Verify Green's Thm (3) & (4)

w/ $R: x^2+y^2 \le a^2$, $C: r(t) = \langle a\cos t, a\sin t\rangle$

$t \in [0,1]$

§16.4

Exercises 16.4

Verifying Green's Theorem

In Exercises 1–4, verify the conclusion of Green's Theorem by evaluating both sides of Equations (3) and (4) for the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. Take the domains of integration in each case to be the disk $R: x^2 + y^2 \le a^2$ and its bounding circle $C: \mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, 0 \le t \le 2\pi$.

1. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
2. $\mathbf{F} = y\mathbf{i}$
3. $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$
4. $\mathbf{F} = -x^2y\mathbf{i} + xy^2\mathbf{j}$

Greens Theorem, $F = \langle P, Q\rangle$

(3) $\oint_C F\cdot T\, ds = \iint_R Q_x - P_y \, dA$

(4) $\oint_C F\cdot n\, ds = \iint_R P_x + Q_y \, dA$

#1 $F = \langle -y, x\rangle$, $r'(t) = \langle -a\sin t, a\cos t\rangle$, $P=-y$, $Q = x$, $Q_x - P_y = 2$

(3) LHS $\oint_C F\cdot T ds = \int_0^{2\pi} \langle -a\sin t, a\cos t\rangle \cdot \langle -a\sin t, a\cos t\rangle dt$

$= \int_0^{2\pi} a^2\sin^2 t + a^2\cos^2 t \, dt = \int_0^{2\pi} a^2 dt = \boxed{2\pi a^2}$

switch to polar

RHS $= \iint_R Q_x - P_y \, dA = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 1-(-1)\, dy\, dx = \int_0^{2\pi}\int_0^a \underbrace{2}_{Q_x - P_y} * r\, dr\, d\theta = \int_0^{2\pi} r^2\Big|_0^a d\theta$

$= \int_0^{2\pi} a^2 \, d\theta = a^2\theta\Big|_0^{2\pi} = \boxed{2a^2\pi}$

So $\boxed{\text{LHS} = \text{RHS}}$ for (3).

(4) $n \sim \langle a\cos t, a\sin t\rangle$ (outward pointing ✓)

LHS $\oint_C F\cdot n\, ds = \int_0^{2\pi} \langle -a\sin t, a\cos t\rangle \cdot \langle a\cos t, a\sin t\rangle dt$

$= \int_0^{2\pi} -a\sin t\cos t + a\sin t\cos t\, dt = \int_0^{2\pi} 0\, dt = \boxed{0}$

RHS $\iint_R P_x + Q_y \, dA = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 0 - 0 \, dy\, dx = \boxed{0}$

$\boxed{\text{LHS} = \text{RHS}}$ so (4) verified ✓

#2 $F = \langle y, 0\rangle$ Curl $F\cdot\hat{k} = 0 - 1 = -1 \neq 0$ so F not conservative ✓

(3) LHS $\oint_C F\cdot T\, ds = \int_0^{2\pi} \langle a\sin t, 0\rangle \cdot \langle -a\sin t, a\cos t\rangle dt = \int_0^{2\pi} -a^2\sin^2 t\, dt$

$= \int_0^{2\pi} -\frac{a^2}{2}(1 - \cos 2t)\, dt = -\frac{a^2}{2}t + \frac{a^2}{4}\sin 2t \Big|_0^{2\pi} = \left(-\frac{a^2}{2}2\pi + \frac{a^2}{4}\cancelto{0}{\sin 2\pi}\right) - \left(-\frac{a^2}{2}*0 + \frac{a^2}{4}\cancelto{0}{\sin 0}\right) = \boxed{-a^2\pi}$

RHS $\iint_R Q_x - P_y \, dA = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 0 - 1 \, dy\, dx = \int_0^{2\pi}\int_0^a -r\, dr\, d\theta = \int_0^{2\pi} -\frac{1}{2}r^2\Big|_0^a d\theta$

$= \int_0^{2\pi} -\frac{1}{2}a^2\, d\theta = -\frac{1}{2}a^2 * 2\pi = \boxed{-a^2\pi}$

Since $\boxed{\text{LHS} = \text{RHS}}$ This verifies (3)

Exercises 16.4

Verifying Green's Theorem

In Exercises 1–4, verify the conclusion of Green's Theorem by evaluating both sides of Equations (3) and (4) for the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. Take the domains of integration in each case to be the disk $R: x^2 + y^2 \le a^2$ and its bounding circle $C: \mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le 2\pi$.

1. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ 2. $\mathbf{F} = y\mathbf{i}$

3. $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$ 4. $\mathbf{F} = -x^2y\mathbf{i} + xy^2\mathbf{j}$

#2 Cont. $F = \langle y, 0\rangle$, $r(t) = \langle a\cos t, a\sin t\rangle$, $t \in [0, 2\pi]$

$r'(t) = \langle -a\sin t, a\cos t\rangle$ so $n \sim \langle a\cos t, a\sin t\rangle$

(4) LHS $\oint_C F \cdot n\, ds = \int_0^{2\pi} \langle a\sin t, 0\rangle \cdot \langle a\cos t, a\sin t\rangle\, dt$

$= \int_0^{2\pi} a^2 \sin t\cos t\, dt = \frac{a^2}{2}\sin^2 t\Big|_0^{2\pi} = \frac{a^2}{2}\left[\sin^2(2\pi) - \sin^2(0)\right] = \boxed{0}$

RHS $\iint_R P_x + Q_y\, dA = \iint_R 0 + 0\, dA = \boxed{0}$ This verifies (4).

#3 $F = \langle 2x, -3y\rangle$ (same r as #1, #2) $\begin{cases} P_y = 0 \\ Q_x = 0 \end{cases}$

$= \langle P, Q\rangle$

u-sub: $\boxed{\begin{matrix} u = \sin t \\ du = \cos t\, dt \end{matrix}}$ $t = 0 \Rightarrow u = 0$, $t = 2\pi \Rightarrow u = 0$

(3) LHS $\oint_C F \cdot T\, ds = \int_0^{2\pi} \langle 2a\cos t, -3a\sin t\rangle \cdot \langle -a\sin t, a\cos t\rangle\, dt = \int_0^{2\pi} -2a^2\sin t\cos t + 3a^2\sin t\cos t\, dt$

$= \int_0^{2\pi} a^2 \sin t\cos t\, dt = \int_0^0 a^2 u\, du = a^2 \frac{1}{2}u^2\Big|_0^0 = 0 - 0 = \boxed{0}$ $\boxed{LHS = RHS}$ so (3) verified

RHS $\iint_R Q_x - P_y\, dA = \int_0^{2\pi}\int_0^a 0 - 0\; r\, dr\, d\theta = \boxed{0}$

(4) Now $P_x = 2$ and $Q_y = -1$ so $P_x + Q_y = -1$ $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$

LHS $\oint_C F \cdot dn\, ds = \int_0^{2\pi} \langle 2a\cos t, -3a\sin t\rangle \cdot \langle a\cos t, a\sin t\rangle\, dt = \int_0^{2\pi} 2a^2\cos^2 t - 3a^2\sin^2 t\, dt$

$= \int_0^{2\pi} 2a^2\cos^2 t - 3a^2(1 - \cos^2 t)\, dt = \int_0^{2\pi} 5a^2\cos^2 t - 3a^2\, dt = \int_0^{2\pi} \frac{5a^2}{2}(1 + \cos 2t) - 3a^2\, dt = \int_0^{2\pi} -\frac{1}{2}a^2 + \frac{5}{2}a^2\cos 2t\, dt$

$= -\frac{1}{2}a^2 t + \frac{5}{4}a^2\sin(2t)\Big|_0^{2\pi} = \left(-\frac{1}{2}a^2 * 2\pi + \cancelto{0}{\frac{5}{4}a^2\sin 4\pi}\right) - \left(-\frac{1}{2}a^2 * 0 + \cancelto{0}{\frac{5}{4}a^2\sin 0}\right) = \boxed{-a^2\pi}$

Déja vu

RHS $\iint_R P_x + Q_y\, dA = \iint_R 2 - 3\; dx\, dy = \iint_R -1\, dA = -\text{Area}(R) = \boxed{-a^2\pi}$ since $\boxed{LHS = RHS}$ this verifies (4)

#4 $F = \langle -x^2 y, xy^2\rangle$ $P_y = -x^2$, $Q_x = y^2$ (F not conservative)

$P_x = -2xy$, $Q_y = 2xy$ so $\operatorname{div} F = P_x + Q_y = 0$ and Flux $= \oint_C F \cdot n\, ds = 0$ (for all loops C).

(3) LHS $\oint_C F \cdot T\, ds = \int_0^{2\pi} \langle -a^3\cos^2 t\sin t, a^3\cos t\sin^2 t\rangle \cdot \langle -a\sin t, a\cos t\rangle\, dt$

$= \int_0^{2\pi} a^4\cos^2 t\sin^2 t + a^4\cos^2 t\sin^2 t\, dt = \int_0^{2\pi} 2a^4\cos^2 t\sin^2 t\, dt = \int_0^{2\pi} 2a^4\cos^2 t(1 - \cos^2 t)\, dt$

$= \int_0^{2\pi} 2a^4\cos^2 t - 2a^4\cos^4 t\, dt = \int_0^{2\pi} a^4(1 + \cos 2t) - \frac{a^4}{2}(1 + \cos 2t)^2\, dt = a^4\int_0^{2\pi} 1 + \cos 2t - \frac{1}{2}(1 + 2\cos 2t + \cos^2 2t)\, dt$

$= a^4\int_0^{2\pi} \frac{1}{2} + 2\cos 2t - \frac{1}{4}(1 + \cos 4t)\, dt = a^4\int_0^{2\pi} \frac{1}{4} + 2\cos 2t - \frac{1}{4}\cos 4t\, dt = a^4\left(\frac{1}{4}t + \sin 2t - \frac{1}{16}\sin 16t\Big|_0^{2\pi}\right) = \frac{a^4}{4} * 2\pi + 0 = \boxed{\frac{a^4\pi}{2}}$

#4 Cont. $F=\langle -x^2y,\ xy^2\rangle$ $P_y=-x^2,\ Q_x=y^2$ & $P_x=-2xy,\ Q_y=2xy$

(3) RHS $\iint_R Q_x-P_y\,dA = \iint_R y^2+x^2\,dx\,dy = \int_0^{2\pi}\int_0^a (r^2\sin^2\theta+r^2\cos^2\theta)\,r\,dr\,d\theta = \int_0^{2\pi} r^3\,dr\,d\theta$

$= \int_0^{2\pi}\frac{1}{4}r^4\Big|_0^a d\theta = \int_0^{2\pi}\frac{a^4}{4}\,d\theta = \frac{a^4}{4}t\Big|_0^{2\pi} = \frac{a^4}{4}*2\pi = \boxed{\frac{a^4\pi}{2}}$

So $\boxed{LHS=RHS}$ which verifies (3)

$u=\cos t$
$du=-\sin t\,dt$

$u=\sin t$
$du=\cos t\,dt$

(4) LHS $\oint_C F\cdot n\,ds = \int_0^{2\pi}\langle -a^3\cos^2t\sin t,\ a^3\cos t\sin^2 t\rangle\cdot\langle a\cos t,\ a\sin t\rangle\,dt = \int_0^{2\pi} -a^4\cos^3t\sin t + a^4\cos t\sin^3 t\,dt$

$= \frac{a^4}{4}\cos^4 t+\frac{a^4}{4}\sin^4 t\Big|_0^{2\pi} = \frac{a^4}{4}\Big[\Big(\cancelto{1}{\cos^4(2\pi)}+\cancelto{0}{\sin^4(2\pi)}\Big)-\Big(\cancelto{1}{\cos^4 0}+\cancelto{0}{\sin^4 0}\Big)\Big] = \boxed{0}$

RHS $\iint_R P_x+Q_y\,dA = \iint_R -2xy+2xy\,dA = \iint_R 0\,dA = \boxed{0}$ this verifies (4) since $\boxed{LHS=RHS}$

Find Flow and Flux using Green's Theorem (G'sT)

Circulation and Flux

In Exercises 5–14, use Green's Theorem to find the counterclockwise circulation and outward flux for the field **F** and curve C.

