§16.1

#1 $r(t) = t\hat{\imath} + (1-t)\hat{\jmath}$

$r(0) = \hat{\jmath}$ and $r(1) = \hat{\imath}$, straight line

So $\boxed{c}$

#2 $r(t) = \langle 1, 1, t \rangle$, $-1 \le t \le 1$

$r(-1) = \langle 1, 1, -1 \rangle$ and $r(1) = \langle 1, 1, 1 \rangle$, straight line

So $\boxed{e}$

#3 $r(t) = \langle 2\cos t, 2\sin t, 0 \rangle$, $0 \le t \le 2\pi$

Goes around the circle of radius $r = 2$ in the floor, centered at $(0,0,0)$, one time.

So $\boxed{g.}$

#4 $r(t) = t\hat{\imath}$, $-1 \le t \le 1$

$r(-1) = -\hat{\imath}$ & $r(1) = \hat{\imath}$, straight line

So $\boxed{a}$

#5 $r(t) = \langle t, t, t \rangle$ $0 \le t \le 2$

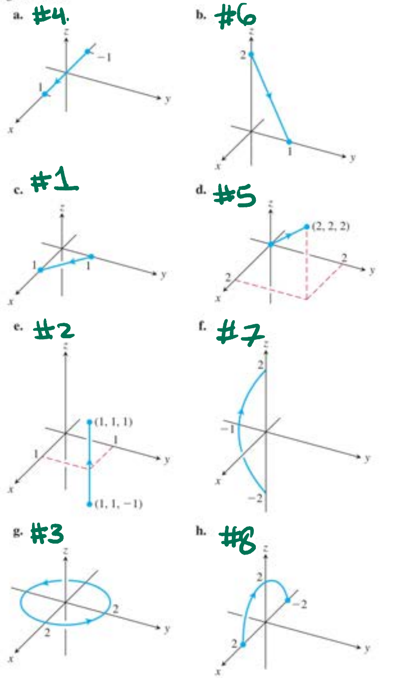
$r(0) = \langle 0, 0, 0 \rangle$ & $r(2) = \langle 2, 2, 2 \rangle$, straight line.

So $\boxed{d}$

Exercises 16.1

Graphs of Vector Equations

Match the vector equations in Exercises 1–8 with the graphs (a)–(h) given here.



1. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1$
2. $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, $-1 \le t \le 1$
3. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$, $0 \le t \le 2\pi$
4. $\mathbf{r}(t) = t\mathbf{i}$, $-1 \le t \le 1$
5. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2$
6. $\mathbf{r}(t) = t\mathbf{j} + (2-2t)\mathbf{k}$, $0 \le t \le 1$
7. $\mathbf{r}(t) = (t^2-1)\mathbf{j} + 2t\mathbf{k}$, $-1 \le t \le 1$
8. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{k}$, $0 \le t \le \pi$

#6 $r(t) = \langle 0, t, 2-2t \rangle$ $0 \le t \le 1$

$r(0) = \langle 0, 0, 2 \rangle$, $r(1) = \langle 0, 1, 0 \rangle$, straight line w/ speed $|r'(t)| = \sqrt{5}$

So $\boxed{b}$

#7 $r(t) = \langle 0, t^2-1, 2t \rangle$ $-1 \le t \le 1$ NOT a straight line.

$r(-1) = \langle 0, 0, -2 \rangle$, $r(1) = \langle 0, 0, 2 \rangle$

So $\boxed{f}$

#8 $r(t) = \langle 2\cos t, 0, 2\sin t \rangle$ $0 \le t \le \pi$, half circle in xz-plane

So $\boxed{h}$

Evaluate

Evaluating Line Integrals over Space Curves

9. Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x = t,\ y = (1-t),\ z = 0$, from $(0, 1, 0)$ to $(1, 0, 0)$.

10. Evaluate $\int_C (x - y + z - 2)\,ds$ where C is the straight-line segment $x = t,\ y = (1-t),\ z = 1$, from $(0, 1, 1)$ to $(1, 0, 1)$.

11. Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2-2t)\mathbf{k},\ 0 \le t \le 1$.

12. Evaluate $\int_C \sqrt{x^2+y^2}\,ds$ along the curve $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k},\ -2\pi \le t \le 2\pi$.

13. Find the line integral of $f(x, y, z) = x + y + z$ over the straight-line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

14. Find the line integral of $f(x, y, z) = \sqrt{3}/(x^2+y^2+z^2)$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 1 \le t \le \infty$.

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|\mathbf{v}(t)|\,dt.$$

#9 $\int_C x+y\,ds$ $\quad C: r(t) = \langle t, 1-t, 0\rangle\ t\in[0,1]$.

$v = r'(t) = \langle 1,-1,0\rangle$

$|v| = \sqrt{2}$

$$= \int_0^1 [t + (1-t)]\sqrt{2}\,dt = \int_0^1 \sqrt{2}\,dt = \sqrt{2}\Big|_0^1 = \boxed{\sqrt{2}}$$

#10 $\int_C x-y+z-2\,ds$ $\quad C: r(t) = \langle t, 1-t, 1\rangle$ so $v = r'(t) = \langle 1,-1,0\rangle,\ |v| = \sqrt{2}$

$t\in[0,1]$

$$= \int_0^1 \left(t - (1-t) + 1 - 2\right)\sqrt{2}\,dt = \int_0^1 (2t-2)\sqrt{2}\,dt = \sqrt{2}t^2 - 2\sqrt{2}t\Big|_0^1 = (\sqrt{2} - 2\sqrt{2}) - 0$$

$$= \boxed{-\sqrt{2}}$$

#12 $\int_C \sqrt{x^2+y^2}\,ds$ $\quad C: r(t) = \langle 4\cos t, 4\sin t, 3t\rangle,\ t\in[-2\pi, 2\pi]$

$v = r'(t) = \langle -4\sin t, 4\cos t, 3\rangle$ $\quad |v|^2 = 16\sin^2 t + 16\cos^2 t + 9 = 25$

So $|v| = 5$

$$= \int_{-2\pi}^{2\pi} \sqrt{16\cos^2 t + 16\sin^2 t} * 5\,dt = \int_{-2\pi}^{2\pi} 20\,dt = 20t\Big|_{-2\pi}^{2\pi} = 20(2\pi - (-2\pi))$$

$$= 20 * 4\pi = \boxed{80\pi}$$

#14 $\int_C \frac{\sqrt{3}}{x^2+y^2+z^2}\,ds$ $\quad C: r(t) = \langle t,t,t\rangle,\ 1 \le t \le \infty$.

$r'(t) = \langle 1,1,1\rangle,\ |v| = \sqrt{3}$

$$= \int_1^\infty \frac{\sqrt{3}}{t^2+t^2+t^2}\sqrt{3}\,dt = \lim_{N\to\infty}\int_1^N \frac{1}{t^2}\,dt = \lim_{N\to\infty} -\frac{1}{t}\Big|_1^N = \lim_{N\to\infty}\left(\frac{-1}{N} - \frac{-1}{1}\right)$$

$$= \lim_{N\to\infty} 1 - \frac{1}{N} = \boxed{1}$$

17. Integrate $f(x, y, z) = (x + y + z)/(x^2 + y^2 + z^2)$ over the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 < a \le t \le b$.

