{4u}}{3}\right]du=\int_{-1}^{2}\frac{3}{2}u\left[\left(\frac{7+4u}{3}+\frac{4+4u}{3}\right)\underbrace{\left(\frac{7+\cancel{4u}}{3}-\frac{4+\cancel{4u}}{3}\right)}_{3/3=1}\right]-4u^2\left[\frac{3}{3}\right]du$$

$$=\int_{-1}^{2}\frac{\cancel{3}}{2}u\frac{1}{\cancel{3}}(11+8u)-4u^2\,du=\int_{-1}^{2}\frac{11u}{2}+4u^2-4u^2\,du=\int_{-1}^{2}\frac{11u}{2}\,du=\frac{11u}{4}\Big|_{-1}^{2}=\frac{11}{2}-\left(\frac{-11}{4}\right)=\boxed{\frac{33}{4}}$$

9. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy=1$, $xy=9$ and the lines $y=x$, $y=4x$. Use the transformation $x=u/v$, $y=uv$ with $u>0$ and $v>0$ to rewrite

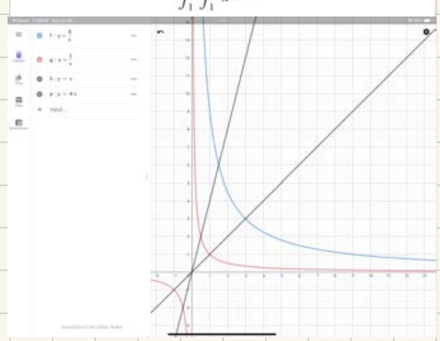
$$\iint_R\left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dx\,dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

10. a. Find the Jacobian of the transformation $x=u$, $y=uv$ and sketch the region G: $1\le u\le 2$, $1\le uv\le 2$, in the uv-plane.

b. Then use Equation (2) to transform the integral

$$\int_1^2\int_1^2 \frac{y}{x}\,dy\,dx$$



#9 $\begin{cases} x=\frac{u}{v} \\ y=uv \end{cases}$ (w/ $u>0$ & $v>0$) R: first quadrant bdd by $xy=1$, $xy=9$, $y=x$, $y=4x$

$xy=\frac{u}{v}*uv=u^2$

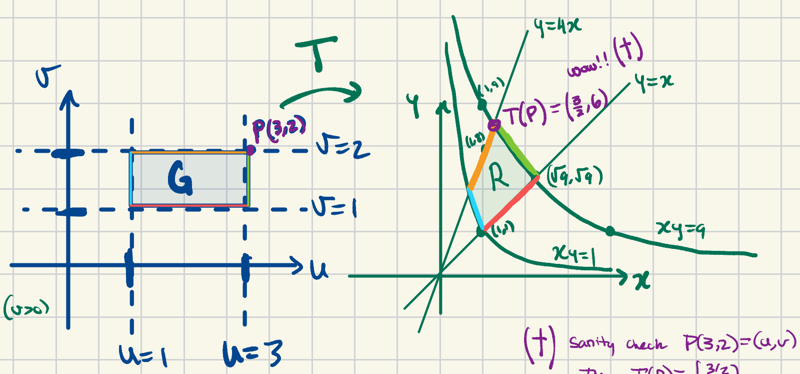
$\frac{y}{x}=\frac{uv}{u/v}=v^2$

@ $xy=1 \Rightarrow u^2=1 \Rightarrow u=1$ $(u>0)$

@ $xy=9 \Rightarrow u^2=9 \Rightarrow u=3$

@ $y=4x \Rightarrow \frac{y}{x}=4 \Rightarrow v^2=4 \Rightarrow v=2$ $(v>0)$

@ $y=x \Rightarrow \frac{y}{x}=1 \Rightarrow v^2=1 \Rightarrow v=1$



(†) Sanity check $P(3,2)=(u,v)$

Then $T(P)=\begin{bmatrix}3/2\\6\end{bmatrix}$

Should be on $xy=9$ ① & $y=4x$ ②

Check $\frac{3}{2}(6)=9$ ✓

$6=4*\frac{3}{2}$ ✓ Wow! cool.