5. $\mathbf{F} = (x-y)\mathbf{i} + (y-x)\mathbf{j}$
C: The square bounded by $x=0,\ x=1,\ y=0,\ y=1$

6. $\mathbf{F} = (x^2+4y)\mathbf{i} + (x+y^2)\mathbf{j}$
C: The square bounded by $x=0,\ x=1,\ y=0,\ y=1$

7. $\mathbf{F} = (y^2-x^2)\mathbf{i} + (x^2+y^2)\mathbf{j}$
C: The triangle bounded by $y=0,\ x=3$, and $y=x$

8. $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$
C: The triangle bounded by $y=0,\ x=1$, and $y=x$

9. $\mathbf{F} = (xy+y^2)\mathbf{i} + (x-y)\mathbf{j}$ **10.** $\mathbf{F} = (x+3y)\mathbf{i} + (2x-y)\mathbf{j}$

#5 $F=\langle x-y,\ y-x\rangle$ $P_y=-1,\ Q_x=-1$ so F conservative
$R=[0,1]\times[0,1]$ $P_x=1,\ Q_y=1.$

Flow $= \oint_C F\cdot T\,ds \overset{G'sT}{=} \iint_R Q_x-P_y\,dA = \iint_R (-1)-(-1)\,dA$

$= \iint_R 0\,dA = \boxed{0.}$ (must be 0 by FToLI's)

Flux $= \oint_C F\cdot n\,ds = \iint_R P_x+Q_y\,dA = \int_0^1\int_0^1 1+1\,dx\,dy = \int_0^1 2x\Big|_0^1 dy = 2y\Big|_0^1 = \boxed{2}$

#6 $F=\langle x^2+4y,\ x+y^2\rangle$ $Q_x=1,\ P_y=4$ (F not conservative)
$R=[0,1]\times[0,1]$ $P_x=2x,\ Q_y=2y$

Flow $= \oint_C F\cdot T\,ds = \iint_R 1-4\,dA = \iint_R -3\,dA = -3*\text{Area}(R) = -3*1 = \boxed{-3}$

Flux $= \oint F\cdot n\,ds = \iint_R 2x+2y\,dA = \int_0^1\int_0^1 2x+2y\,dx\,dy = \int_0^1 x^2+2xy\Big|_0^1 dy = \int_0^1 1+2y\,dy$
$= y+y^2\Big|_0^1 = (1+1)-(0+0) = \boxed{2}$

#7 $F=\langle y^2-x^2,\ x^2+y^2\rangle$ $Q_x=2x,\ P_y=2y$ (F not conservative)
$R:\ x\in[0,3],\ y\in[0,x]$ $P_x=-2x,\ Q_y=2y$

Flow $= \oint_C F\cdot T\,ds = \iint_R 2x-(2y)\,dA = \int_0^3\int_0^x 2x-2y\,dy\,dx = \int_0^3 2xy-y^2\Big|_0^x dx = \int_0^3 2x^2-x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = \boxed{9}$

Flux $= \oint_C F\cdot n\,ds = \iint_R -2x+2y\,dA = \int_0^3\int_0^x -2x+2y\,dy\,dx = \int_0^3 -2xy+y^2\Big|_0^x dx = \int_0^3 -x^2\,dx = \frac{-1}{3}x^3\Big|_0^3 = \boxed{-9}$

Find Flow & Flux using Green's Theorem

#8 $F = \langle x+y, -x^2-y^2 \rangle$ $\quad Q_x = -2x,\ P_y = 1 \quad$ (F not conservative)

$R: x\in[0,3], y\in[0,x] \quad P_x = 1,\ Q_y = -2y$

$$\text{Flow} = \oint_C F\cdot T\, ds \overset{G'sT}{=} \iint_R -2x - 1\, dA = \int_0^3\int_0^x -2x-1\, dy\, dx = \int_0^3 -2xy - y\Big|_0^x dx$$

$$= \int_0^3 -2x^2 - x\, dx = -\tfrac{2}{3}x^3 - \tfrac{1}{2}x^2\Big|_0^3 = \left[-18 - \tfrac{9}{2}\right] - [0-0] = \boxed{-45/2}$$

$$\text{Flux} = \oint_C F\cdot n\, ds = \iint_R 1 + (-2y)\, dA = \int_0^3\int_0^x 1-2y\, dy\, dx$$

$$= \int_0^3 y - y^2\Big|_0^x dx = \int_0^3 x - x^2\, dx = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\Big|_0^3 = \tfrac{9}{2} - 9 = \boxed{-9/2}$$

Circulation and Flux

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C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

6. $\mathbf{F} = (x^2 + 4y)\mathbf{i} + (x + y^2)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

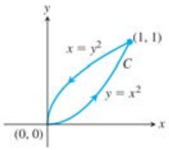
7. $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$

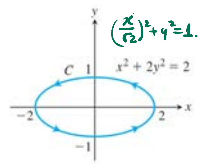
C: The triangle bounded by $y = 0$, $x = 3$, and $y = x$

8. $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$

C: The triangle bounded by $y = 0$, $x = 1$, and $y = x$

9. $\mathbf{F} = (xy + y^2)\mathbf{i} + (x - y)\mathbf{j}$ $\quad$ **10.** $\mathbf{F} = (x + 3y)\mathbf{i} + (2x - y)\mathbf{j}$





#9 $F = \langle xy + y^2, x - y\rangle$ $\quad Q_x = 1,\ P_y = x + 2y \quad$ (F not conservative)

$R: x\in[0,1], y\in[x^2, \sqrt{x}] \quad P_x = y,\ Q_y = -1$

$$\text{Flow} = \oint_C F\cdot T\, ds = \iint_R 1-(x+2y)\, dA = \int_0^1\int_{x^2}^{\sqrt{x}} 1 - x - 2y\, dy\, dx = \int_0^1 y - xy - y^2\Big|_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 (\sqrt{x} - x\sqrt{x} - x) - (x^2 - x^3 - x^4)\, dx = \tfrac{2}{3}x^{3/2} - \tfrac{2}{5}x^{5/2} - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5\Big|_0^1$$

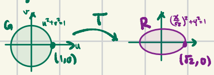
$$= \left(\tfrac{2}{3} - \tfrac{2}{5} - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}\right) - 0 = \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{4} = \frac{20-12-15}{3*5*4} = \boxed{\frac{-7}{60}}$$

$$\text{Flux} = \oint_C F\cdot n\, ds = \iint_R y + (-1)\, dA = \int_0^1\int_{x^2}^{\sqrt{x}} y - 1\, dy\, dx = \int_0^1 \tfrac{1}{2}y^2 - y\Big|_{x^2}^{\sqrt{x}} = \int_0^1 \left(\tfrac{1}{2}x - \sqrt{x}\right) - \left(\tfrac{1}{2}x^4 - x^2\right) dx$$

$$= \tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2} - \tfrac{1}{10}x^5 + \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{4} - \tfrac{2}{3} - \tfrac{1}{10} + \tfrac{1}{3} = \tfrac{1}{4} - \tfrac{1}{3} - \tfrac{1}{10} = \frac{30-40-12}{120} = \frac{-22}{120} = \boxed{\frac{-11}{60}}$$

#10 $F = \langle x + 3y, 2x - y\rangle$ $\quad Q_x = 2,\ P_y = 3 \quad$ (F not conservative)

$R: \frac{x^2}{2} + y^2 \le 1 \quad P_x = 1,\ Q_y = -1$

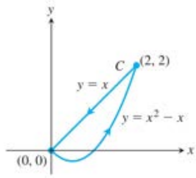


$T\begin{bmatrix} u \\ v\end{bmatrix} = \begin{bmatrix}\sqrt{2}u \\ v\end{bmatrix}$ $\quad$ Jacobian $|\det DT| = \left|\det\begin{bmatrix}\sqrt{2} & 0 \\ 0 & 1\end{bmatrix}\right| = \sqrt{2}$

$$\text{Flow} = \oint_C F\cdot T\, ds = \iint_R 2-3\, dA = \iint_R -1\, dA = \iint_G -1 * \sqrt{2}\, dA = -\sqrt{2}\,\text{Area}(G) = \boxed{-\sqrt{2}\pi}$$

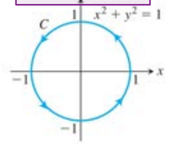
$$\text{Flux} = \oint_C F\cdot n\, ds = \iint_R 1 + (-1)\, dA = \iint_R 0\, dA = \boxed{0}$$

11. $\mathbf{F} = x^3y^2\mathbf{i} + \frac{1}{2}x^4y\,\mathbf{j}$



12. $\mathbf{F} = \frac{x}{1+y^2}\mathbf{i} + (\tan^{-1} y)\mathbf{j}$

Flow ONLY



Find Flow and Flux using G's T

(F conservative!)

