MATH 2551 Midterm 3
VERSION A
Spring 2026
COVERS SECTIONS 15.5-15.8, 16.1-16.8

Full name: Key ______________________ **GT ID:** ____________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	6
4	10
5	10
6	10
7	10
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If C_1 and C_2 are two curves with the same starting point and ending point, then $\int_{C_1} \nabla f \cdot \mathbf{T}\, ds = \int_{C_2} \nabla f \cdot \mathbf{T}\, ds$.

   ● **TRUE** ○ **FALSE**

2. (2 points) If C is a curve with parametrization $\mathbf{r}_1(t) = \langle x(t), y(t)\rangle$ with $t \in [a,b]$, then the parametrization $\mathbf{r}_2(t) = \mathbf{r}_1(-t)$ with $t \in [b,a]$ is also a parametrization of C with the opposite orientation. [A]

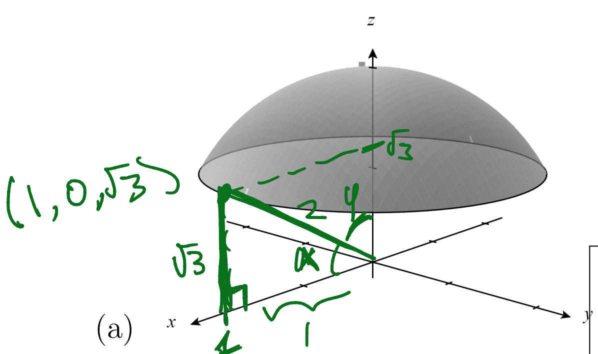
   ○ **TRUE** ● **FALSE**

3. (6 points) Let S be the surface which is the cap of a sphere of radius 2 units with $z \geq \sqrt{3}$. The point $P(1,0,\sqrt{3})$ is on S at the intersection of the sphere and the plane. [AN]

   (a) Sketch the point P on the image provided below.

   (b) Find the spherical coordinates (ρ, φ, θ) of the point $P(1,0,\sqrt{3})$.

   (c) Find a parametrization $r(u,v)$ of S. *Hint: spherical coordinates.*

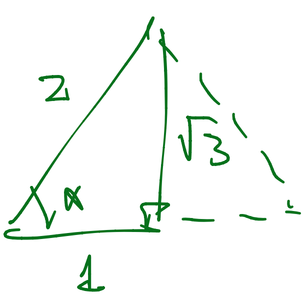


(a)

(b) = $(2, \pi/6, 0)$

(c) $r(u,v) = \langle 2\sin u\cos v, 2\sin u\sin v, 2\cos u\rangle$

$u \in [0, \pi/6]$ $v \in [0, 2\pi)$



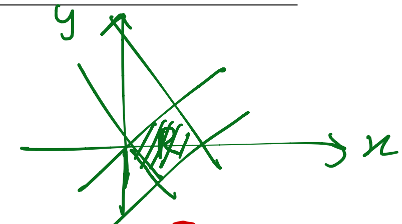
$x = \rho\sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$
$z = \rho\cos\varphi$

$\alpha = \tan^{-1}(\sqrt{3}) = \pi/3$

so $\varphi = \pi/2 - \pi/3 = \underline{\pi/6}$

$r(u,v) = \langle 2\sin u\cos v, 2\sin u\sin v, 2\cos u\rangle$

$u \in [0, \pi/6]$ $v \in [0, 2\pi)$



4. (10 points) In this problem you will compute the integral

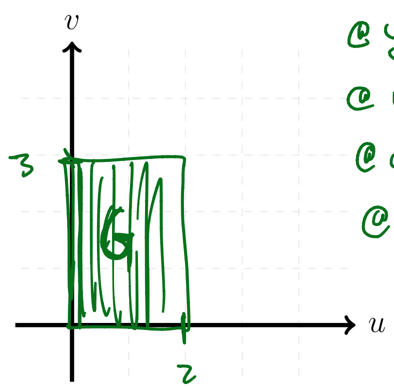
$$M = \iint_R \overset{u}{(3x+y)}\overset{v}{(x-y)}\,dx\,dy$$

for the region R ~~in the first quadrant~~ bounded by the lines $y=-3x$, $y=-3x+\cancel{4}\,2$, $y=x$, $y=x-3$ using the transformation $u=3x+y$, $v=x-y$. [AJN]

(a) On the axes provided, sketch the new region of integration G in the uv-plane.

(b) Solve the transformation equations for x and y in terms of u and v and compute the Jacobian determinant, i.e., find $\mathbf{T}(u,v)$ and $|\det D\mathbf{T}(u,v)|$.

(c) Evaluate the double integral using a change of variables and your results from parts (a) and (b) to find the value of M.



