MATH 2551 J/HP Midterm 2
VERSION A
Spring 2026
COVERS SECTIONS 12.1-12.5, 13.1-13.4, 14.1-14.2

Full name: Key **GT ID:**

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. **I do not have a phone within reach**, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	2
4	5
5	5
6	4
7	6
8	8
9	8
10	8
Total:	50

For T/F problems choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If $f(x, y)$ is differentiable at the point $(2, 1)$, then the vector

$$\langle f_x(2,1), f_y(2,1), -1\rangle$$

is normal to the plane tangent to the graph of $z = f(x, y)$ at the point $(2, 1)$.

● **TRUE** ○ **FALSE**

$z = f(x,y) \Rightarrow F(x,y,z) = f(x,y) - z = 0$

$\Rightarrow \nabla F = \langle f_x, f_y, -1\rangle$ normal to $F = 0$

2. (2 points) Suppose $w = f(x, y)$ is a differentiable function and $x(t), y(t)$ are both differentiable. Give an equation for the formula for $\frac{dw}{dt}$. [AN]

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

3. (2 points) Suppose S is a surface $F(x, y, z) = 7$ and $P(2, -1, 3)$ is a point on S. If $\nabla F(P) = \langle 4, 5, -6\rangle$ then what is the equation for the tangent plane of S at $P(2, -1, 3)$? [AN]

$$4(x-2) + 5(y+1) - 6(z-3) = 0$$

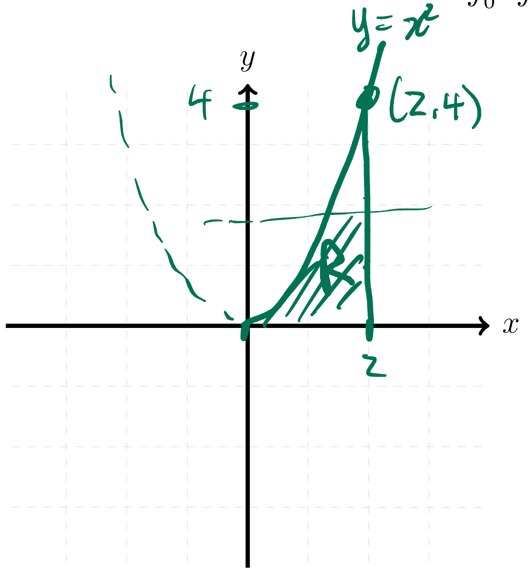
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

is tangent plane eqn for $F(x,y,z) = k$

@ $P(x_0, y_0, z_0)$ w/ normal $\nabla F = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

4. (5 points) Sketch R the region of integration and switch the order of integration. That is, rewrite the integral in the order $dx\,dy$. *Do not evaluate!* [AN]

$$\int_0^2 \int_0^{x^2} \cos(x^3)\; dy\; dx$$

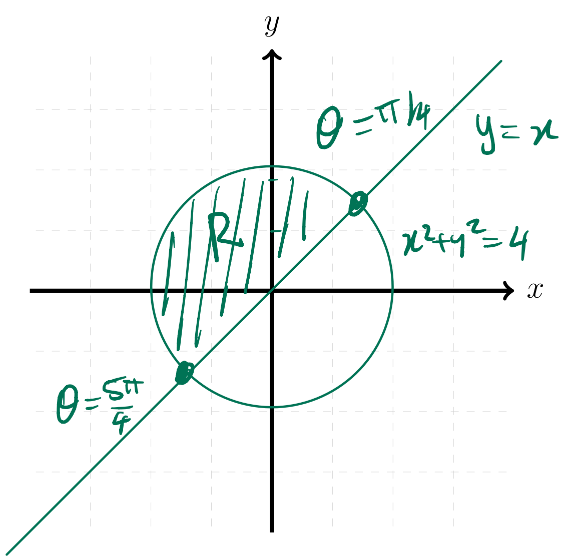


$y = x^2 \iff x = \sqrt{y} \quad (x \geq 0)$

$$\boxed{\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\, dx\, dy}$$

5. (5 points) Sketch R the region of integration and set up an integral in polar coordinates for evaluating $\iint_R f(x,y)\; dA$. Note that R is the region inside the circle of radius $r = 2$ which is over the line $y = x$. *Do not evaluate!* [AN]