#11 $F = \langle x^3y^2, \frac{1}{2}x^4y \rangle$ $\quad Q_x = 2x^3y,\ P_y = 2x^3y$

R: $x \in [0,2),\ y \in [x^2 - x, x]$ $\quad P_x = 3x^2y^2,\ Q_y = \frac{1}{2}x^4$

$$\text{Flow} = \oint_C F \cdot T\, ds = \iint_R 2x^3y - 2x^3y\, dA = \iint_R 0\, dA = \boxed{0}\ \checkmark$$

$$\text{Flux} = \oint_C F \cdot n\, ds = \iint_R 3x^2y^2 + \frac{1}{2}x^4\, dA = \int_0^2 \int_{x^2-x}^{x} 3x^2y^2 + \frac{1}{2}x^4\, dy\, dx = \int_0^2 x^2y^3 + \frac{1}{2}x^4y \Big|_{x^2-x}^{x} dx$$

$$= \int_0^2 x^2(x^3 - (x^2-x)^3) + \frac{1}{2}x^4(x - (x^2 - x))\, dx = \int_0^2 x^5(1 - (x-1)^3) + \frac{1}{2}x^5(2 - x)\, dx$$

binomial coeff.
1
1 1
1 2 1
1 3 3 1

$$= \int_0^2 x^5(1 - (x^3 - 3x^2 + 3x - 1)) + x^5 - \frac{1}{2}x^6\, dx = \int_0^2 -x^8 + 3x^7 - 3x^6 + 2x^5 + x^5 - \frac{1}{2}x^6\, dx$$

$$= \int_0^2 -x^8 + 3x^7 - \frac{7}{2}x^6 + 3x^5\, dx = -\frac{1}{9}x^9 + \frac{3}{8}x^8 - \frac{1}{2}x^7 + \frac{1}{2}x^6 \Big|_0^2 = -\frac{1}{9}2^9 + \frac{3}{8}2^8 - \frac{1}{2}2^7 + \frac{1}{2}2^6 - 0$$

$$= -\frac{2^9}{9} + \underbrace{3 * 2^5 - 2^6 + 2^5}_{4*2^5 = 2^7} = 2^6\left(\frac{-8}{9} + 2 - 1\right) = 2^6\left(\frac{1}{9}\right) = \boxed{64/9}$$

#12 $F = \langle \frac{x}{1+y^2}, \tan^{-1}(y) \rangle$ $\quad Q_x = 0,\ P_y = \frac{-2xy}{(1+y^2)^2}$

R: unit circle @ (0,0) $\quad P_x = \frac{1}{1+y^2},\ Q_y = \frac{1}{1+y^2}$

$u = 1 + y^2$
$du = 2y\, dy$

$$\text{Flow} = \oint_C F \cdot T\, ds = \iint_R \frac{2xy}{(1+y^2)^2}\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2xy}{(1+y^2)^2}\, dy\, dx = \int_{-1}^{1} \frac{x}{1+y^2} \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

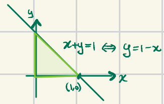
$$= \int_{-1}^{1} \frac{x}{1 + (1-x^2)} - \frac{x}{1 + (1-x^2)}\, dx = \int_{-1}^{1} 0\, dx = \boxed{0}$$

$$\text{Flux} = \oint_C F \cdot n\, ds = \iint_R \frac{1}{1+y^2} + \frac{1}{1+y^2}\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{1+y^2}\, dy\, dx = \int_{-1}^{1} 2\tan^{-1}(\sqrt{1-x^2}) - 2\tan^{-1}(-\sqrt{1-x^2})\, dx$$

$$= \cdots = \boxed{2\pi}$$ but really NASTY.

(??) Flow only

Evaluate



Using Green's Theorem

Apply Green's Theorem to evaluate the integrals in Exercises 21–24.

21. $\oint_C (y^2\,dx + x^2\,dy)$

C: The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$

22. $\oint_C (3y\,dx + 2x\,dy)$

C: The boundary of $0 \le x \le \pi$, $0 \le y \le \sin x$

23. $\oint_C (6y + x)\,dx + (y + 2x)\,dy$

C: The circle $(x-2)^2 + (y-3)^2 = 4$

24. $\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy$

C: Any simple closed curve in the plane for which Green's Theorem holds

$F = \langle y^2, x^2 \rangle \qquad Q_x = 2x,\ P_y = 2y$

#21 $\oint_C y^2\,dx + x^2\,dy \qquad C$ bounding R: $x \in [0,1]\ y \in [0, 1-x]$

$$\overset{G'sT}{=} \iint_R 2x - 2y\,dA = \int_0^1 \int_0^{1-x} 2x - 2y\,dy\,dx = \int_0^1 2xy - y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 2x(1-x) - (1-x)^2\,dx = \int_0^1 (1-x)(2x - (1-x))\,dx$$

$$= \int_0^1 -(x-1)(3x-1)\,dx = \int_0^1 -(3x^2 - x - 3x + 1)\,dx$$

$$= \int_0^1 -3x^2 + 4x - 1\,dx = -x^3 + 2x^2 - x \Big|_0^1 = (-1+2-1) - 0 = \boxed{0}$$

$F = \langle 2x + y^2, 2xy + 3y \rangle \qquad Q_x = 2y,\ P_y = 2y \qquad F$ conservative!

#24 $$\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy = \iint_R Q_x - P_y\,dA = \iint_R 0 - 0\,dA = \boxed{0}$$

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Use the Green's Theorem area formula given above to find the areas of the regions enclosed by the curves in Exercises 25–28.

25. The circle $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $\quad 0 \le t \le 2\pi$

26. The ellipse $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j}$, $\quad 0 \le t \le 2\pi$

#26 Find Area(R) using Greens Thm

R: ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad C: r(t) = \langle a\cos t, b\sin t \rangle$

$t \in [0, 2\pi]$

$r'(t) = \langle -a\sin t, b\cos t \rangle$

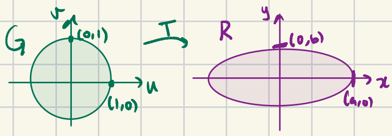
$$\text{Area } R = \iint_R 1\,dA = \iint \underbrace{\tfrac{1}{2}}_{Q_x} - \underbrace{\left(\tfrac{-1}{2}\right)}_{P_y} dA \overset{G'sT}{=} \oint_C \left\langle -\tfrac{1}{2}y, \tfrac{1}{2}x \right\rangle \cdot dr = \int_0^{2\pi} \left\langle -\tfrac{b}{2}\sin t, \tfrac{a}{2}\cos t \right\rangle \cdot \langle -a\sin t, b\cos t \rangle\,dt$$

$$= \int_0^{2\pi} \frac{ab}{2}\sin^2 t + \frac{ab}{2}\cos^2 t\,dt = \int_0^{2\pi} \frac{ab}{2}\,dt = \frac{ab}{2}t \Big|_0^{2\pi} = \frac{ab}{2} * 2\pi = \boxed{ab\pi}$$

Check using Jacobians

$T\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au \\ bv \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ Jacobian det

$|\det DT| = \left|\det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right| = ab$



$$\iint_R 1\,dA = \iint_G 1 * ab\,dA = ab\pi~~^2~~ \checkmark$$

§ 16.5

Exercises 16.5

Finding Parametrizations

In Exercises 1–16, find a parametrization of the surface. (There are many correct ways to do these, so your answers may not be the same as those in the back of the book.)

1. The paraboloid $z = x^2 + y^2, z \le 4$
2. The paraboloid $z = 9 - x^2 - y^2, z \ge 0$
3. **Cone frustum** The first-octant portion of the cone $z = \sqrt{x^2 + y^2}/2$ between the planes $z = 0$ and $z = 3$
4. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$
5. **Spherical cap** The cap cut from the sphere $x^2 + y^2 + z^2 = 9$ by the cone $z = \sqrt{x^2 + y^2}$
6. **Spherical cap** The portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$
7. **Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 3$ between the planes $z = \sqrt{3}/2$ and $z = -\sqrt{3}/2$
8. **Spherical cap** The upper portion cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$

Find a parametrization
(several correct answers are possible)

#1. $S: z = x^2 + y^2,\ z \le 4$ use cylindrical coords

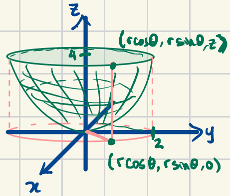
$x = r\cos\theta$

$y = r\sin\theta$

$z = x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

& $z \le 4$ if $r \in [0,2]$

So $\boxed{\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle \quad \theta \in [0,2\pi],\ r \in [0,2]}$

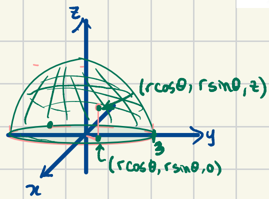


#2 $S: z = 9 - x^2 - y^2,\ z \ge 0$

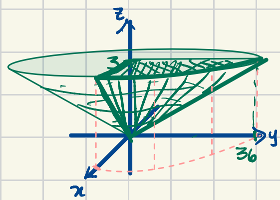
$x = r\cos\theta$

$y = r\sin\theta$

$z = 9 - (x^2+y^2) = 9 - r^2,\ z \ge 0$ if $r \in [0,3]$



$\boxed{\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 9 - r^2\rangle,\ \theta \in [0,2\pi],\ r \in [0,3]}$



#3 $S: z = \dfrac{\sqrt{x^2+y^2}}{2},\ 0 \le z \le 3$ First octant so $x \ge 0$ & $y \ge 0$

Cylindrical coords $x = r\cos\theta$

$y = r\sin\theta$

$z = \dfrac{\sqrt{x^2+y^2}}{2} = r/2$

& $0 \le z \le 3$ if $r \in [0,6]$

$\boxed{\text{So } \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r/2\rangle \text{ and } \theta \in [0, \pi/2],\ r \in [0,6]}$

#4 $S: z = 2\sqrt{x^2+y^2},\ 3 \le z \le 4$

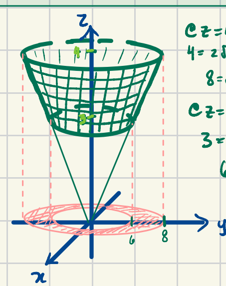
$x = r\cos\theta$

$y = r\sin\theta$

$z = 2r$ since $r^2 = x^2+y^2$

$3 \le z \le 4 \Rightarrow r \in [6,8]$

So $\boxed{\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle \text{ with } \theta \in [0,2\pi] \text{ and } r \in [6,8]}$



@ $z = 4$

$4 = 2\sqrt{x^2+y^2}$ @ $x = 0$

$8 = \sqrt{y^2} \Rightarrow 8 = y$

@ $z = 3$

$3 = 2\sqrt{x^2+y^2}$ @ $x = 0$

$6 = \sqrt{y^2} \Rightarrow 6 = y$

#5 S: over cone $z = \sqrt{x^2+y^2}$ & on sphere $x^2+y^2+z^2 = 9$

Sphere in spherical coords is just $\rho = 3$, $\varphi \in [0,\pi]$, $\theta \in [0,2\pi]$.

We want part of sphere "above" $\varphi = \pi/4$, so $\varphi \in [0, \pi/4]$.