18. Integrate $f(x, y, z) = -\sqrt{x^2+z^2}$ over the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

#17 $\int_C \frac{x+y+z}{x^2+y^2+z^2}\,ds$ $\quad C: r(t) = \langle t,t,t\rangle,\ t\in[a,b]$

$|v| = |r'(t)| = \sqrt{3}$

$$= \int_a^b \frac{t+t+t}{t^2+t^2+t^2}\sqrt{3} = \int_a^b \frac{3t}{3t^2}\sqrt{3}\,dt = \int_a^b \frac{\sqrt{3}}{t}\,dt = \sqrt{3}\ln t\Big|_a^b = \boxed{\sqrt{3}(\ln(b) - \ln(a))}$$

Evaluate

17. Integrate $f(x, y, z) = (x + y + z)/(x^2 + y^2 + z^2)$ over the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 < a \le t \le b$.

18. Integrate $f(x, y, z) = -\sqrt{x^2 + z^2}$ over the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, \quad 0 \le t \le 2\pi.$$

#18 $\displaystyle\int_C -\sqrt{x^2+z^2}\,ds$ C: $r(t) = \langle 0, a\cos t, a\sin t\rangle$ $t \in [0, 2\pi]$

$|v|^2 = |r'(t)|^2 = a^2\sin^2 t + a^2\cos^2 t = a^2$ So $|v| = a$

$\sin t \geq 0,\ t \in [0,\pi]$ $\sin t \leq 0,\ t\in[\pi, 2\pi]$ $\sin(t+\pi) = -\sin t$

$$= \int_0^{2\pi} -\sqrt{0 + a^2\sin^2 t} * a\,dt = \int_0^{\pi} -a^2 \sin t\,dt + \int_\pi^{2\pi} a^2 \sin t\,dt = \int_0^{\pi} -2a^2\sin t\,dt$$

$$= 2a^2 \cos t \Big|_0^{\pi} = 2a^2(-1-1) = \boxed{-4a^2}$$

Line Integrals over Plane Curves

19. Evaluate $\int_C x\,ds$, where C is

a. the straight-line segment $x = t$, $y = t/2$, from $(0, 0)$ to $(4, 2)$.

b. the parabolic curve $x = t$, $y = t^2$, from $(0, 0)$ to $(2, 4)$.

20. Evaluate $\int_C \sqrt{x + 2y}\,ds$, where C is

a. the straight-line segment $x = t$, $y = 4t$, from $(0, 0)$ to $(1, 4)$.

b. $C_1 \cup C_2$; C_1 is the line segment from $(0, 0)$ to $(1, 0)$ and C_2 is the line segment from $(1, 0)$ to $(1, 2)$.

21. Find the line integral of $f(x, y) = ye^{x^2}$ along the curve $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j}$, $-1 \le t \le 2$.

22. Find the line integral of $f(x, y) = x - y + 3$ along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

23. Evaluate $\displaystyle\int_C \frac{x^2}{y^{4/3}}\,ds$, where C is the curve $x = t^2$, $y = t^3$, for $1 \le t \le 2$.

24. Find the line integral of $f(x, y) = \sqrt{y}/x$ along the curve $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $1/2 \le t \le 1$.

#19 $\displaystyle\int_C x\,ds$ $C_1: r_1(t) = \langle t, t/2\rangle$ $0 \le t \le 4$

$C_2: r_2(t) = \langle t, t^2\rangle$ $0 \le t \le 2$

$|v_1| = \sqrt{5}/2$

$|v_2| = \sqrt{1+4t^2}$

(a) $\displaystyle\int_0^4 t * \frac{\sqrt5}{2}\,dt = \frac{\sqrt5}{4}t^2\Big|_0^4 = \boxed{4\sqrt5}$

(b) $\displaystyle\int_0^2 t\sqrt{1+4t^2}\,dt = \int_1^{17} \frac{1}{8}u^{1/2}\,du$

$u = 1+4t^2$
$du = 8t\,dt$
$\frac{1}{8}du = dt$

$t=0 \Rightarrow u=1$ & $t=2 \Rightarrow u=17$

$$= \frac{1}{8} * \frac{2}{3}u^{3/2}\Big|_1^{17} = \boxed{\frac{1}{12}17^{3/2} - \frac{1}{12}}$$

$r'(t) = \langle 1, 4\rangle$, $|v| = \sqrt{17}$

#20 $\displaystyle\int_C \sqrt{x+2y}\,ds$ $C: r(t) = \langle t, 4t\rangle$ $t\in[0,1]$

$$= \int_0^1 \sqrt{t+8t}\,\sqrt{17}\,dt = \int_0^1 3\sqrt{17}\,t^{1/2}\,dt = \cancel{3}\sqrt{17} * \frac{2}{\cancel{3}}t^{3/2}\Big|_0^1 = \boxed{2\sqrt{17}}$$

#21 $\displaystyle\int_C ye^{x^2}\,ds$ $C: \langle 4t, -3t\rangle$ $t\in[-1,2]$ $r'(t) = \langle 4, -3\rangle$

$|v| = \sqrt{16+9} = 5$

$$= \int_{-1}^{2} -3t e^{16t^2} * 5\,dt = \int_{16}^{64} \frac{-15}{32}e^u\,du = \frac{-15}{32}e^u\Big|_{16}^{64}$$

$u = 16t^2$
$du = 32t\,dt$
$\frac{1}{32}du = t\,dt$

$t=-1 \Rightarrow u=16$
$t=2 \Rightarrow u=64$

$$= \boxed{\frac{-15}{32}\left(e^{64} - e^{16}\right)}$$

§16.2

Exercises 16.2

Vector Fields

Find the gradient fields of the functions in Exercises 1–4.

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
2. $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$
3. $g(x, y, z) = e^z - \ln(x^2 + y^2)$
4. $g(x, y, z) = xy + yz + xz$
5. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the property that $\mathbf{F}$ points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to the origin. (The field is not defined at $(0, 0)$.)
6. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0, 0)$ and that at any other point (a, b), $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

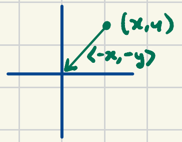
#2 Find $F = \nabla f$

$f(x,y,z) = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2)$

So $\nabla f = \left\langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \right\rangle$

and so gradient field is $\boxed{F(x,y,z) = \left\langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \right\rangle}$

#5 $F(x,y)$ should point towards the origin.