And $T\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}u/v\\uv\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$ And $DT=\begin{bmatrix}x_u & x_v\\ y_u & y_v\end{bmatrix}$

So $DT=\begin{bmatrix}1/v & -u/v^2\\ v & u\end{bmatrix}$ and $|\det DT|=\left|\frac{u}{v}-\frac{-u}{v^{\cancel{2}}}*\cancel{v}\right|=\frac{2u}{v}$. $(u>0, v>0)$.

$u\in[1,3]$, $v\in[1,2]$

And $$\iint_R f(x,y)\,dx\,dy=\iint_G f(T(u,v))\,|\det DT|\,du\,dv=\int_1^3\int_1^2\left(\sqrt{v^2}+\sqrt{u^2}\right)*\frac{2u}{v}\,dv\,du$$

$$=\int_1^3\int_1^2 2u+\frac{2u^2}{v}\,dv\,du=\int_1^3 2uv+2u^2\ln v\Big|_1^2\,du=\int_1^3 2u(2-1)+2u^2(\ln 2-\cancel{\ln 1})\,du=u^2+\frac{2\ln 2}{3}u^3\Big|_1^3$$

$$=(9-1)+\frac{2\ln 2}{3}(3^3-1)=8+18\ln 2-\frac{2}{3}\ln 2=\boxed{8+\frac{52}{3}\ln 2}$$

10 Find Jacobian of T, Sketch G & R, and Integrate

10. a. Find the Jacobian of the transformation $x = u$, $y = uv$ and sketch the region G: $1 \le u \le 2$, $1 \le uv \le 2$, in the uv-plane.
b. Then use Equation (2) to transform the integral
$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx$$
into an integral over G, and evaluate both integrals.

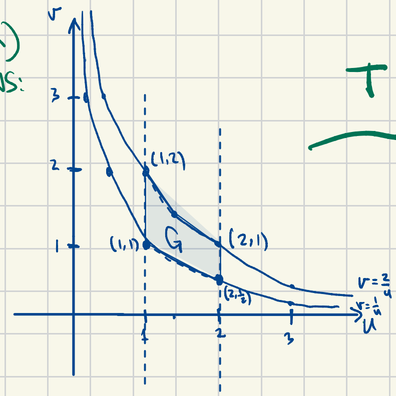
$\begin{cases} x = u \\ y = uv \end{cases}$ & $\begin{matrix} x \in [1,2] \\ y \in [1,2] \end{matrix} \Rightarrow \begin{cases} u = x \\ v = y/x \end{cases}$ & $\begin{matrix} u \in [1,2] \\ v \in [1/u, 2/u] \end{matrix}$ ↗ $\left(\begin{matrix} \text{Since } y = uv \\ @\, y = 1 \Rightarrow 1 = uv \\ @\, y = 2 \Rightarrow 2 = uv \end{matrix} \right)$

(a) ANS:

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u \\ uv \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{So } DT = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & u \end{bmatrix}$$

$$|\det DT| = |u| = u \quad (u > 0)$$



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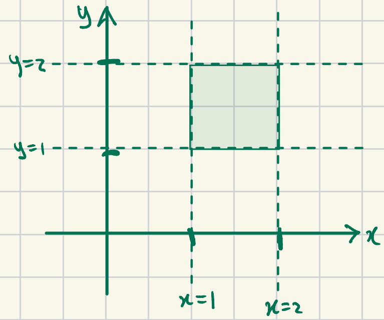


original

(b) $V = \int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx = \int_1^2 \frac{1}{2x} y^2 \Big|_1^2 dx = \int_1^2 \frac{1}{2x}(4-1)\, dx = \int_1^2 \frac{3}{2} \cdot \frac{1}{x}\, dx = \frac{3}{2} \ln(x) \Big|_1^2 = \boxed{\frac{3}{2}\ln 2} - 0$

and $V = \int_1^2 \int_{1/u}^{2/u} v * u\, dv\, du = \int_1^2 \frac{1}{2} u v^2 \Big|_{1/u}^{2/u} du = \int_1^2 \frac{1}{2} \cancel{u} \overbrace{\left(\frac{4}{u^{\cancel{2}}} - \frac{1}{u^{\cancel{2}}}\right)}^{3} = \int_1^2 \frac{3}{2} \cdot \frac{1}{u}\, du = \frac{3}{2} \ln u \Big|_1^2 = \boxed{\frac{3}{2}\ln 2}$ ✓

12. The area of an ellipse The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G: $u^2 + v^2 \le 1$ in the uv-plane. Find the area this way.