$x = \rho\sin\varphi\cos\theta$

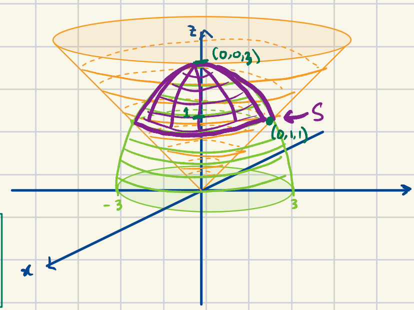
$y = \rho\sin\varphi\sin\theta$

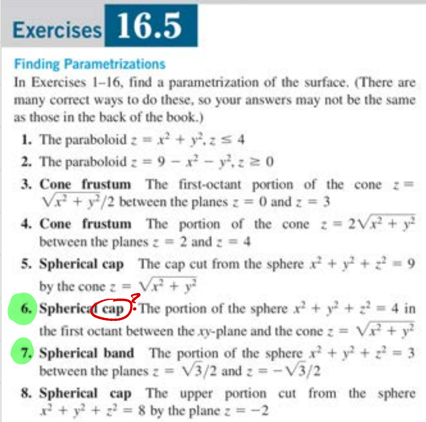
$z = \rho\cos\varphi$

And $\rho = 3$

So

$\boxed{r(\varphi,\theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi\rangle \quad \varphi \in [0,\pi/4],\ \theta \in [0,2\pi]}$




Exercises 16.5

Finding Parametrizations

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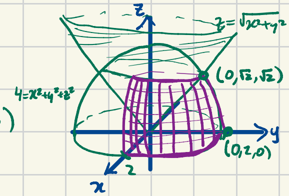
2. The paraboloid $z = 9 - x^2 - y^2, z \ge 0$
3. **Cone frustum** The first-octant portion of the cone $z = \sqrt{x^2+y^2}/2$ between the planes $z = 0$ and $z = 3$
4. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z = 2$ and $z = 4$
5. **Spherical cap** The cap cut from the sphere $x^2 + y^2 + z^2 = 9$ by the cone $z = \sqrt{x^2+y^2}$
6. **Spherical cap** ? The portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy-plane and the cone $z = \sqrt{x^2+y^2}$
7. **Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 3$ between the planes $z = \sqrt{3}/2$ and $z = -\sqrt{3}/2$
8. **Spherical cap** The upper portion cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$

Find a parametrization (several correct answers are possible)

#6 S: between xy-plane $(z=0)$ and cone $z=\sqrt{x^2+y^2}$ in first octant $(x\ge 0, y\ge 0)$ an on sphere $x^2+y^2+z^2=4$



intersection $\underset{\text{on cone}}{z=\sqrt{x^2+y^2}}$ and $\underset{\text{on sphere}}{z=\sqrt{4-x^2-y^2}}$

so $x^2+y^2 = 4-x^2-y^2 \Rightarrow x^2+y^2=2$

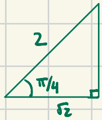
@ $x=0$, $y=\sqrt2 \Rightarrow z=\sqrt2$

$\underset{\text{cartesian}}{(0,\sqrt2,\sqrt2)_c} = \underset{\text{spherical}}{(2,\pi/4,\pi/2)_s}$

$\rho=\sqrt{0^2+2+2}=2$

$\theta=\pi/2$ (above + y-axis)

$\varphi = \pi/2-\pi/4=\pi/4$



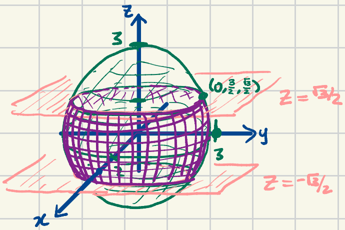
So $\rho=2$ (on sphere)

$\varphi\in[\pi/4,\pi/2]$ (from $(0,\sqrt2,\sqrt2)$ to floor)

and $\theta\in[0,\pi/2]$ (first octant)

Hence $\boxed{r(\varphi,\theta)=\langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi\rangle \quad \varphi\in[\pi/4,\pi/2] \text{ and } \theta\in[0,\pi/2]}$

#7 S: portion of sphere $x^2+y^2+z^2=3$ with $-\sqrt3/2 \le z \le \sqrt3/2$.



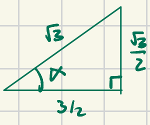
So $\rho=\sqrt3$ radius of sphere, $\theta\in[0,2\pi]$.

And intersection of plane $z=\sqrt3/2$ and sphere is @ $z=\sqrt3/2$ $\quad 3=x^2+y^2+(\sqrt3/2)^2 \Rightarrow 3-\frac34=x^2+y^2 \Rightarrow x^2+y^2=(\frac32)^2$

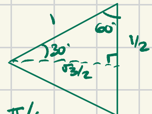
So radius of circle of intersection is $r=3/2$

$\underset{\text{cartesian}}{(0,\frac32,\frac{\sqrt3}{2})_c} = (\sqrt3,\pi/3,\pi/2)_s$

$\sqrt{\frac94+\frac34}=\sqrt{12/4}=\sqrt3$ ✓



sim to



So $\alpha=30°=\pi/6$

First φ: $\varphi=\pi/2-\alpha=\frac{3\pi}{6}-\frac{\pi}{6}=\frac{2\pi}{6}=\pi/3$

last φ: $\pi-\pi/3=2\pi/3$

So $\rho=\sqrt3$ (on sphere)

$\theta\in[0,2\pi]$ (full rotation for θ)

$\varphi\in[\pi/3,2\pi/3]$ (portion above $z=-\sqrt3/2$ & below $z=\sqrt3/2$)

$$\boxed{r(\varphi,\theta)=\langle\sqrt3\sin\varphi\cos\theta,\ \sqrt3\sin\varphi\sin\theta,\ \sqrt3\cos\varphi\rangle \quad \varphi\in[\pi/3,2\pi/3],\ \theta\in[0,2\pi]}$$

Find $SA = \iint_S 1\, d\sigma$

Surface Area of Parametrized Surfaces

17. Tilted plane inside cylinder The portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

18. Plane inside cylinder The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$

19. Cone frustum The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$

20. Cone frustum The portion of the cone $z = \sqrt{x^2 + y^2}/3$ between the planes $z = 1$ and $z = 4/3$

21. Circular cylinder band The portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$

22. Circular cylinder band The portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$

23. Parabolic cap The cap cut from the paraboloid $z = 2 - x^2 - y^2$ by the cone $z = \sqrt{x^2 + y^2}$

24. Parabolic band The portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

25. Sawed-off sphere The lower portion cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$

26. Spherical band The portion of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = -1$ and $z = \sqrt{3}$

#17 S: portion of $y+2z=2$ inside $x^2+y^2=1$

On plane of $z = 1-\frac{1}{2}y$ & inside cylinder of $x^2+y^2=r^2 \le 1$

$\Rightarrow z = 1 - \frac{1}{2} r\sin\theta$

So $S: \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1-\frac{1}{2}r\sin\theta\rangle$

$R: r\in[0,1]$ and $\theta\in[0,2\pi]$

$\vec{r}_r = \langle \cos\theta, \sin\theta, -\frac{1}{2}\sin\theta\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, -\frac{1}{2}r\cos\theta\rangle$

$$\vec{r}_r\times\vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta \\ -r\sin\theta & r\cos\theta & -\frac{1}{2}r\cos\theta \end{vmatrix} = \left\langle -\tfrac{1}{2}r\sin\theta\cos\theta + \tfrac{1}{2}r\sin\theta\cos\theta, -\left(-\tfrac{1}{2}r\cos^2\theta - \tfrac{1}{2}r\sin^2\theta\right), r\cos^2\theta + r\sin^2\theta \right\rangle$$

$$= \langle 0, \tfrac{1}{2}r, r\rangle$$

So $\|\vec{r}_r\times\vec{r}_\theta\|^2 = \frac{1}{4}r^2 + r^2 = \frac{5}{4}r^2$ and $\|\vec{r}_r\times\vec{r}_\theta\| = \frac{\sqrt{5}}{2}r$

Then $SA = \iint_S 1\, dS = \iint_R \frac{\sqrt{5}}{2} r\, dA = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}}{2} r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{5}}{4} r^2 \Big|_0^1 d\theta = \int_0^{2\pi} \frac{\sqrt{5}}{4}\, d\theta$

$= \frac{\sqrt{5}}{4} * 2\pi = \boxed{\frac{\sqrt{5}\pi}{2}}$

#18 S: portion of $z=-x$ inside $x^2+y^2=4$. $(x,y,z)\in S$ if $z = -r\cos\theta$, $r^2 \le 4$.

$S: \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, -r\cos\theta\rangle$ and $R: r\in[0,2]$, $\theta\in[0,2\pi]$.

$\vec{r}_r = \langle \cos\theta, \sin\theta, -\cos\theta\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, r\sin\theta\rangle$

$$\vec{r}_r\times\vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$$= \langle r\sin^2\theta + r\cos^2\theta, -(r\sin\theta\cos\theta - r\sin\theta\cos\theta), -r\sin^2\theta - r\cos^2\theta\rangle$$

$= \langle r, 0, -r\rangle$ So $\|\vec{r}_r\times\vec{r}_\theta\| = \sqrt{2r^2} = \sqrt{2}\, r$

$$SA = \iint_S 1\, d\sigma = \int_0^{2\pi}\int_0^2 \sqrt{2}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} r^2 \Big|_0^2 d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} * 2^2\, d\theta = \int_0^{2\pi} 2\sqrt{2}\, d\theta$$

$= 2\sqrt{2}\,\theta \Big|_0^{2\pi} = 2\sqrt{2} * 2\pi - 0 = \boxed{4\sqrt{2}\pi}$

18. Plane inside cylinder The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$

19. Cone frustum The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$

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26. Spherical band The portion of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = -1$ and $z = \sqrt{3}$

Find $SA = \iint_S 1\, d\sigma$

#19 S: on $z = 2\sqrt{x^2+y^2}$ w/ $2 \le z \le 6$

Then using cylindrical coords, $x^2+y^2=r^2$ and $z = 2r$ and $z \in [2,6]$.

$\Rightarrow R$: $r \in [1,3]$, $\theta \in [0, 2\pi]$.

S: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle$

R: $r \in [1,3]$, $\theta \in [0,2\pi]$.

$\vec{r}_r = \langle \cos\theta, \sin\theta, 2\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle -2r\cos\theta, -(2r\sin\theta), r\cos^2\theta + r\sin^2\theta\rangle$$

$$= \langle -2r\cos\theta, -2r\sin\theta, r\rangle$$

So $\|\vec{r}_r \times \vec{r}_\theta\|^2 = 4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2$

$= 4r^2 + r^2 = 5r^2 \Rightarrow \|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{5}\, r$

So $SA = \iint_S 1\, d\sigma = \iint_R \sqrt{5}\, r\, dA = \int_0^{2\pi}\int_1^3 \sqrt{5}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{5}}{2} r^2 \Big|_1^3 d\theta = \int_0^{2\pi} \frac{\sqrt{5}}{2}(9-1)\, d\theta$

$= \int_0^{2\pi} 4\sqrt{5}\, d\theta = 4\sqrt{5} * 2\pi - 0 = \boxed{8\sqrt{5}\pi}$

#20 S: portion of $z = \frac{\sqrt{x^2+y^2}}{3}$ w/ $1 \le z \le 4/3$. So $z = r/3$, $r \in [3,4]$.

and S: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r/3\rangle$, R: $r \in [3,4]$ and $\theta \in [0, 2\pi]$.