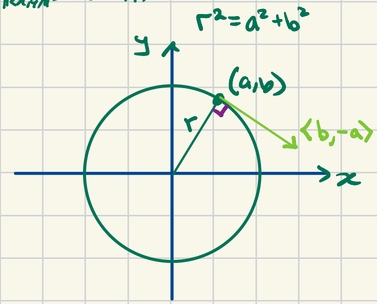
Try $F_1(x,y) = -\langle x,y\rangle = \langle -x,-y\rangle$

and $\|\langle x,y\rangle\| = \sqrt{x^2+y^2}$

So $F_2(x,y) = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector pointing towards the origin.

To make $\|F(x,y)\|$ the reciprocal of $\|\langle x,y\rangle\|^2$ we need to multiply $\frac{1}{\|\langle x,y\rangle\|^2} * F_2(x,y)$.

So $\boxed{F(x,y) = \left\langle \frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}} \right\rangle}$



#6 Based on diagram we can pick (and check).

$F(x,y) = \langle y, -x\rangle$

(1) $F(0,0) = \vec{0}$ ✓

(2) $F(a,b) = \langle b,-a\rangle$ is tangent to $x^2+y^2=r^2$ at the point (a,b), where $r = \|\langle a,b\rangle\| = \sqrt{a^2+b^2}$ ✓

(3) $\|F(a,b)\| = \|\langle b,-a\rangle\| = \sqrt{b^2+a^2}$ ✓

Line Integrals of Vector Fields

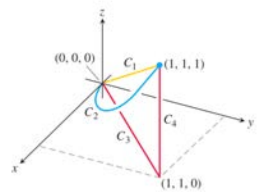
In Exercises 7–12, find the line integrals of $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \le t \le 1$

c. The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$

7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$
9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$
10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
11. $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$
12. $\mathbf{F} = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$



Compute Work done (ie line integral of f)

$$\boxed{\text{Work} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

a. $r_1(t) = \langle t,t,t\rangle$, $t\in[0,1]$

$r_1'(t) = \langle 1,1,1\rangle$

b. $r_2(t) = \langle t,t^2,t^4\rangle$, $t\in[0,1]$

$r_2'(t) = \langle 1,2t,4t^3\rangle$

#7 $F(x,y,z) = \langle 3y, 2x, 4z\rangle$

(a) $\text{Work} = \int_0^1 \langle 3t,2t,4t\rangle \cdot \langle 1,1,1\rangle\, dt$

$= \int_0^1 3t + 2t + 4t\, dt = \int_0^1 9t\, dt = \frac{9}{2}t^2 \Big|_0^1 = \boxed{9/2}$

(b) $\text{Work} = \int_0^1 \langle 3t^2, 2t, 4t^4\rangle \cdot \langle 1, 2t, 4t^3\rangle\, dt$

$= \int_0^1 3t^2 + 4t^2 + 16t^7\, dt = \int_0^1 7t^2 + 16t^7\, dt = \frac{7}{3}t^3 + 2t^8 \Big|_0^1 = \left(\frac{7}{3}+2\right) - 0 = \boxed{13/3}$

Compute Work done (ie line integral of f)

$$\boxed{\text{Work} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

a. $r_1(t) = \langle t, t, t \rangle,\ t \in [0,1]$

$r_1'(t) = \langle 1, 1, 1 \rangle$

b. $r_2(t) = \langle t, t^2, t^4 \rangle,\ t \in [0,1]$

$r_2'(t) = \langle 1, 2t, 4t^3 \rangle$

Line Integrals of Vector Fields

In Exercises 7–12, find the line integrals of $\mathbf{F}$ from (0, 0, 0) to (1, 1, 1) over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \le t \le 1$

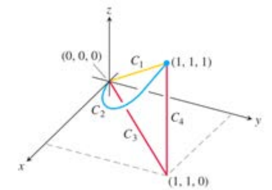
c. The path $C_3 \cup C_4$ consisting of the line segment from (0, 0, 0) to (1, 1, 0) followed by the segment from (1, 1, 0) to (1, 1, 1)

7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$ **8.** $\mathbf{F} = [1/(x^2+1)]\mathbf{j}$

9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$ **10.** $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

11. $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$

12. $\mathbf{F} = (y+z)\mathbf{i} + (z+x)\mathbf{j} + (x+y)\mathbf{k}$



#8 $F(x,y,z) = \langle 0, \frac{1}{1+x^2}, 0 \rangle$

a. $\text{Work} = \int_0^1 \langle 0, \frac{1}{1+t^2}, 0 \rangle \cdot \langle 1,1,1 \rangle\, dt = \int_0^1 \frac{1}{1+t^2}\, dt$

$= \tan^{-1}(t) \Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0$

$= \boxed{\frac{\pi}{4}}$

$x(t) = t$

b. $\text{Work} = \int_0^1 \langle 0, \frac{1}{1+t^2}, 0 \rangle \cdot \langle 1, 2t, 4t^3 \rangle\, dt$

$= \int_0^1 \frac{1}{1+t^2} * 2t\, dt = \int_1^2 \frac{1}{u}\, du = \ln u \Big|_1^2 = \ln 2 - \ln 1 = \boxed{\ln 2}$

#10 $F(x,y,z) = \langle xy, yz, xz \rangle$

a. $\text{Work} = \int_0^1 \langle t^2, t^2, t^2 \rangle \cdot \langle 1,1,1 \rangle\, dt = \int_0^1 3t^2\, dt = t^3 \Big|_0^1 = \boxed{1}$

b. $\text{Work} = \int_0^1 \langle \underset{}{\overset{x*y}{t * t^2}}, \overset{y*z}{t^2 * t^4}, \overset{x*z}{t * t^4} \rangle \cdot \langle 1, 2t, 4t^3 \rangle\, dt = \int_0^1 t^3 + 2t^7 + 4t^8\, dt$

$= \frac{1}{4}t^4 + \frac{1}{4}t^8 + \frac{4}{9}t^9 \Big|_0^1 = \left(\frac{1}{4} + \frac{1}{4} + \frac{4}{9}\right) - 0 = \frac{1}{2} + \frac{4}{9} = \frac{9+8}{18} = 27/18 = \boxed{3/2}$

#12 $F(x,y,z) = \langle x+z, z+x, x+y \rangle$

a. $\text{Work} = \int_0^1 \langle t+t, t+t, t+t \rangle \cdot \langle 1,1,1 \rangle\, dt = \int_0^1 6t\, dt = 3t^2 \Big|_0^1 = \boxed{1}$

b. $\text{Work} = \int_0^1 \langle t + t^4, t^4 + t, t + t^2 \rangle \cdot \langle 1, 2t, 4t^3 \rangle\, dt = \int_0^1 (t+t^4) + 2t(t+t^4) + 4t^3(t+t^2)\, dt$

$= \int_0^1 t + t^4 + 2t^2 + 2t^5 + 4t^4 + 4t^5\, dt = \int_0^1 t + 2t^2 + 5t^4 + 6t^5\, dt$

$= \frac{1}{2}t^2 + \frac{2}{3}t^3 + t^5 + t^6 \Big|_0^1 = \frac{1}{2} + \frac{2}{3} + 1 + 1 = 2 + \frac{3+4}{6} = \boxed{19/6}$

Work

In Exercises 19–22, find the work done by **F** over the curve in the direction of increasing t.

19. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$
 $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$

20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$
 $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$

21. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
 $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2\pi$

22. $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$
 $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$

Find Work done by F over r

#19 $F(x,y,z) = \langle xy, y, -yz \rangle$

$r(t) = \langle t, t^2, t \rangle \quad , t \in [0,1]$

$r'(t) = \langle 1, 2t, 1 \rangle$

$$\boxed{\text{Work} = \int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt}$$

$$\text{Work} = \int_0^1 \langle t \cdot t^2, t^2, -t^2 \cdot t \rangle \cdot \langle 1, 2t, 1 \rangle\, dt = \int_0^1 t^3 + 2t^3 - t^3\, dt = \int_0^1 2t^3\, dt$$

$$= \frac{1}{2} t^4 \Big|_0^1 = \frac{1}{2} - 0 = \boxed{1/2}$$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$\sin^2 t = \frac{1-\cos 2t}{2}$

#20 $F(x,y,z) = \langle 2y, 3x, x+y \rangle \quad , \quad r(t) = \langle \cos t, \sin t, t/6 \rangle \quad , \quad 0 \le t \le 2\pi$

$r'(t) = \langle -\sin t, \cos t, 1/6 \rangle$

$$\text{Work} = \int_0^{2\pi} \langle 2\sin t, 3\cos t, \cos t + \sin t \rangle \cdot \langle -\sin t, \cos t, 1/6 \rangle\, dt = \int_0^{2\pi} -2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\, dt$$

$$= \int_0^{2\pi} -2\left(\frac{1-\cos 2t}{2}\right) + 3\left(\frac{1+\cos 2t}{2}\right) + \frac{1}{6}\cos 2t + \frac{1}{6}\sin 2t\, dt = \int_0^{2\pi} -1 + \cos 2t + \frac{3}{2} + \frac{3}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\, dt$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\, dt = \frac{1}{2}t + \frac{5}{4}\sin 2t + \frac{1}{6}\sin t - \frac{1}{6}\cos t \Big|_0^{2\pi}$$

$$= \left(\frac{1}{2}(2\pi) + \frac{5}{4}\cancelto{0}{\sin 4\pi} + \frac{1}{6}\cancelto{0}{\sin 2\pi} - \frac{1}{6}\cos t\right) - \left(\frac{1}{2}(0) + \frac{5}{4}\cancelto{0}{\sin 0} + \frac{1}{6}\cancelto{0}{\sin 0} - \frac{1}{6}\cos 0\right) = \pi - \frac{1}{6} + \frac{1}{6} = \boxed{\pi}$$

u-sub
$u = \cos t$
$du = -\sin t\, dt$

#22 $F(x,y,z) = \langle 6z, y^2, 12x \rangle \quad , \quad r(t) = \langle \sin t, \cos t, t/6 \rangle \quad , \quad 0 \le t \le 2\pi$

$r'(t) = \langle \cos t, -\sin t, 1/6 \rangle$

$$\text{Work} = \int_0^{2\pi} \langle 1, \cos^2 t, 12\sin t \rangle \cdot \langle \cos t, -\sin t, 1/6 \rangle\, dt = \int_0^{2\pi} \cos t - \sin t \cos^2 t + 2\sin t\, dt$$

$$= \sin t - \frac{1}{3}\cos^3 t - 2\cos t \Big|_0^{2\pi} = \left(\cancelto{0}{\sin 2\pi} - \frac{1}{3}\cos^3 2\pi - 2\cos 2\pi\right) - \left(\cancelto{0}{\sin 0} - \frac{1}{3}\cos^3 0 - 2\cos 2\pi\right)$$

$$= -\frac{1}{3} - 2 + \frac{1}{3} + 2 = \boxed{0}$$

Evaluate the line integral along C

Line Integrals in the Plane

23. Evaluate $\int_C xy\,dx + (x+y)\,dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

24. Evaluate $\int_C (x-y)\,dx + (x+y)\,dy$ counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

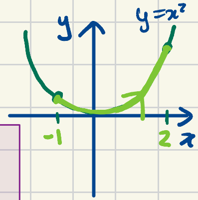
25. Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ for the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(4, 2)$ to $(1, -1)$.

26. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

#23 $\quad F = \langle xy,\ x+y\rangle$

$r(t) = \langle t, t^2\rangle,\ t\in[-1,2]$

$$\boxed{\int_C P\,dx + Q\,dy = \int_C F\cdot dr,\quad F = \langle P, Q\rangle}$$



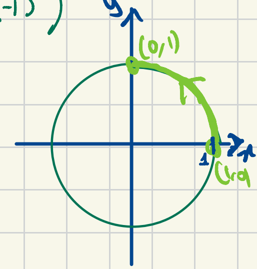
$r'(t) = \langle 1, 2t\rangle$

$$\begin{array}{r} 18 \\ *\ \ 4 \\ \hline 40+32=72 \end{array}$$

$$\int_C xy\,dx + (x+y)\,dy = \int_{-1}^{2} \langle t*t^2,\ t+t^2\rangle\cdot\langle 1, 2t\rangle\,dt = \int_{-1}^{2} t^3 + 2t^2 + 2t^3\,dt$$

$$= \int_{-1}^{2} 3t^3 + 2t^2\,dt = \frac{3}{4}t^4 + \frac{2}{3}t^3 \Big|_{-1}^{2} = \left(\frac{3}{4}(2)^4 + \frac{2}{3}(2)^3\right) - \left(\frac{3}{4}(-1)^4 + \frac{2}{3}(-1)^3\right)$$

$$= 12 + \frac{16}{3} - \frac{3}{4} + \frac{2}{3} = 18 - \frac{3}{4} = \boxed{69/4}$$



#26 $\quad F = \langle y, -x\rangle,\ r(t) = \langle \cos t, \sin t\rangle \quad t\in[0, \pi/2]$

$r'(t) = \langle \sin t, -\cos t\rangle$

$$\int F\cdot dr = \int_0^{\pi/2} \langle \sin t, -\cos t\rangle\cdot\langle \sin t, -\cos t\rangle\,dt = \int_0^{\pi/2} \sin^2 t + \cos^2 t\,dt = \int_0^{\pi/2} 1\,dt$$

$$= t\Big|_0^{\pi/2} = \pi/2 - 0 = \boxed{\pi/2}$$

Find Work done by F over C.

Work, Circulation, and Flux in the Plane

27. **Work** Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from (1, 1) to (2, 3).

28. **Work** Find the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from (2, 0) to itself.