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4,\; y \geq x\right\}, \text{ and } f(x,y) = y.$$



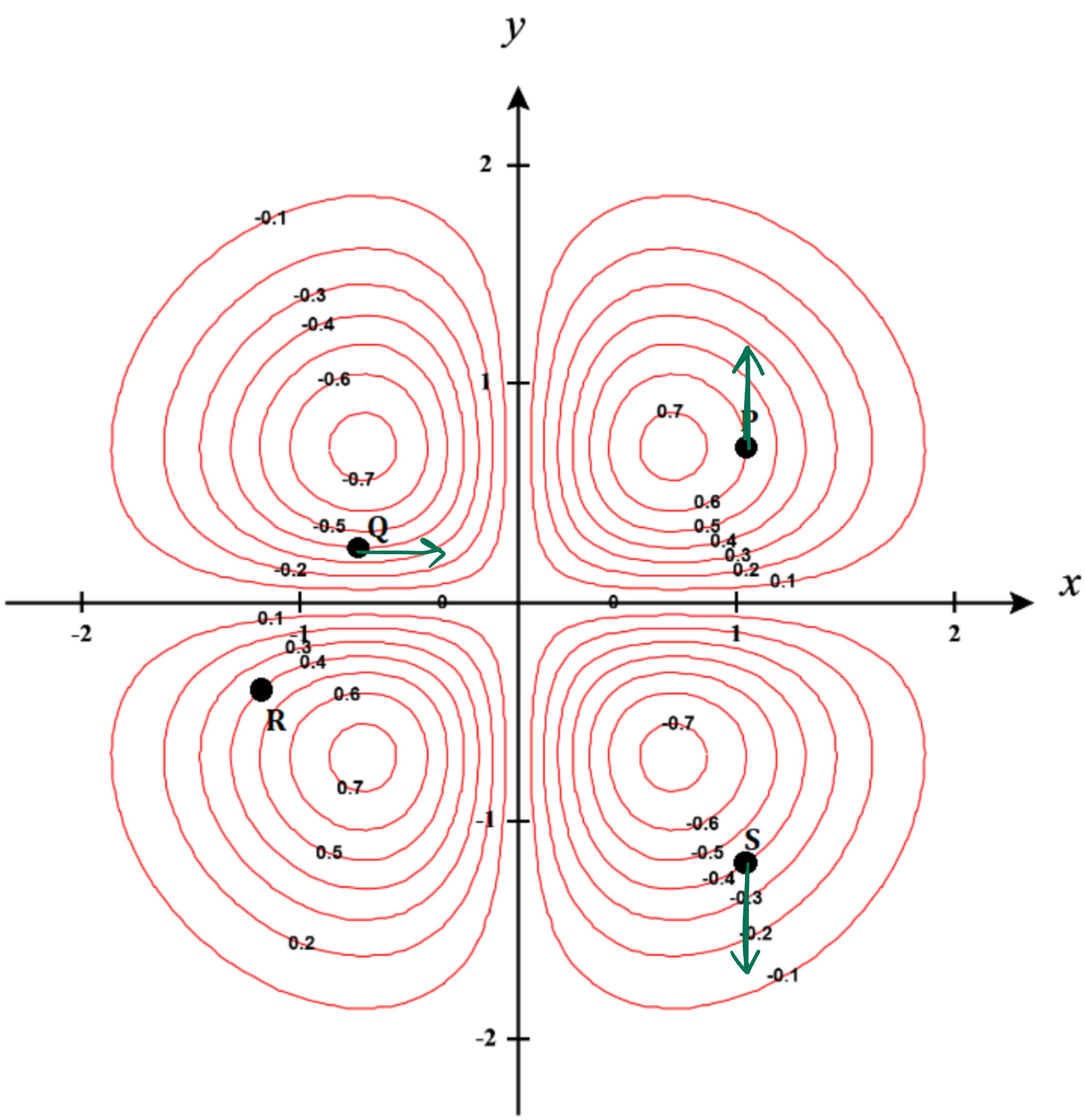
Volume= $\boxed{\int_{\pi/4}^{5\pi/4} \int_0^2 r\sin\theta * r\; dr\, d\theta}$