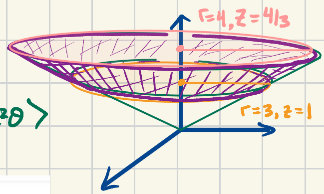
$\vec{r}_r = \langle \cos\theta, \sin\theta, 1/3\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1/3 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

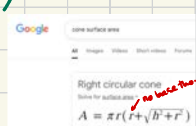
$$= \langle -\tfrac{1}{3}r\cos\theta, -(\tfrac{1}{3}r\sin\theta), r\cos^2\theta + r\sin^2\theta\rangle$$

$$= \langle -\tfrac{1}{3}r\cos\theta, -\tfrac{1}{3}r\sin\theta, r\rangle$$



So $\|\vec{r}_r \times \vec{r}_\theta\|^2 = \frac{1}{9}r^2\cos^2\theta + \frac{1}{9}r^2\sin^2\theta + r^2$

$= \frac{1}{9}r^2 + r^2 = \frac{10}{9}r^2 \Rightarrow \|\vec{r}_r \times \vec{r}_\theta\| = \frac{\sqrt{10}}{3} r$



A(large cone) $= 4\pi\sqrt{16/9+16}$

A(small cone) $= 3\pi\sqrt{1+9}$

difference $= \pi\left[\frac{4\sqrt{10}*4}{3} - 3\sqrt{10}\right]$

$= \pi\left[\frac{16\sqrt{10}}{3} - \frac{9\sqrt{10}}{3}\right]$

$= \boxed{\frac{7\sqrt{10}}{3}\pi}$ ✓

So $SA = \iint_S 1\, d\sigma = \iint_R \frac{\sqrt{10}}{3} r\, dA = \int_0^{2\pi}\int_3^4 \frac{\sqrt{10}}{3} r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{10}}{6} r^2 \Big|_3^4 d\theta$

$= \int_0^{2\pi} \frac{\sqrt{10}}{6}(16-9)\, d\theta = \int_0^{2\pi} \frac{7\sqrt{10}}{6}\, d\theta = \frac{7\sqrt{10}}{6} * 2\pi - 0 = \boxed{\frac{7\sqrt{10}}{3}\pi}$

Find $SA = \iint_S 1\, d\sigma$

18. **Plane inside cylinder** The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$
19. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$
20. **Cone frustum** The portion of the cone $z = \sqrt{x^2 + y^2}/3$ between the planes $z = 1$ and $z = 4/3$
21. **Circular cylinder band** The portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$
22. **Circular cylinder band** The portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$
23. **Parabolic cap** The cap cut from the paraboloid $z = 2 - x^2 - y^2$ by the cone $z = \sqrt{x^2 + y^2}$
24. **Parabolic band** The portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$
25. **Sawed-off sphere** The lower portion cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$
26. **Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = -1$ and $z = \sqrt{3}$

#24 S: on $z = x^2 + y^2$ between $z \in [1, 4]$.

cylindrical coords $z = r^2$, $r \in [1, 2]$

$S: \vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$, $R: r \in [1,2], \theta \in [0, 2\pi]$

$\vec{r}_r = \langle \cos\theta, \sin\theta, 2r \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle -2r^2\cos\theta, 2r^2\sin\theta, r\cos^2\theta + r\sin^2\theta \rangle$$

So $\|\vec{r}_r \times \vec{r}_\theta\|^2 = 4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2$

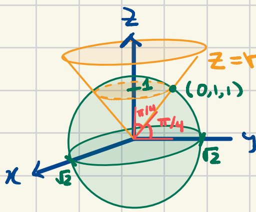
and $\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$

$$SA = \iint_S 1\, d\sigma = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\, dr\, d\theta = \int_0^{2\pi} \frac{1}{8}\frac{2}{3}u^{3/2}\Big|_5^{17} = \boxed{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}$$

$$\boxed{\begin{aligned} u &= 4r^2+1 \\ du &= 8r\, dr \\ \tfrac{1}{8}du &= r\, dr \end{aligned}}$$

$r = 1 \Rightarrow u = 5$

$r = 2 \Rightarrow u = 17$



#25 S: on $x^2 + y^2 + z^2 = 2$ and below $z = \sqrt{x^2+y^2}$

intersection $x^2 + y^2 + (x^2 + y^2) = 2$

$\Rightarrow x^2 + y^2 = 1 \Rightarrow z = 1$

cartesian $(0,1,1)_c = (\sqrt{2}, \pi/2, \pi/4)_s$ spherical coord

Note standard measure element $d\sigma = \|\vec{r}_\varphi \times \vec{r}_\theta\|\, d\varphi\, d\theta = \rho^2 \sin\varphi\, d\varphi\, d\theta$ (*)

So $S: \vec{r}(\varphi, \theta) = \langle \sqrt{2}\sin\varphi\cos\theta, \sqrt{2}\sin\varphi\sin\theta, \sqrt{2}\cos\varphi \rangle$

$R: \varphi \in [\pi/4, \pi]$ and $\theta \in [0, 2\pi]$.

$\vec{r}_\varphi = \langle \sqrt{2}\cos\varphi\cos\theta, \sqrt{2}\cos\varphi\sin\theta, -\sqrt{2}\sin\varphi \rangle$

$\vec{r}_\theta = \langle -\sqrt{2}\sin\varphi\sin\theta, \sqrt{2}\sin\varphi\cos\theta, 0 \rangle$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sqrt{2}\cos\varphi\cos\theta & \sqrt{2}\cos\varphi\sin\theta & -\sqrt{2}\sin\varphi \\ -\sqrt{2}\sin\varphi\sin\theta & \sqrt{2}\sin\varphi\cos\theta & 0 \end{vmatrix} = \langle 2\sin^2\varphi\cos\theta, -2\sin^2\varphi\sin\theta, \overbrace{2\sin\varphi\cos\varphi\cos^2\theta + 2\sin\varphi\cos\varphi\sin^2\theta}^{1} \rangle$$

$$\|\vec{r}_\varphi \times \vec{r}_\theta\|^2 = \overbrace{4\sin^4\varphi\cos^2\theta + 4\sin^4\sin^2\theta}^{1} + 4\sin^2\varphi\cos^2\varphi = 4\sin^4\varphi + 4\sin^2\varphi\cos^2\varphi$$

$$= 4\sin^2\varphi(\sin^2\varphi + \cos^2\varphi) = 4\sin^2\varphi.$$ So $\|\vec{r}_\varphi \times \vec{r}_\theta\| = 2\sin\varphi$ (*) ✓

$$SA = \iint_S 1\, ds = \int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi} -2\cos\varphi\Big|_{\pi/4}^{\pi} d\theta = \int_0^{2\pi} -2(\overbrace{\cos\pi - \cos\pi/4}^{-1 - \sqrt{2}/2})\, d\theta = \int_0^{2\pi}(2+\sqrt{2})\, d\theta$$

$$= \boxed{2\pi(2+\sqrt{2})}$$

Evaluate the surface integral.

16.6

Exercises 16.6

Surface Integrals of Scalar Functions

In Exercises 1–8, integrate the given function over the given surface.

1. **Parabolic cylinder** $G(x, y, z) = x$, over the parabolic cylinder $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$
2. **Circular cylinder** $G(x, y, z) = z$, over the cylindrical surface $y^2 + z^2 = 4$, $z \ge 0$, $1 \le x \le 4$
3. **Sphere** $G(x, y, z) = x^2$, over the unit sphere $x^2 + y^2 + z^2 = 1$
4. **Hemisphere** $G(x, y, z) = z^2$, over the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$
5. **Portion of plane** $F(x, y, z) = z$, over the portion of the plane $x + y + z = 4$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane
6. **Cone** $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$
7. **Parabolic dome** $H(x, y, z) = x^2\sqrt{5 - 4z}$, over the parabolic dome $z = 1 - x^2 - y^2$, $z \ge 0$
8. **Spherical cap** $H(x, y, z) = yz$, over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

#1 $G(x,y,z)=x$ over S: $y=x^2$, $0\le x\le 2$, $0\le z\le 3$.

Parametrize S : $\vec{r}(s,t)=\langle s,s^2,t\rangle$
R : $s\in[0,2]$, $t\in[0,3]$

Then $\vec{r}_s=\langle 1,2s,0\rangle$, $\vec{r}_t=\langle 0,0,1\rangle$

$$\vec{r}_s\times\vec{r}_t=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2s&0\\0&0&1\end{vmatrix}=\langle 2s,-1,0\rangle$$

So $\|\vec{r}_s\times\vec{r}_t\|=\sqrt{1+4s^2}$ and $M=\iint_S G\,d\sigma=\int_0^3\int_0^2 s\sqrt{1+4s^2}\,ds\,dt$

u-sub: $u=1+4s^2$, $du=8s\,ds$

$$=\int_0^3 \frac{1}{8}*\frac{2}{3}(1+4s^2)^{3/2}\Big|_0^2\,dt=\int_0^3\frac{1}{12}(17^{3/2}-1)\,dt=\boxed{\frac{1}{4}(17^{3/2}-1)}$$

#2 S: $y^2+z^2=4$, $z\ge 0$, $1\le x\le 4$. use cylindrical coords @ $r=2$ $\begin{cases}x=t\\y=2\cos\theta\\z=2\sin\theta\end{cases}$

$\theta\in[0,\pi]$ $(z\ge 0)$
$t\in[1,4]$

S: $\vec{r}(t,\theta)=\langle t,2\cos\theta,2\sin\theta\rangle$, R: $t\in[1,4]$, $\theta\in[0,\pi]$

$\vec{r}_t=\langle 1,0,0\rangle$, $\vec{r}_\theta=\langle 0,-2\sin\theta,2\cos\theta\rangle$

$$\vec{r}_t\times\vec{r}_\theta=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&0&0\\0&-2\sin\theta&2\cos\theta\end{vmatrix}=\langle 0,-2\cos\theta,-2\sin\theta\rangle$$

So $\|\vec{r}_t\times\vec{r}_\theta\|=\sqrt{4\cos^2\theta+4\sin^2\theta}=2$

Then $M=\iint_S G\,d\sigma=\int_0^\pi\int_1^4 t*2\,dt\,d\theta=\int_0^\pi t^2\Big|_1^4\,d\theta=\int_0^\pi 15\,d\theta=15\theta\Big|_0^\pi=\boxed{15\pi}$

#3 S: unit sphere $\rho=1$ then surface measure is the standard spherical coords element $d\sigma=\|\vec{r}_\varphi\times\vec{r}_\theta\|=\rho^2\sin\varphi\,d\varphi\,d\theta$

$r(\varphi,\theta)=\langle\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi\rangle$
R: $\varphi\in[0,\pi]$, $\theta\in[0,2\pi]$

So $M=\iint_S G\,d\sigma=\int_0^{2\pi}\int_0^\pi(\sin\varphi\cos\theta)^2\sin\varphi\,d\varphi\,d\theta=\int_0^{2\pi}\int_0^\pi(1-\cos^2\varphi)\cos^2\theta\sin\varphi\,d\varphi$

u-sub: $u=\cos\varphi$, $du=-\sin\varphi\,d\varphi$; $\varphi=0\Rightarrow u=1$, $\varphi=\pi\Rightarrow u=-1$

$$=\int_0^{2\pi}\int_1^{-1}-(1-u^2)\cos^2\theta\,du\,d\theta=\int_0^{2\pi}\int_{-1}^1(1-u^2)\cos^2\theta\,du\,d\theta=\int_0^{2\pi}\cos^2\theta\left(u-\frac{1}{3}u^3\Big|_{-1}^1\right)d\theta=\int_0^{2\pi}2\cos^2\theta\left(1-\frac{1}{3}\right)d\theta$$

$$=\frac{4}{3}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta=\frac{2}{3}\left(\theta-\frac{1}{2}\cos 2\theta\Big|_0^{2\pi}\right)=\frac{2}{3}\left[\left(2\pi-\frac{1}{2}\right)-\left(0-\frac{1}{2}\right)\right]=\boxed{\frac{4\pi}{3}}$$