29. **Circulation and flux** Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

a. The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

30. **Flux across a circle** Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

#28 $F = \nabla f$ where $f(x,y) = (x+y)^2$

$r(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi$

✱ (By FToLI (§16.3) $\int_C F \cdot dr = f(0) - f(0) = 0 - 0 = \boxed{0}$)

$F = \nabla f = \langle 2(x+y), 2(x+y) \rangle$

$r'(t) = \langle -\sin t, \cos t \rangle$

$$W = \int_0^{2\pi} \langle 2(\cos t + \sin t), 2(\cos t + \sin t) \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} -2\cancel{\sin t\cos t} - 2\sin^2 t + 2\cos^2 t + 2\cancel{\sin t\cos t} = \int_0^{2\pi} -2\left(\frac{1-\cos 2t}{2}\right) + 2\left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \int_0^{2\pi} -1 + \cos 2t + 1 + \cos 2t \, dt = \int_0^{2\pi} 2\cos 2t \, dt = \sin 2t \Big|_0^{2\pi} = \sin 4\pi - \sin 0 = 0 - 0 = \boxed{0}$$

✱ had to be zero by FToLI (§16.3)

Find Circulation and Flux

$F_1 = \langle x, y \rangle$ and $F_2 = \langle -y, x \rangle$

#29 a. $r(t) = \langle \cos t, \sin t \rangle \quad t \in [0, 2\pi]$; $r'(t) = \langle -\sin t, \cos t \rangle$

$n \sim \langle \cos t, \sin t \rangle$

Note $F_1 = \nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 + C\right)$ so by FToLI ✱

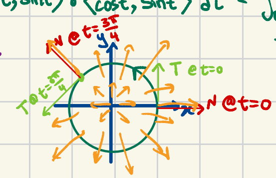
a. ① $\text{Flow} = \int_C F_1 \cdot dr = \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle \sin t, -\cos t \rangle \, dt = \int_0^{2\pi} \sin t\cos t - \sin t\cos t \, dt$

$= \int_0^{2\pi} 0 \, dt = \boxed{0}$

$\text{Flux} = \int_C F_1 \cdot n \, ds = \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle \cos t, \sin t \rangle \, dt = \int_0^{2\pi} \cos^2 t + \sin^2 t \, dt$

outward pointing $\vec{n}$

$= \int_0^{2\pi} 1 \, dt = t \Big|_0^{2\pi} = 2\pi - 0 = \boxed{2\pi}$



② $\text{Flow} = \int_C F_2 \cdot dr = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt$

$= \int_0^{2\pi} 1 \, dt = t \Big|_0^{2\pi} = 2\pi - 0 = \boxed{2\pi}$

$\text{Flux} = \int_C F_2 \cdot n \, ds = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle \cos t, \sin t \rangle \, dt = \int_0^{2\pi} -\sin t\cos t + \sin t\cos t \, dt$

feel like something is going on here... maybe?

$= \int_0^{2\pi} 0 \, dt = \boxed{0}$

Work, Circulation, and Flux in the Plane

27. **Work** Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from (1, 1) to (2, 3).

28. **Work** Find the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from (2, 0) to itself.

29. **Circulation and flux** Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

a. The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

30. **Flux across a circle** Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

#29 b. (cont.)

Note $F_1 = \nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 + C\right)$

$F_1 = \langle x, y \rangle$ and $F_2 = \langle -y, x \rangle$

$r(t) = \langle \cos t, 4\sin t \rangle \quad t \in [0, 2\pi]$

$r'(t) = \langle -\sin t, 4\cos t \rangle, \quad n \sim \langle 4\cos t, \sin t \rangle$

① $\text{Flow} = \int_C F_1 \cdot dr = \int_0^{2\pi} \langle \cos t, 4\sin t \rangle \cdot \langle -\sin t, 4\cos t \rangle dt$

$= \int_0^{2\pi} -\sin t\cos t + 16\sin t\cos t \, dt = \int_0^{2\pi} 15 \sin t \cos t \, dt$

$u = \sin t$
$du = \cos t \, dt$

$= \frac{15}{2}\sin^2 t \Big|_0^{2\pi} = \frac{1}{2}\sin^2(2\pi) - \frac{1}{2}\sin^2 0 = 0 - 0 = \boxed{0}$

* had to happen by FToLI. since $F = \nabla f$ & C closed.

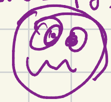
$\text{Flux} = \int_C F_1 \cdot dr = \int_0^{2\pi} \langle \cos t, 4\sin t \rangle \cdot \langle 4\cos t, \sin t \rangle dt = \int_0^{2\pi} 4\cos^2 t + 4\sin^2 t \, dt$

$= \int_0^{2\pi} 4 \, dt = 4t\Big|_0^{2\pi} = 8\pi - 0 = \boxed{8\pi}$

② $\text{Flow} = \int_C F_2 \cdot dr = \int_0^{2\pi} \langle -4\sin t, \cos t \rangle \cdot \langle -\sin t, 4\cos t \rangle dt = \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t \, dt$

$= \int_0^{2\pi} 4 \, dt = 4t\Big|_0^{2\pi} = 8\pi - 0 = \boxed{8\pi}$

Waaait a minute...
(looks at previous page)
?!..



Some kind of dot product magic...

$\text{Flux} = \int_C F_2 \cdot n \, ds = \int_0^{2\pi} \langle -4\sin t, \cos t \rangle \cdot \langle 4\cos t, \sin t \rangle dt$

$= \int_0^{2\pi} -4\sin t\cos t + \sin t \cos t \, dt = \int_0^{2\pi} -3\sin t \cos t \, dt = -\frac{3}{2}\sin^2 t\Big|_0^{2\pi}$

$= -\frac{3}{2}\left(\cancelto{0}{\sin 2\pi} - \cancelto{0}{\sin 0}\right) = \boxed{0}$

Something like
$T: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by 180°
& $T(F, C) = \int_C T(F) \cdot dr$

Then $T'(F, C) + T^{\circ}(F, C) = 0$?

Find circulation

#38 b. $F(x,y)=\langle y,\ x+2y\rangle$, $r(t)=\langle \cos t, \sin t\rangle$ $t\in[0,2\pi]$

$r'(t)=\langle -\sin t, \cos t\rangle$

Flow $= \int_C F\cdot dr = \int_0^{2\pi} \langle \sin t, \cos t+2\sin t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt$

$= \int_0^{2\pi} -\sin^2 t + \cos^2 t + 2\sin t\cos t \, dt$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$\cos^2 t = \frac{1+\cos 2t}{2}$

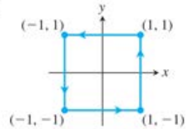
$2\sin t\cos t = \sin(2t)$

$= \int_0^{2\pi} \cos 2t + \sin 2t \, dt = \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \Big|_0^{2\pi}$

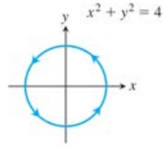
$= \left(\frac{1}{2}\cancelto{0}{\sin 4\pi} - \frac{1}{2}\cancelto{1}{\cos 4\pi}\right) - \left(\frac{1}{2}\cancelto{0}{\sin 0} - \frac{1}{2}\cancelto{1}{\cos 0}\right) = -\frac{1}{2} + \frac{1}{2} = \boxed{0}$

38. Find the circulation of the field $\mathbf{F} = y\mathbf{i} + (x+2y)\mathbf{j}$ around each of the following closed paths.