6. (4 points) In this problem, you will work with the function $f : \mathbb{R}^2 \to \mathbb{R}$ whose contour plot near the origin is shown below. [AN]

   (a) Determine the sign $(+, -, 0)$ of the directional derivative at the point S in the direction $\langle 0, -2 \rangle$. $\boxed{+}$

   (b) Determine the sign $(+, -, 0)$ of the directional derivative at the point Q in the direction $\langle 1, 0 \rangle$. $\boxed{0}$

   (c) Draw a non-zero vector $\vec{u}$ at the point P for which $D_{\vec{u}} f(P) = 0$ the directional derivative of f in the direction of $\vec{u}$ equals 0 at the point P.



7. (6 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = e^{xy}$ and $\mathbf{r}(s, t) : \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$\mathbf{r}(s,t) = \begin{bmatrix} s/t \\ s+t \end{bmatrix}.$$

   (a) Find Df.
   (b) Find $D\mathbf{r}$.
   (c) Evaluate $\mathbf{r}(2, 1)$ and $D\mathbf{r}|_{(s,t)=(2,1)}$.
   (d) Finally, use the Chain Rule to evaluate $D(f(\mathbf{r}(s, t)))|_{(s,t)=(2,1)}$.
   Please put the answer to part (d) in the box below.

[AJN]

(a)
$$Df = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} ye^{xy} & xe^{xy} \end{bmatrix}$$

(b)
$$Dr = \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} = \begin{bmatrix} 1/t & -s/t^2 \\ 1 & 1 \end{bmatrix}$$

(c) $r(2,1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad Dr(2,1) = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

(d) If $g = f \circ r$ then $Dg = Df \ast Dr$

@ $(s,t) = (2,1)$ then $(x,y) = r(2,1) = (2,3)$

and $Df(2,3) = \begin{bmatrix} 3e^6 & 2e^6 \end{bmatrix}$

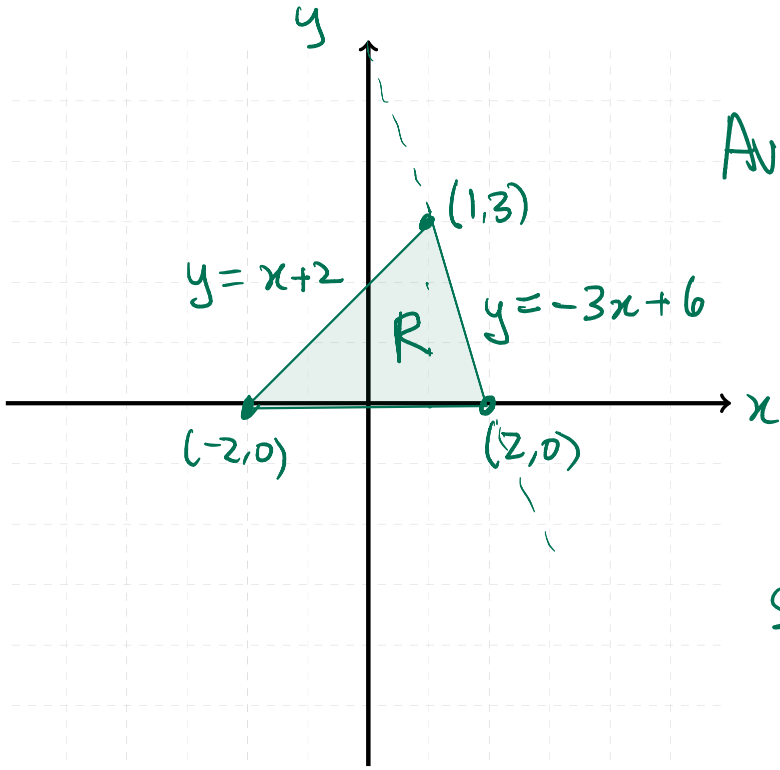
So

$$Dg(2,1) = \begin{bmatrix} 3e^6 & 2e^6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 5e^6 & -4e^6 \end{bmatrix}$$