Evaluate the surface integral

#5 S: on $x+y+z=4$ over $x\in[0,1]$, $y\in[0,1]$

$r(s,t)=\langle s,t,4-s-t\rangle$

R: $s\in[0,1]$, $t\in[0,1]$

Standard element for ⊛ $z=f(x,y)$ is

$d\sigma = \|\vec r_s \times \vec r_t\| = \sqrt{1+f_x^2+f_y^2}$

$r_s=\langle 1,0,-1\rangle$

$r_t=\langle 0,1,-1\rangle$

$r_s\times r_t = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ 1 & 0 & -1 \\ 0 & 1 & -1\end{vmatrix} = \langle 1,1,1\rangle$ so $\|r_s\times r_t\| = \sqrt3$

$$M=\iint_S F\,d\sigma = \int_0^1\int_0^1 (4-s-t)\sqrt3\,dt\,ds = \sqrt3\int_0^1 4t-st-\tfrac12 t^2\Big|_0^1 ds$$

$$=\sqrt3\int_0^1 \left(4-s-\tfrac12\right)-0\,ds = \sqrt3\int_0^1 \tfrac72 - s\,ds = \sqrt3\left(\tfrac72 s-\tfrac12 s^2\Big|_0^1\right)$$

$$=\sqrt3\left[\left(\tfrac72-\tfrac12\right)-0\right] = \boxed{3\sqrt3}$$

Find Flux of F over S.

#19 S: on $z=4-y^2$ between $x=0, x=1$ and above $z=0$

so $r(s,t)=\langle s,t,4-t^2\rangle$, R: $s\in[0,1]$, $t\in[-2,2]$

$r_s=\langle 1,0,0\rangle$, $r_t=\langle 0,1,-2t\rangle$ $r_s\times r_t = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ 1 & 0 & 0 \\ 0 & 1 & -2t\end{vmatrix} = \langle 0,2t,1\rangle$

So $\|r_s\times r_t\| = \sqrt{1+4t^2}$

matches ⊛ above ✓

$$\text{So Flux}=\iint_S F\cdot n\,d\sigma = \int_0^1\int_{-2}^2 \langle (4-t^2)^2, s, -3(4-t^2)\rangle\cdot\langle 0,2t,1\rangle\,dt\,ds$$

$$=\int_0^1\int_{-2}^2 2st-3(4-t^2)\,dt\,ds = \int_0^1 \cancel{st^2}^{\,0} - 12t + t^3\Big|_{-2}^{2} ds = \int_0^1 2(-24+8)\,ds$$

$$=\int_0^1 -32\,ds = -32s\Big|_0^1 = \boxed{-32}$$

#20 S: on $y=x^2$, $-1\le x\le 1$, between $z=0$ and $z=2$

$r(s,t)=\langle s,s^2,t\rangle$, R: $s\in[-1,1]$, $t\in[0,2]$

$r_s=\langle 1,2s,0\rangle$, $r_t=\langle 0,0,1\rangle$ $r_s\times r_t = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ 1 & 2s & 0 \\ 0 & 0 & 1\end{vmatrix} = \langle 2s,-1,0\rangle$ $F(r(s,t))=\langle 0,s^2,-st\rangle$

$$\text{Flux}=\int_{-1}^1\int_0^2 \langle 0,s^2,-st\rangle\cdot\langle 2s,-1,0\rangle\,dt\,ds = \int_{-1}^1\int_0^2 -s^2\,dt\,ds = \int_{-1}^1 -2s^2\,ds = -\tfrac23 s^3\Big|_{-1}^1$$

odd function from $-a$ to a.

$$=2\left(-\tfrac23(1)\right) = \boxed{-4/3}$$

Exercises 16.6

Surface Integrals of Scalar Functions

In Exercises 1–8, integrate the given function over the given surface.

1. **Parabolic cylinder** $G(x, y, z) = x$, over the parabolic cylinder $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$
2. **Circular cylinder** $G(x, y, z) = z$, over the cylindrical surface $y^2 + z^2 = 4$, $z \ge 0$, $1 \le x \le 4$
3. **Sphere** $G(x, y, z) = x^2$, over the unit sphere $x^2 + y^2 + z^2 = 1$
4. **Hemisphere** $G(x, y, z) = z^2$, over the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$
5. **Portion of plane** $F(x, y, z) = z$, over the portion of the plane $x + y + z = 4$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane
6. **Cone** $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$
7. **Parabolic dome** $H(x, y, z) = x^2\sqrt{5 - 4z}$, over the parabolic dome $z = 1 - x^2 - y^2$, $z \ge 0$
8. **Spherical cap** $H(x, y, z) = yz$, over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

Finding Flux or Surface Integrals of Vector Fields

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ across the surface in the specified direction.

19. **Parabolic cylinder** $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$
20. **Parabolic cylinder** $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ outward (normal away from the yz-plane) through the surface cut from the parabolic cylinder $y = x^2$, $-1 \le x \le 1$, by the planes $z = 0$ and $z = 2$
21. **Sphere** $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin
22. **Sphere** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ in the direction away from the origin

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$$

Find the Flux of F over S

Finding Flux or Surface Integrals of Vector Fields

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ across the surface in the specified direction.

19. **Parabolic cylinder** $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$
20. **Parabolic cylinder** $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ outward (normal away from the yz-plane) through the surface cut from the parabolic cylinder $y = x^2$, $-1 \le x \le 1$, by the planes $z = 0$ and $z = 2$
21. **Sphere** $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin
22. **Sphere** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ in the direction away from the origin

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$$

#21 S: Sphere of radius $\rho = a$, First octant so $x \ge 0$, $y \ge 0$, $z \ge 0$

$r(\varphi,\theta) = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi\rangle$

R: $\theta \in [0, \pi/2]$, $\varphi \in [0, \pi/2]$.

$r_\varphi = \langle a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi\rangle$

$r_\theta = \langle -a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0\rangle$ $\quad F(r(\varphi,\theta)) = \langle 0, 0, a\cos\varphi\rangle$

$$r_\varphi \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \langle a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, \overbrace{a^2\sin\varphi\cos\varphi\cos^2\theta + a^2\sin\varphi\cos\varphi\sin^2\theta}^{1}\rangle$$

$$= \langle a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi\rangle$$

(*) u-sub: $\boxed{u = \cos\varphi \quad du = -\sin\varphi\, d\varphi}$

So $\text{Flux} = \int_0^{\pi/2}\int_0^{\pi/2} \langle 0,0,a\cos\varphi\rangle \cdot \langle a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi\rangle \, d\varphi\, d\theta$

$$= \int_0^{\pi/2}\int_0^{\pi/2} a^3\sin\varphi\cos^2\varphi\, d\varphi\, d\theta \overset{*}{=} \int_0^{\pi/2} a^3 * \frac{-1}{3}\cos^3\varphi \Big|_0^{\pi/2} d\theta = \int_0^{\pi/2} -\frac{a^3}{3}\left(\cancel{\cos^3\left(\tfrac{\pi}{2}\right)}^{0} - \cancel{\cos^3(0)}^{1}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{a^3}{3}\, d\theta = \frac{a^3}{3}\theta \Big|_0^{\pi/2} = \frac{a^3}{3} * \frac{\pi}{2} = \boxed{\frac{a^3 * \pi}{6}}$$

23. **Plane** $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ upward across the portion of the plane $x + y + z = 2a$ that lies above the square $0 \le x \le a$, $0 \le y \le a$, in the xy-plane
24. **Cylinder** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = a$
25. **Cone** $\mathbf{F} = xy\mathbf{i} - z\mathbf{k}$ outward (normal away from the z-axis) through the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$
26. **Cone** $\mathbf{F} = y^2\mathbf{i} + xz\mathbf{j} - \mathbf{k}$ outward (normal away from the z-axis) through the cone $z = 2\sqrt{x^2 + y^2}$, $0 \le z \le 2$
27. **Cone frustum** $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$ outward (normal away from the z-axis) through the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$
28. **Paraboloid** $\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (normal away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$

#25 S: on $z = \sqrt{x^2+y^2}$ w/ $z \in [0,1]$ use cylindrical coords

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ $\quad$ R: $r \in [0,1]$, $\theta \in [0, 2\pi]$

$\vec{r}_r = \langle \cos\theta, \sin\theta, 1\rangle$,

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$ $\quad F(\vec{r}(r,\theta)) = \langle r^2\cos\theta\sin\theta, 0, -r\rangle$

Flip order

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -(r\sin\theta), r\cos^2\theta + r\sin^2\theta\rangle = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

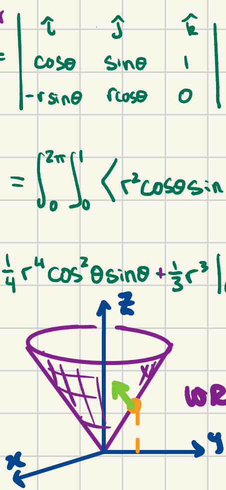
at $\langle x,y,z\rangle$ $n \sim \langle -x,-y,z\rangle$ Flip sign!

u-sub: $\boxed{u = \cos\varphi \quad du = -\sin\varphi\, d\varphi}$

So

$$\text{Flux} = \int_0^{2\pi}\int_0^1 \langle r^2\cos\theta\sin\theta, 0, -r\rangle \cdot \langle +r\cos\theta, +r\sin\theta, -r\rangle\, dr\, d\theta = \int_0^{2\pi}\int_0^1 + r^3\cos^2\theta\sin\theta + r^2\, dr\, d\theta$$

$$= \int_0^{2\pi} +\frac{1}{4}r^4\cos^2\theta\sin\theta + \frac{1}{3}r^3 \Big|_0^1 d\theta = \int_0^{2\pi} +\frac{1}{4}\cos^2\theta\sin\theta + \frac{1}{3}\, d\theta = \frac{-1}{12}\cos^3\theta + \frac{1}{3}\theta \Big|_0^{2\pi} = \left(\cancel{\frac{-1}{12}\cos^3(2\pi)} + \frac{2\pi}{3}\right) - \left(\cancel{\frac{-1}{12}\cos^3(0)} + 0\right)$$



WRONG DIRECTION!

$$= \boxed{\frac{+2\pi}{3}}$$ ✓

Use S'sT to Calculate circulation

§16.7

Exercises 16.7

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction. P Q R

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$

   C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above

2. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$

   C: The circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above

3. $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$

   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

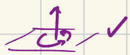
#1. C: $4x^2+y^2=4$ in the xy-plane, CCW from above

$\Rightarrow x^2+\frac{y^2}{4}=1 \Rightarrow x=\cos\theta,\ y=2\sin\theta,\ \theta\in[0,2\pi]$

S: $r(t,\theta)=\langle t\cos\theta, 2t\sin\theta, 0\rangle$ R: $t\in[0,1],\ \theta\in[0,2\pi]$.