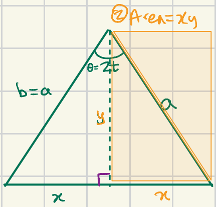
a.



b.



c. Use any closed path different from parts (a) and (b).



① Area = ab sin θ ✓ so,

Area $= \frac{1}{2} a \cdot a \cdot \sin 2t$

$x = a\sin t$

$y = a\cos t$

② Area $= x \cdot y = a\sin t \, a\cos t$

$\Rightarrow \frac{1}{2}a^2 \sin 2t = a^2 \sin t\cos t$

$\Rightarrow \sin 2t = 2\sin t\cos t$ ✓

Which vector fields are conservative?

§16.3

Exercises 16.3

Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

1. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
2. $\mathbf{F} = (y\sin z)\mathbf{i} + (x\sin z)\mathbf{j} + (xy\cos z)\mathbf{k}$
3. $\mathbf{F} = y\mathbf{i} + (x+z)\mathbf{j} - y\mathbf{k}$
4. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
5. $\mathbf{F} = (z+y)\mathbf{i} + z\mathbf{j} + (y+x)\mathbf{k}$
6. $\mathbf{F} = (e^x\cos y)\mathbf{i} - (e^x\sin y)\mathbf{j} + z\mathbf{k}$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$

THM: F conservative

① iff $\int_C F\cdot dr$ path indep.

② iff $F = \nabla f$

③ iff $\oint_C F\cdot dr = 0$

$F = \langle M, N, P\rangle$ conservative on simply connected D

④ iff $P_y = N_z$, $M_z = P_x$, $N_x = M_y$

#1 $F(x,y,z) = \langle yz, xz, xy\rangle$

Notice $F = \nabla f$ where $f(x,y,z) = xyz$. Conservative by ②

Can also check ③ since $\mathbb{R}^3$ is simply connected.

$F = \langle M, N, P\rangle$, $M = yz$, $N = xz$, $P = xy$

$M_y = z$ $N_x = z$ $P_x = y$

$M_z = y$ $N_z = x$ $P_y = x$

And $M_y = N_x$ ✓ $M_z = P_x$ ✓ $N_z = P_y$ ✓

So Conservative on $\mathbb{R}^3$ by ④

#2 $F(x,y,z) = \langle y\sin z, x\sin z, xy\cos z\rangle$

If $f = xy\sin z$ then $\nabla f = F$

So F is conservative by ②

#3 $F(x,y,z) = \langle y, x+z, -y\rangle$

Show $\oint_C F\cdot dr \neq 0$ for some closed loop

e.g. $C: r(t) = \langle \cos t, 0, \sin t\rangle$ $0 \le t \le 2\pi$

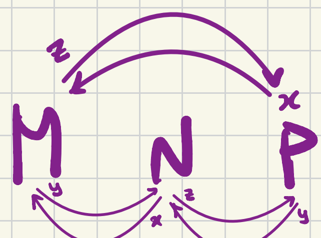
Then $\oint_C F\cdot dr = \int_0^{2\pi} \langle 0, \cos t + \sin t, 0\rangle \cdot \langle -\sin t, 0, \cos t\rangle\, dt$

$= \int_0^{2\pi} 0\, dt = 0$ dang it. Try $C: r(t) = \langle \cos t, \sin t, 0\rangle$

then $\oint_C F\cdot dr = \int_0^{2\pi} \langle \sin t, \cos t, -\sin t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$

$= \int_0^{2\pi} -\sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} 2\cos^2 t - 1\, dt = \int_0^{2\pi} 1 + \cos 2t - 1\, dt$

$= \int_0^{2\pi} \cos 2t\, dt = -\frac{1}{2}\sin 2t\Big|_0^{2\pi} = \frac{-1}{2}(\sin 4\pi - \sin 0) = 0 - 0 = 0$ ☹



$1 = \sin^2 t + \cos^2 t$

$\Rightarrow -\sin^2 t = \cos^2 t - 1$

$\cos^2 t = \frac{1 + \cos 2t}{2}$

§16.3

Exercises 16.3

Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

1. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
2. $\mathbf{F} = (y\sin z)\mathbf{i} + (x\sin z)\mathbf{j} + (xy\cos z)\mathbf{k}$
3. $\mathbf{F} = y\mathbf{i} + (x+z)\mathbf{j} - y\mathbf{k}$
4. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
5. $\mathbf{F} = (z+y)\mathbf{i} + z\mathbf{j} + (y+x)\mathbf{k}$
6. $\mathbf{F} = (e^x\cos y)\mathbf{i} - (e^x\sin y)\mathbf{j} + z\mathbf{k}$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$

Decide if F is conservative or not.

#3 $F(x,y,z) = \langle y, x+z, -y \rangle$

Show $\oint_C F\cdot dr \neq 0$ for some closed loop C.

Try $C: r(t) = \langle 0, \cos t, \sin t \rangle$, $t \in [0, 2\pi]$.

Then $\oint_C F\cdot dr = \int_0^{2\pi} \langle \cos t, \sin t, -\cos t \rangle \cdot \langle 0, -\sin t, \cos t \rangle \, dt$

$$= \int_0^{2\pi} -\sin^2 t - \cos^2 t \, dt = \int_0^{2\pi} (-1)\, dt = -1 \Big|_0^{2\pi} = \underline{\underline{-2\pi}}$$

Since $\oint_C F\cdot dr$ for C a closed loop,

F is $\boxed{\text{NOT conservative by ③}}$

THM: F conservative

① iff $\int_C F\cdot dr$ path indep.

② iff $F = \nabla f$

③ iff $\oint_C F\cdot dr = 0$

#5 $F(x,y,z) = \langle z+y, z, y+x \rangle$

Curl test: $\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z+y & z & y+x \end{vmatrix} = \langle 1-1, -1+1, 0-1 \rangle$

$\boxed{\text{By ④ Curl test, F is } \underline{\underline{\text{NOT}}} \text{ conservative.}}$ $= \langle 0, 0, -1 \rangle \neq \vec{0}$

Curl test: $F = \langle P, Q, R \rangle$ conservative on simply connected D

④ iff $\text{Curl}(F) = \langle 0,0,0 \rangle$

where $\text{Curl } F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

iff $Q_z = R_y$
$P_z = R_x$
$Q_x = P_y$

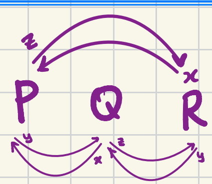
Can verify ③ using a loop contained in $(\text{Span Curl}(F))^\perp$