12 Evaluate $A = \iint_R 1\, dx\, dy$ where R: bdd by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Use $T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} au \\ bv \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ so G: bdd by $u^2 + v^2 = 1$. <u>Note</u> $DT = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$|\det DT| = ab \quad \left(\begin{matrix} a > 0 \\ b > 0 \end{matrix}\right)$

Then $\iint_R 1\, dx\, dy = \iint_G 1\, |DT| * du\, dv = \iint_G 1\, ab\, du\, dv = ab * \text{Area}\, G$

$= \boxed{ab * \pi}$ ✓

13. Use the transformation in Exercise 2 to evaluate the integral
$$\int_0^{2/3} \int_y^{2-2y} (x + 2y) e^{(y-x)}\, dx\, dy$$
by first writing it as an integral over a region G in the uv-plane.

14. Use the transformation $x = u + (1/2)v$, $y = v$ to evaluate the integral
$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2}\, dx\, dy$$
by first writing it as an integral over a region G in the uv-plane.

13 The transformation from # 2 was

$\begin{cases} u = x + 2y \\ v = x - y \end{cases}$ $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ w/ Jacobian

$$|\det DT| = \left| \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \right| = |-3| = 3$$

And inverse:

$\begin{cases} x = \frac{1}{3}u + \frac{2}{3}v \\ y = \frac{1}{3}u - \frac{1}{3}v \end{cases}$ $T^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 1/3\, u + 2/3\, v \\ 1/3\, u - 1/3\, v \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

So if R: $y \in [0, 2/3]$ $x \in [y, 2-2y]$

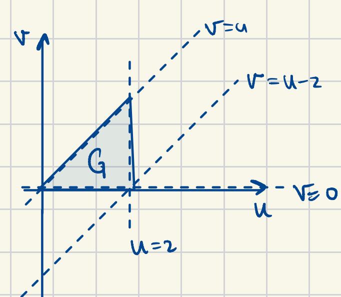
then G: $u \in [0, 2]$, $v \in [0, u]$

$@\, y = 0 \Rightarrow 0 = \frac{1}{3}u - \frac{1}{3}v \Rightarrow v = u$

$@\, y = \frac{2}{3} \Rightarrow \frac{2}{3} = \frac{1}{3}u - \frac{1}{3}v \Rightarrow v = u - 2$

$@\, x = y \Rightarrow \cancel{\frac{1}{3}u} + \frac{2}{3}v = \cancel{\frac{1}{3}u} - \frac{1}{3}v \Rightarrow v = 0$

$@\, x = 2 - 2y \Rightarrow \frac{1}{3}u + \frac{2}{3}v = 2 - \left(\frac{2}{3}u - \frac{2}{3}v\right) \Rightarrow u = 2$



$\int x e^{-x} dx = -xe^{-x} + \int e^{-x} dx$
$= -xe^{-x} - e^{-x} + C$
$= -(1+x)e^{-x} + C$

IBP BOX
$u = x$ $dv = e^{-x} dx$
$du = dx$ $v = -e^{-x}$

$V = \int_0^2 \int_0^u u e^{-v} * \frac{1}{3}\, dv\, du = \int_0^2 -\frac{1}{3} u e^{-v} \Big|_0^u du = \int_0^2 -\frac{1}{3} u e^{-u} + \frac{1}{3} u\, du = -\frac{1}{3}\left(-(u+1)e^{-u} - \frac{1}{2}u^2 \Big|_0^2\right)$

$= -\frac{1}{3}\left[\left(-3e^{-2} - \frac{4}{2}\right) - (-1 - 0)\right] = \frac{-1}{3}\left(-1 - 3e^{-2}\right) = \boxed{3e^{-2} + \frac{1}{3}}$