$r_t=\langle\cos\theta, 2\sin\theta, 0\rangle$, $r_\theta=\langle -t\sin\theta, 2t\cos\theta, 0\rangle$

$$r_t\times r_\theta=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos\theta & 2\sin\theta & 0\\ -t\sin\theta & 2t\cos\theta & 0\end{vmatrix}=\langle 0,0,2t\cos^2\theta+2t\sin^2\theta\rangle=\langle 0,0,2t\rangle$$

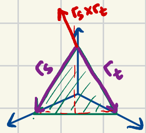
pointing up ✓  ✓

$F=\langle x^2, 2x, z^2\rangle$

$\operatorname{Curl} F=\nabla\times F=\langle R_y-Q_z, P_z-R_x, Q_x-P_y\rangle=\langle 0-0, 0-0, 2-0\rangle=\langle 0,0,2\rangle$

$$\text{Flow}=\oint_C F\cdot dr \overset{S'sT}{=} \iint_S \nabla\times F\cdot n\, d\sigma=\int_0^{2\pi}\int_0^1\langle 0,0,2\rangle\cdot\langle 0,0,2t\rangle\,dt\,d\theta=\int_0^{2\pi}\int_0^1 4t\,dt\,d\theta$$

$$=\int_0^{2\pi} 2t^2\Big|_0^1 d\theta=\int_0^{2\pi} 2\,d\theta=2(2\pi-0)=\boxed{4\pi}$$



#3 C: boundary of $x+y+z=1$ in first octant $(x\geq 0, y\geq 0, z\geq 0)$

S: $\vec{r}(s,t)=\langle s, t, 1-s-t\rangle$, R: $s\in[0,1],\ t\in[0,1-s]$

$$F=\langle \overset{P}{y}, \overset{Q}{xz}, \overset{R}{x^2}\rangle$$

$r_s=\langle 1,0,-1\rangle$ $r_t=\langle 0,1,-1\rangle$
$$r_s\times r_t=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & -1\\ 0 & 1 & -1\end{vmatrix}=\langle 1,1,1\rangle$$
outward pointing ✓

$\nabla\times F=\langle R_y-Q_z, P_z-R_x, Q_x-P_y\rangle=\langle 0-x, 0-2x, z-1\rangle=\langle -x,-2x,z-1\rangle$

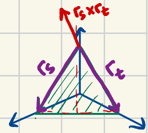
$$\text{Flow}=\oint_C F\cdot dr \overset{S'sT}{=} \iint_S \nabla\times F\cdot n\,d\sigma=\int_0^1\int_0^{1-s}\langle -s,-2s,(1-s-t)-1\rangle\cdot\langle 1,1,1\rangle\,dt\,ds$$

$$=\int_0^1\int_0^{1-s} -s-2s-s-t\,dt\,ds=-\int_0^1\int_0^{1-s} +4s+t\,dt\,ds=-\int_0^1 4st+\tfrac{1}{2}t^2\Big|_0^{1-s} ds$$

$$=-\int_0^1 4s(1-s)+\tfrac{1}{2}(1-s)^2\,ds=-\int_0^1 4s-4s^2+\tfrac{1}{2}s^2-s+\tfrac{1}{2}\,ds=\int_0^1 \tfrac{7}{2}s^2-3s-\tfrac{1}{2}\,ds=\tfrac{7}{6}s^3-\tfrac{3}{2}s^2-\tfrac{1}{2}s\Big|_0^1$$

$$=\frac{7}{6}-\frac{3}{2}-\frac{1}{2}=\frac{7}{6}-\frac{12}{6}=\boxed{\frac{-5}{6}}$$

Find Circulation using S's T



Exercises 16.7

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$
   C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above
2. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$
   C: The circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above
3. $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$
   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above
4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$
   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above
5. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$
   C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counterclockwise when viewed from above
6. $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$
   C: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above

#4 C: boundary of $x+y+z=1$, $x\geq 0, y\geq 0, z\geq 0$.

$S: \vec{r}(s,t) = \langle s, t, 1-s-t\rangle$, $R: s\in[0,1], t\in[0,1-s]$

$$r_s = \langle 1,0,-1\rangle \quad r_t = \langle 0,1,-1\rangle \quad r_s\times r_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \langle 1,1,1\rangle \text{ outward pointing } \checkmark$$

$$F = \langle \overset{P}{y^2+z^2}, \overset{Q}{x^2+z^2}, \overset{R}{x^2+y^2}\rangle$$

$$\nabla\times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 2y-2z, 2z-2x, 2x-2y\rangle$$

So

$$\text{Flow} = \oint_C F\cdot dr \overset{S's T}{=} \int_0^1\int_0^{1-s} \langle 2y-2z, 2z-2x, 2x-2y\rangle \cdot \langle 1,1,1\rangle \, dt\, ds$$

$$= \int_0^1\int_0^{1-s} \cancel{2y}-\cancel{2z}+\cancel{2z}-\cancel{2x}+\cancel{2x}-\cancel{2y}\, dt\,ds = \int_0^1\int_0^{1-s} 0\, dt\, ds = \boxed{0}$$

#5 C: boundary of $x\in[-1,1], y\in[-1,1]$ $\quad F = \langle y^2+z^2, x^2+z^2, x^2+y^2\rangle$

$S: r(s,t) = \langle s,t,0\rangle$ $s\in[-1,1]$, $t\in[-1,1]$ $\quad \nabla\times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 2y-2z, 2z-2x, 2x-2y\rangle$

$$r_s = \langle 1,0,0\rangle \quad r_t = \langle 0,1,0\rangle \quad r_s\times r_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{vmatrix} = \langle 0,0,1\rangle$$

$$\text{So Flow} = \oint_C F\cdot dr \overset{S's T}{=} \int_0^1\int_0^1 \langle 2t, -2s, 2s-2t\rangle \cdot \langle 0,0,1\rangle\, ds\, dt = \int_0^1\int_0^1 2s-2t\, ds\, dt$$

$$= \int_0^1 s^2 - 2ts\Big|_0^1 dt = \int_0^1 1-2t\, dt = t - t^2\Big|_0^1 = 1-1 = \boxed{0}$$

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

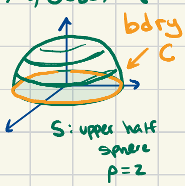
13. $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$
    S: $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (4 - r^2)\mathbf{k}$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$
14. $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$
    S: $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9 - r^2)\mathbf{k}$, $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$
15. $\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$
    S: $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$
16. $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$
    S: $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (5 - r)\mathbf{k}$, $0 \leq r \leq 5$, $0 \leq \theta \leq 2\pi$

Use S's T to calculate Flux of $\nabla\times F$ across S.

#13 $S: \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4-r^2\rangle$ $\quad R: r\in[0,2], \theta\in[0,2\pi]$

$C: \vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0\rangle$, $\theta\in[0,2\pi]$

bdry circle of radius 2 in xy-plane



$\vec{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0\rangle$

$F = \langle 2z, 3x, 5y\rangle$

$$\text{So Flux thru } S = \iint_S \nabla\times F\cdot n\, d\sigma = \oint_C F\cdot dr$$

$$= \int_0^{2\pi} \langle 0, 6\cos\theta, 10\sin\theta\rangle\cdot\langle -2\sin\theta, 2\cos\theta, 0\rangle\, d\theta = \int_0^{2\pi} 12\cos^2\theta\, d\theta = \int_0^{2\pi} 6(1+\cos 2\theta)\, d\theta$$

$$= 6\theta + 3\sin 2\theta\Big|_0^{2\pi} = (6(2\pi) + 3\sin 4\pi) - (6(0) + 3\sin 0) = \boxed{12\pi}$$

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field **F** across the surface S in the direction of the outward unit normal **n**.

13. $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (4 - r^2)\mathbf{k}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

14. $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9 - r^2)\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

15. $\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

16. $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (5 - r)\mathbf{k}$, $0 \le r \le 5$, $0 \le \theta \le 2\pi$

Use S'sT to Evaluate Flux of $\nabla \times F$ across S

top half of sphere $\rho = 3$

#14 $S: \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 9 - r^2\rangle$ $R: r \in [0,3], \theta \in [0, 2\pi]$

$C: \vec{r}(\theta) = \langle 3\cos\theta, 3\sin\theta, 0\rangle$ circle w/ radius 3 in xy-plane

$\vec{r}'(\theta) = \langle -3\sin\theta, 3\cos\theta, 0\rangle, \theta \in [0, 2\pi]$

$F = \langle y - z, z - x, x + z\rangle$

$$\text{Flux} = \iint_S \nabla \times F \cdot n \, d\sigma \overset{\text{S'sT}}{=} \oint_C F \cdot dr = \int_0^{2\pi} \langle 3\sin\theta, -3\cos\theta, 3\cos\theta\rangle \cdot \langle -3\sin\theta, 3\cos\theta, 0\rangle \, d\theta$$

$$= \int_0^{2\pi} -9\sin^2\theta - 9\cos^2\theta \, d\theta = -9\int_0^{2\pi} 1 \, d\theta = -9\theta \Big|_0^{2\pi} = -9(2\pi - 0) = \boxed{-18\pi}$$

Calculate div F

If $F=\langle P,Q\rangle$, then $\operatorname{div} F = P_x + Q_y$

§16.8

Exercises 16.8

Calculating Divergence

In Exercises 1–4, find the divergence of the field.

1. The spin field in Figure 16.12
2. The radial field in Figure 16.11
3. The gravitational field in Figure 16.8 and Exercise 38a in Section 16.3
4. The velocity field in Figure 16.13

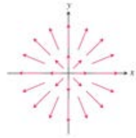


FIGURE 16.11 The radial field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of position vectors of points in the plane. Notice the convention that an arrow is drawn with its tail, not its head, at the point where $\mathbf{F}$ is evaluated.

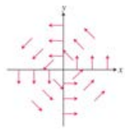


FIGURE 16.12 A "spin" field of rotating unit vectors $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2+y^2)^{1/2}$ in the plane. The field is not defined at the origin.