$r(t) = \langle \cos t, \sin t, 0 \rangle$ $t \in [0, 2\pi]$, $r'(t) = \langle -\sin t, \cos t, 0 \rangle$

$$\oint_C F\cdot dr = \int_0^{2\pi} \langle \sin t, 0, \cos t + \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t \, dt = \int_0^{2\pi} -\frac{1-\cos 2t}{2} \, dt$$

$$= \int_0^{2\pi} -\frac{1}{2} + \frac{1}{2}\cos 2t \, dt = -\frac{1}{2}t + \frac{1}{4}\sin 2t \Big|_0^{2\pi} = \underline{\underline{-\pi \neq 0}}$$ ✓



#6 $F(x,y,z) = \langle e^x \cos y, -e^x \sin y, z \rangle = \langle P, Q, R \rangle$

Curl test: $\text{Curl } F = \langle R_y - Q_z, -R_x + P_z, Q_x - P_y \rangle$

$$= \langle 0-0, 0-0, -e^x \sin y - (-e^x \sin y) \rangle = \langle 0,0,0 \rangle$$ ✓

$\boxed{\text{by ④ Curl test, F is conservative}}$

⊛ For 2D-vector fields Curl test is just $Q_x = P_y$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$
9. $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$
10. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
11. $\mathbf{F} = (\ln x + \sec^2(x+y))\mathbf{i} + \left(\sec^2(x+y) + \frac{y}{y^2+z^2}\right)\mathbf{j} + \frac{z}{y^2+z^2}\mathbf{k}$
12. $\mathbf{F} = \frac{y}{1+x^2y^2}\mathbf{i} + \left(\frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}\right)\mathbf{j} + \left(\frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}\right)\mathbf{k}$

#6 cont. $F(x,y,z) = \langle e^x \cos y, -e^x \sin y, z \rangle = \langle P, Q, R \rangle$

Can verify using ② and find potential function f such that $\nabla f = \langle f_x, f_y, f_z \rangle = F = \langle P, Q, R \rangle$

Start with $f_x = e^x \cos y$

$\Rightarrow f = \int e^x \cos y \, dx = e^x \cos y + C(y,z)$

So $\frac{\partial}{\partial y} f = -e^x \sin y + C_y(y,z) = Q = -e^x \sin y$

So $C_y(y,z) = 0$ and $C(y,z) = C(z)$ only

Now

$\frac{\partial}{\partial z} f = 0 + C_z = z \Rightarrow C(z) = \frac{1}{2} z^2 + C.$ So $\underline{f(x,y,z) = e^x \cos y + \frac{1}{2} z^2 + C}$ ✓

Find potential function f s.t. $\nabla f = F$

#8 $F(x,y,z) = \langle y+z, x+z, x+y \rangle = \langle P, Q, R \rangle$

$f_x = P \Rightarrow f = \int y+z \, dx = (y+z)x + C(y,z)$

So $f_y = x + C_y(y,z) = Q = x+z$. So $C_y(y,z) = z$ and $C(y,z) = \int z \, dy = yz + C(z)$.

Now $f_z = \frac{\partial}{\partial z}\left((y+z)x + yz + C(z)\right) = x + y + C'(z) = R = x+y$. So $C'(z) = 0$ and $C(z) = C$.

Hence $f(x,y,z) = (y+z)x + yz + C$ $\boxed{f(x,y,z) = xy + xz + yz + C}$

#9 $F(x,y,z) = \langle e^{y+2z}, xe^{y+2z}, 2xe^{y+2z} \rangle$

$f_x = P \Rightarrow f = \int e^{y+2z} \, dx = x e^{y+2z} + C(y,z)$

So $f_y = \frac{\partial}{\partial y}\left(x e^{y+2z} + C(y,z)\right) = xe^{y+2z} + C_y(y,z) = Q = xe^{y+2z} \Rightarrow C_y(y,z) = 0$

$\Rightarrow C(y,z) = C(z)$.

Next $f_z = \frac{\partial}{\partial z}\left(xe^{y+2z} + C(z)\right) = 2xe^{y+2z} + C_z(z) = R = 2xe^{y+2z}$

$\Rightarrow C(z) = C$.

So

$$\boxed{f(x,y,z) = xe^{y+2z} + C}$$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$
9. $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$
10. $\mathbf{F} = (y\sin z)\mathbf{i} + (x\sin z)\mathbf{j} + (xy\cos z)\mathbf{k}$
11. $\mathbf{F} = (\ln x + \sec^2(x+y))\mathbf{i} + \left(\sec^2(x+y) + \frac{y}{y^2+z^2}\right)\mathbf{j} + \frac{z}{y^2+z^2}\mathbf{k}$
12. $\mathbf{F} = \frac{y}{1+x^2y^2}\mathbf{i} + \left(\frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}\right)\mathbf{j} + \left(\frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}\right)\mathbf{k}$

Find potential function f s.t. $\nabla f = F$

#10 $F(x,y,z) = \langle y\sin z,\ x\sin z,\ xy\cos z\rangle = \langle P,Q,R\rangle$

$$f_x = P \Rightarrow f = \int y\sin z\,dx = xy\sin z + C(y,z)$$

So $f_y = \frac{\partial}{\partial y}\left(xy\sin z + C(y,z)\right) = x\sin z + C_y(y,z) = Q$

$$= x\sin z \Rightarrow C(y,z) = C(z)$$

Next $f_z = \frac{\partial}{\partial z}\left(xy\sin z + C(z)\right) = xy\cos z + C'(z) = R = xy\cos z \Rightarrow C(z) = C.$

So $\boxed{f(x,y,z) = xy\sin z + C}$

$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx$

IBP $\int u\,dv = uv - \int v\,du$

$= x\ln x - x + C$

$u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x}dx$ $\quad v = x$

#11 $F(x,y,z) = \left\langle \ln x + \sec^2(x+y),\ \sec^2(x+y) + \frac{y}{y^2+z^2},\ \frac{z}{y^2+z^2}\right\rangle = \langle P,Q,R\rangle$

$$f_x = P \Rightarrow f = \int \ln x + \sec^2(x+y)\,dx = x\ln x - x + \tan(x+y) + C(y,z).$$

So $f_y = \frac{\partial}{\partial y}\left(x\ln x - x + \tan(x+y) + C(y,z)\right) = 0 + \sec^2(x+y) + C_y(y,z) = Q = \sec^2(x+y) + \frac{y}{y^2+z^2}$

$$\Rightarrow C_y(y,z) = \frac{y}{y^2+z^2} \Rightarrow C(y,z) = \int \frac{y}{y^2+z^2}\,dy = \frac{1}{2}\ln(y^2+z^2) + C(z)$$

u-sub

$u = y^2+z^2$

$du = 2y\,dy$

Next $f_z = \frac{\partial}{\partial z}\left(x\ln x - x + \tan(x+y) + \frac{1}{2}\ln(y^2+z^2) + C(z)\right) = 0 + 0 + \frac{z}{y^2+z^2} + C'(z) = R = \frac{z}{y^2+z^2}$