(d) $\boxed{\begin{bmatrix} 5e^6 & -4e^6 \end{bmatrix}}$

8. (8 points) Sketch the region R including labels for the axes, and labels for each boundary curve and each intersection point between curves. Then, compute the average value of $f(x,y) = y$ over the region R which is the triangle with vertices $(-2,0)$, $(2,0)$, and $(1,3)$.

[AJN]



$\text{Avg } f = \boxed{1}$

Area $R = \frac{1}{2}(\text{base})(\text{height})$
$= \frac{1}{2}(4)(3) = 6$

So R region
$y \in [0,3]$ and
$x \in [y-2, -\frac{1}{3}y+2]$

left bdry $y = x+2 \Leftrightarrow x = y-2$

right bdry $y = -3x+6 \Leftrightarrow -3x = y-6 \Rightarrow x = -\frac{1}{3}y+2$

$$\text{Vol} = \iint_R f(x,y)\, dA = \int_0^3 \int_{y-2}^{-\frac{1}{3}y+2} y\, dx\, dy$$

$$= \int_0^3 yx \Big|_{y-2}^{-\frac{1}{3}y+2} dy = \int_0^3 y\left(-\frac{1}{3}y+2\right) - y(y-2)\, dy$$

$$= \int_0^3 -\frac{1}{3}y^2 + 2y - y^2 + 2y\, dy = \int_0^3 -\frac{4}{3}y^2 + 4y\, dy$$

$$= -\frac{4}{9}y^3 + 2y^2 \Big|_0^3 = -\frac{4}{9}(3)^3 + 2(3)^2 = 18 - 12 = 6$$

So Avg f
$= \frac{1}{6} * 6$
$= \boxed{1}$

9. (8 points) Find all critical points and use the Hessian matrix to classify the critical points of the function. [AJN]

$$f(x, y) = x^2 + xy + 3x + 2y + 5$$

Solve $\nabla f = 0$

$$\nabla f = \begin{bmatrix} 2x+y+3 \\ x+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} ① \ 2x+y+3=0 \\ ② \ x+2=0 \end{cases}$$

From ② $x=-2$, sub into ① $2(-2)+y+3=0$

$\Rightarrow y=1$.

Only one crit pt @ $(-2, 1)$.

$$Hf = D^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \det Hf = -1 < 0$$

and $f_{xx} = 2 > 0$ so $\boxed{(-2, 1) \text{ is a saddle}}$

10. (8 points) Use the method of Lagrange Multipliers to find the maximum and minimum value of $f(x, y) = x + 2y$ subject to the constraint $x^2 + y^2 = 5$. [AJN]

Solve L-G eqns $\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$ $g(x,y) = x^2 + y^2$

$\nabla f = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ So solve $\begin{cases} ① \; 1 = \lambda 2x \\ ② \; 2 = \lambda 2y \\ ③ \; x^2 + y^2 = 5 \end{cases}$

Divide ② by 2, get $1 = \frac{\lambda 2y}{2} = \lambda 2x \Rightarrow \lambda y = \lambda 2x.$

So either $\overset{\text{I.}}{\lambda = 0}$ or $\overset{\text{II.}}{y = 2x}$. (case I. impossible since ① & ② false)

So $y = 2x$. sub into ③.

$x^2 + (2x)^2 = 5 \Rightarrow 5x^2 = 5 \Rightarrow x = \pm 1.$

get $(\pm 1, \pm 2)$ as crit pts.

(x,y)	$f(x,y)$
$(1,2)$	$1 + 2(2) = 5$
$(-1,-2)$	$-1 + 2(-2) = -5$

Max value 5 @ $(1,2)$

Min value -5 @ $(-1,-2)$

FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b\rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $\quad dA = r\ dr\ d\theta$

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