#1 $F = \left\langle \dfrac{-y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$

$$P = \frac{-y}{\sqrt{x^2+y^2}},\quad P_x = \frac{\partial}{\partial x}\left(-y(x^2+y^2)^{-1/2}\right) = \frac{y}{2}(x^2+y^2)^{-3/2} * 2x = \frac{xy}{(x^2+y^2)^{3/2}}$$

$$Q = \frac{x}{\sqrt{x^2+y^2}},\quad Q_y = \frac{\partial}{\partial y}\left(x(x^2+y^2)^{-1/2}\right) = \frac{-x}{2}(x^2+y^2)^{-3/2} * 2y = \frac{-xy}{(x^2+y^2)^{3/2}}$$

So $\operatorname{div} F = P_x + Q_y = \dfrac{xy}{(x^2+y^2)^{3/2}} + \dfrac{-xy}{(x^2+y^2)^{3/2}} = \boxed{0.}$

#3 $F = \dfrac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

38. Gravitational field

a. Find a potential function for the gravitational field
$$\mathbf{F} = -GmM\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$
(G, m, and M are constants).

b. Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is
$$GmM\left(\frac{1}{s_2}-\frac{1}{s_1}\right).$$

b. FToLI
$$W = \int_C F\cdot dr = \int_A^B F\cdot dr$$
$$= f(B) - f(A)$$
$$= \frac{-1}{s_2} + \frac{1}{s_1}$$

Note $f(x,y,z) = -\dfrac{1}{(x^2+y^2+z^2)^{1/2}}$ is potential fn. for F,

so $\nabla f = F$ ⊛

$$P_x = \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$Q_y = \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} = \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$R_z = \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$$

So $\operatorname{div} F = \nabla\cdot\nabla f = \langle f_{xx}, f_{yy}, f_{zz}\rangle =: \nabla^2 f$

called the Laplacian of f

$$P_x = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{(x^2+y^2+z^2)^{3/2} - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{1/2} * 2x}{(x^2+y^2+z^2)^3}$$

$$= \frac{\cancel{(x^2+y^2+z^2)^{1/2}}\left[(x^2+y^2+z^2) - 3x^2\right]}{(x^2+y^2+z^2)^{\cancel{3}\,5/2}} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}},$$

Sim. for y, z by symmetry of F

So $\operatorname{div} F = P_x + Q_y + R_z = \dfrac{1}{(x^2+y^2+z^2)^{5/2}} = \left(\cancel{-2x^2} + \cancel{y^2} + \cancel{z^2} + \cancel{x^2} - \cancel{2y^2} + \cancel{z^2} + \cancel{x^2} + \cancel{y^2} - \cancel{2z^2}\right)$

$= \boxed{0}$

IDEA: Alternate method using the CHAIN RULE

Set $r = \sqrt{x^2+y^2+z^2}$ so that $f(r) = \dfrac{-1}{r}$ the potential function for F, $\nabla f = F$. ⊛

$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Then $\dfrac{\partial f}{\partial x} = \dfrac{df}{dr} * \dfrac{\partial r}{\partial x} = \dfrac{1}{r^2} * \dfrac{x}{(x^2+y^2+z^2)^{1/2}} = \dfrac{x}{r^3}$, Sim. $\dfrac{\partial f}{\partial y} = \dfrac{y}{r^3}$, $\dfrac{\partial f}{\partial z} = \dfrac{z}{r^3}$

and $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\dfrac{x}{r^3} = \dfrac{r^3(1) - x\,3r^2\frac{\partial r}{\partial x}}{r^6} = \dfrac{r^3 - 3r^{\cancel{2}}x\cdot\frac{x}{\cancel{r}}}{r^6} = \dfrac{r^2-3x^2}{r^5}$,

Sim $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{r^2-3y^2}{r^5}$, $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{r^2-3z^2}{r^5}$

So $\nabla^2 f = \operatorname{div}\nabla f = \dfrac{1}{r^5}\left(r^2-3x^2+r^2-3y^2+r^2-3z^2\right)$

$= \dfrac{1}{r^5}\left(3r^2 - 3\underbrace{(x^2+y^2+z^2)}_{r^2}\right) = \dfrac{1}{r^5}(3r^2-3r^2) = \boxed{0}$

Calculating Flux Using the Divergence Theorem

In Exercises 5–16, use the Divergence Theorem to find the outward flux of **F** across the boundary of the region D.

5. Cube $\mathbf{F} = (y-x)\mathbf{i} + (z-y)\mathbf{j} + (y-x)\mathbf{k}$

D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

6. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

a. Cube D: The cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

b. Cube D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

c. Cylindrical can D: The region cut from the solid cylinder $x^2 + y^2 \le 4$ by the planes $z = 0$ and $z = 1$

7. Cylinder and paraboloid $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$

D: The region inside the solid cylinder $x^2 + y^2 \le 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$

Divergence Theorem

$$\boxed{\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV}$$

Use Divergence Theorem to calculate Flux across S

#5 $F = \langle y-x,\ z-y,\ y-x\rangle$ $D: x\in[-1,1],\ y\in[-1,1],\ z\in[-1,1]$.

S: boundary of D.

$$\text{Flux} = \iint_S F\cdot n\,d\sigma \overset{DT}{=} \iiint_D \nabla\cdot F\,dV.$$

$P_x = -1$, $Q_y = -1$, $R_z = 0$ $\text{div}\,F = P_x + Q_y + R_z = -2$

$$\text{Flux} \overset{DT}{=} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -2\,dV = -2 * \text{Vol}(\text{cube}) = -2 * 2^3 = \boxed{-16}$$

#6 $F = \langle x^2, y^2, z^2\rangle$ $D_1: x\in[0,1],\ y\in[0,1],\ z\in[0,1]$. $\text{div}\,F = 2x+2y+2z$

(a)
$$\text{Flux} \overset{DT}{=} \iiint_{D_1} 2x+2y+2z\,dV = \int_0^1\int_0^1\int_0^1 2x+2y+2z\,dz\,dy\,dx = \int_0^1\int_0^1 2xz+2yz+z^2\Big|_0^1\,dy\,dx$$

$$= \int_0^1\int_0^1 2x+2y+\tfrac{1}{2}\,dy\,dx = \int_0^1 2xy+y^2+1y\Big|_0^1\,dx = \int_0^1 2x+1+1\,dx = x^2+2x\Big|_0^1 = (1+2)-(0-0) = \boxed{3}$$

(b) Same F, $D_2: x,y,z\in[-1,1]$.

$$\text{Flux} \overset{DT}{=} \iiint_{D_2} 2x+2y+2z\,dV = \int_{-1}^1\int_{-1}^1\int_{-1}^1 2x+2y+2z\,dz\,dy\,dx = \int_{-1}^1\int_{-1}^1 2xz+2yz+z^2\Big|_{-1}^1\,dy\,dx$$

$$= \int_{-1}^1\int_{-1}^1 (2x+2y+\cancel{1}) - (-2x-2y+\cancel{1})\,dy\,dx = \int_{-1}^1\int_{-1}^1 4x+4y\,dy\,dx = \int_{-1}^1 4xy+2y^2\Big|_{-1}^1\,dx$$

$$= \int_{-1}^1 (4x+\cancel{2}) - (-4x+\cancel{2})\,dx = \int_{-1}^1 8x\,dx = 4x^2\Big|_{-1}^1 = 4-4 = \boxed{0}$$

(c) Same F, $D_3: x\in[-2,2],\ y\in[-\sqrt{4-x^2},\sqrt{4-x^2}],\ z\in[0,1]$. , D: Solid cylinder w/ $r=[0,2], \theta\in[0,2\pi], z\in[0,1]$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\text{Flux} \overset{DT}{=} \iiint_D 2x+2y+2z\,dV = \int_0^{2\pi}\int_0^2\int_0^1 (2r\cos\theta + 2r\sin\theta + 2z)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \underbrace{2r^2\cos\theta z + 2r^2\sin\theta z}_{1} + r*\tfrac{1}{2}2z^2\Big|_0^1\,dr\,d\theta = \int_0^{2\pi}\int_0^2 2r^2+r\,dr\,d\theta = \int_0^{2\pi} \tfrac{2}{3}r^3+\tfrac{1}{2}r^2\Big|_0^2\,d\theta = \int_0^{2\pi} \tfrac{16}{3}+2\,d\theta$$

$$= \int_0^{2\pi} \tfrac{22}{3}\,d\theta = \tfrac{22}{3}\theta\Big|_0^{2\pi} = \tfrac{22}{3}*2\pi = \boxed{\tfrac{44\pi}{3}}$$

b. Cube D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

c. Cylindrical can D: The region cut from the solid cylinder $x^2 + y^2 \le 4$ by the planes $z = 0$ and $z = 1$

7. Cylinder and paraboloid $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$

D: The region inside the solid cylinder $x^2 + y^2 \le 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$

8. Sphere $\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$

D: The solid sphere $x^2 + y^2 + z^2 \le 4$

9. Portion of sphere $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$

D: The region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$

10. Cylindrical can $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z)\mathbf{j} + 4x^2y^3\mathbf{k}$

D: The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$

Find the outward Flux of F over D.

#7. $F = \langle y, xy, -z \rangle$

D: inside $x^2+y^2 \le 4$ with $0 \le z \le x^2+y^2$. $P_x = 0$

in cylindrical coords $r^2 \le 4$ and $0 \le z \le r^2$ $Q_y = x$

So D: $r \in [0,2]$, $\theta \in [0, 2\pi]$ and $z \in [0, r^2]$. $R_z = -1$

$$\text{Flux} = \iint_S F \cdot n \, d\sigma \overset{DT}{=} \iiint_D \text{div} F \, dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} (0 + x + (-1)) * r \, dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_0^{r^2} (r^2\cos\theta - r)\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2\cos\theta\, z - rz \Big|_0^{r^2} dr\,d\theta = \int_0^{2\pi}\int_0^2 r^4\cos\theta - r^3\,dr\,d\theta = \int_0^{2\pi} \tfrac{1}{5}r^5\cos\theta - \tfrac{1}{4}r^4\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{2^5}{5}\cos\theta - \frac{16}{4}\,d\theta = \frac{2^5}{5}\sin\theta - 4\pi\theta\Big|_0^{2\pi} = (0 - 8\pi) - (0-0) = \boxed{-8\pi}$$

#8 $F = \langle x^2, xz, 3z \rangle$ D: Solid sphere $x^2+y^2+z^2 \le 4$ so D: $\rho \le 2$, $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$

$P_x = 2x$, $Q_y = 0$, $R_z = 3$ So $\text{div} F = 2x+3$ in spherical coord $\text{div}F = 2\rho\sin\varphi\cos\theta + 3$.

$$\text{So Flux} = \iint_S F \cdot n\,d\sigma \overset{DT}{=} \iiint_D \text{div}F\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^2 (2\rho\sin\varphi\cos\theta + 3)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^2 2\rho^3\sin^2\varphi\cos\theta + 3\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{2}\rho^4\sin^2\varphi\cos\theta + \rho^3\sin\varphi\Big|_0^2 d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 8\sin^2\varphi\cos\theta + 8\sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi} 4(1-\cos 2\varphi)\cos\theta + 8\sin\varphi\,d\varphi\,d\theta$$

$$= \int_0^{2\pi} 4\cos\theta\,\varphi - 2\sin 2\varphi\cos\theta - 8\cos\varphi\Big|_0^{\pi} d\theta = \int_0^{2\pi} [4\pi\cos\theta - 0 - 8(-1)] - [0 - 0 - 8]$$

$$= \int_0^{2\pi} 4\pi\cos\theta + 16\,d\theta = 4\pi\sin\theta + 16\theta\Big|_0^{2\pi} = (4\pi(0) + 16 \times 2\pi) - (0+0) = \boxed{32\pi}$$