So $\boxed{f(x,y,z) = x\ln x - x + \tan(x+y) + \frac{1}{2}\ln(y^2+z^2) + C}$ $\Rightarrow C(z) = C$

#12 $F(x,y,z) = \left\langle \frac{y}{1+x^2y^2},\ \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}},\ \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}\right\rangle = \langle P,Q,R\rangle$

$\int \frac{1}{1+a^2x^2}\,dx = \frac{1}{a}\tan^{-1}(ax) + C$

check $\left(\frac{1}{a}\tan^{-1}(ax)\right)' = \frac{1}{a}\cdot a\cdot\frac{1}{1+a^2x^2}$ ✓

$$f_x = P \Rightarrow f = \int \frac{y}{1+x^2y^2}\,dx = \tan^{-1}(xy) + C(y,z)$$

So $f_y = \frac{\partial}{\partial y}\left(\tan^{-1}(xy) + C(y,z)\right) = \frac{x}{1+x^2y^2} + C_y(y,z) = Q = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}} \Rightarrow C(y,z) = \int \frac{z}{\sqrt{1-y^2z^2}}\,dy$

Next $f_z = \frac{\partial}{\partial z}\left(\tan^{-1}(xy) + \sin^{-1}(y,z) + C(z)\right)$ $\quad \Rightarrow C(y,z) = \sin^{-1}(yz) + C(z).$

$$= 0 + \frac{y}{\sqrt{1-y^2z^2}} + C'(z) = R = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \Rightarrow C'(z) = \frac{1}{z} \Rightarrow C(z) = \int \frac{1}{z}\,dz = \ln z + C.$$

So $\boxed{f(x,y,z) = \tan^{-1}(xy) + \sin^{-1}(y,z) + \ln z + C}$

Whew!

Exact Differential Forms

In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.

13. $\int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$

14. $\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$

15. $\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$

16. $\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1+z^2}\,dz$

17. $\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz$

Show that $P\,dx + Q\,dy + R\,dz$ is exact, then evaluate the straight line integral.

Defn.

The differential form $P\,dx + Q\,dy + R\,dz$ is exact iff $F = \langle P, Q, R\rangle$ is conservative

#13 $F = \langle 2x, 2y, 2z\rangle = \langle P, Q, R\rangle$ and $\text{Curl } F = \langle 0,0,0\rangle$, since $P = P(x)$, $Q = Q(y)$, $R = R(z)$

Now $f = x^2 + y^2 + z^2$ w/ $\nabla f = F$

So $\int_{(0,0,0)}^{(2,3,-6)} F \cdot dr = f(2,3,-6) - f(0,0,0) = [2^2 + 3^2 + (-6)^2] - 0 = 4 + 9 + 36 = \boxed{49}$

#14 $F = \langle yz, xz, xy\rangle$ has $\text{Curl}(F) = \langle x - x, -y - y, z - z\rangle = \langle 0,0,0\rangle$

So F is exact

and potential $f = xyz$ So $\int_{(1,1,2)}^{(3,5,0)} F \cdot dr = f(1,1,2) - f(3,5,0) = (3*5*0) - (1*1*2) = \boxed{-2}$

#15 $F = \langle 2xy, x^2 - z^2, -2yz\rangle$ has $\text{Curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 - z^2 & -2yz \end{vmatrix} = \langle -2z - (-2z), -0 + 0, 2x - 2x\rangle = \langle 0,0,0\rangle$ ✓

and potential $f = x^2 y - z^2 y$

So $\int_{(0,0,0)}^{(1,2,3)} F \cdot dr = f(1,2,3) - f(0,0,0) = [1^2 * 2 - 3^2 * 2] - [0 - 0] = 2 - 18 = \boxed{-16}$

#16 $F = \langle 2x, -y^2, \frac{-4}{1+z^2}\rangle = \langle P, Q, R\rangle$ $\text{Curl } F = \langle 0,0,0\rangle$ since $P = P(x)$, $Q = Q(y)$, $R = R(z)$.

and potential $f = x^2 - \frac{1}{3}y^3 - 4\tan^{-1} z$ So $\int_{(0,0,0)}^{(3,3,1)} F \cdot dr = f(3,3,1) - f(0,0,0)$

$= \left(3^2 - \frac{1}{3}3^3 - 4\tan^{-1}(1)\right) - (0 - 0 - \xcancel{4\tan^{-1} 0}^{\,0}) = 9 - 9 - 4\frac{\pi}{4} = \boxed{-\pi}$

Finding Potential Functions to Evaluate Line Integrals

Although they are not defined on all of space R^3, the fields associated with Exercises 18–22 are conservative. Find a potential function for each field and evaluate the integrals as in Example 6.

18. $\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}dz$

19. $\int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}dy + 2z\ln y\,dz$

20. $\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

21. $\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}dz$

22. $\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$

Find potential Function & evaluate the straight line integral.

#18 $F = \langle 2\cos y,\ \frac{1}{y} - 2x\sin y,\ \frac{1}{z} \rangle = \langle P, Q, R \rangle$

$f_x = P \Rightarrow f = \int 2\cos y\,dx = 2x\cos y + C(y,z)$

and $f_y = -2x\sin y + C_y(y,z) = Q \Rightarrow C = \ln y + C(z)$

then $f_z = \frac{1}{z} \Rightarrow C(z) = \ln z + C.$

So $f = 2x\cos y + \ln y + \ln z$, $\nabla f = F$ ✓

So $\int_{(0,2,1)}^{(1,\pi/2,2)} F \cdot dr = f(1,\pi/2,2) - f(0,2,1) = \left[2\cancel{\cos\frac{\pi}{2}}^{\,0} + \ln(\pi/2) + \ln 2\right] - \left[0\cos 2 + \ln 2 + \cancel{\ln 1}^{\,0}\right]$

$= \ln\pi - \ln 2 + \cancel{\ln 2} - \cancel{\ln 2}$

$= \boxed{\ln\pi - \ln 2}$

#20 $F = \langle 2x\ln y - yz,\ \frac{x^2}{y} - xz,\ -xy \rangle = \langle P, Q, R \rangle$

$f_x = P \Rightarrow f = \int 2x\ln y - yz\,dx = x^2\ln y - xyz + C(y,z)$

and $f_y = Q \Rightarrow \frac{x^2}{y} - xz + C_y(y,z) = \frac{x^2}{y} - xz \Rightarrow C(y,z) = C(z)$

Then $f_z = R \Rightarrow 0 - xy + C'(z) = -xy \Rightarrow C(z) = C.$

Set $C=0$

So $f = x^2\ln y - xyz + \cancel{C}$

Then $\int_{(1,2,1)}^{(2,1,1)} F \cdot dr = f(1,2,1) - f(2,1,1) = [1\ln 2 - 1*2*1] - [2\cancel{\ln 1}^{\,0} - 2*1*1]$

$= \ln 2 - 2 + 2$

$= \boxed{\ln 2}$