@ $y=-3x \Rightarrow 3x+y=0 \Rightarrow u=0$

@ $y=-3x+2 \Rightarrow 3x+y=2 \Rightarrow u=2$

@ $y=x \Rightarrow x-y=0 \Rightarrow v=0$

@ $y=x-3 \Rightarrow x-y=3 \Rightarrow v=3$

$\mathbf{T}(u,v)=$ $\boxed{\begin{bmatrix} \frac14 u + \frac14 v \\ \frac14 u - \frac34 v \end{bmatrix}}$

$|\det D\mathbf{T}(u,v)| = \boxed{1/4}$

$\begin{cases} u = 3x+y \\ v = x-y \end{cases}$ $\quad T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix}$

$x = v + y = v + (u - 3x)$

$\Rightarrow 4x = u + v \Rightarrow x = \frac14 u + \frac14 v$

$y = x - v = \frac14 u + \frac14 v - v = \frac14 u - \frac34 v$

$$DT = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \frac14 u + \frac14 v \\ \frac14 u - \frac34 v \end{bmatrix}$$

$\det DT = -3/16 - 1/16 = -4/16 = -1/4$

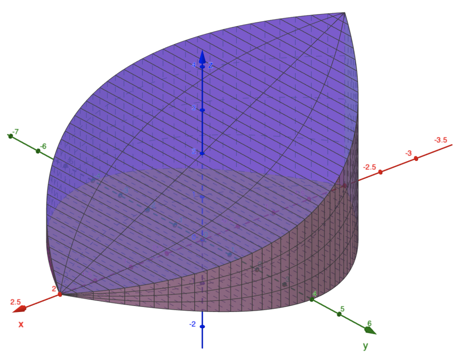
$$M = \int_0^3\int_0^2 uv * \frac14\,du\,dv = \frac14\int_0^3 \frac12 u^2 * v\Big|_0^2\,dv$$

$$= \frac14\int_0^3 2v\,dv = \frac14 v^2\Big|_0^3 = \boxed{9/4}$$

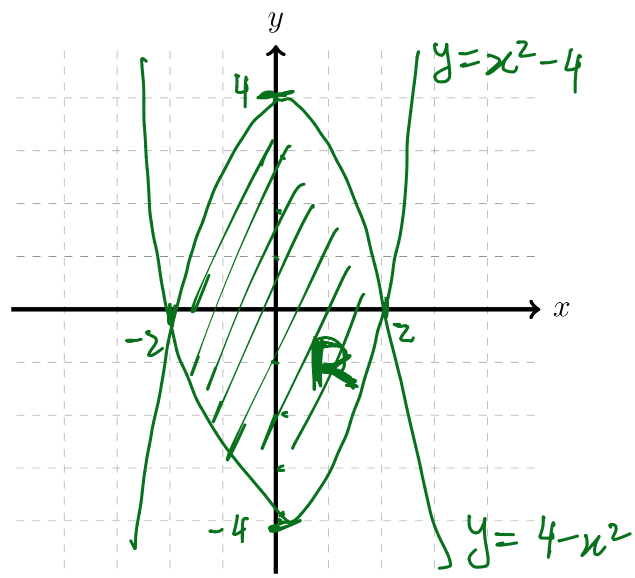
5. (10 points) In this problem you will compute the mass of the solid D pictured below, which occupies the region which is bounded below by $z = 0$, bounded above by the plane $z = 2-x$, and with base R in the xy-plane between the curves $y = 4 - x^2$ and $y = x^2 - 4$. [AJN]

   (a) Sketch the region R bounded by curves $y = 4 - x^2$ and $y = x^2 - 4$ in the xy-plane. Determine the points of intersection of the two curves.

   (b) Find the volume of D bounded by the surfaces $y = 4 - x^2$, $y = x^2 - 4$, $z = 0$ and $z = 2 - x$ in 3-dimensional space.



(a)



(b) $\boxed{128/3}$

$$V = \iiint_D 1 \, dV$$

$$= \int_{-2}^{2} \int_{x^2-4}^{4-x^2} \int_0^{2-x} 1 \, dz \, dy \, dx$$

$$= \int_{-2}^{2} \int_{x^2-4}^{4-x^2} 2-x \, dy \, dx = \int_{-2}^{2} 2y - xy \Big|_{x^2-4}^{4-x^2} dx$$

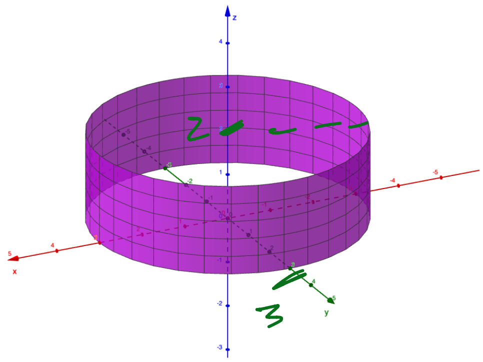
$$= \int_{-2}^{2} (2-x)\Big[\overbrace{4-x^2-(x^2-4)}^{8-2x^2}\Big] dx = \int_{-2}^{2} \overbrace{16-4x^2}^{\text{even}} \overbrace{\cancel{-8x+2x^3}}^{\text{odd}} \, dx$$

$$= 2\left(16x - \frac{4}{3}x^3\right)\Big|_0^2 = 2\left(32 - \frac{32}{3}\right) = 2\left(64/3\right) = \boxed{128/3}$$

6. (10 points) This problem will have you compute the flux of the vector field $\mathbf{F} = \langle x, y, z^2 \rangle$ through S the open ended circular cylinder of radius 3 and height 2 with its base on the xy-plane and centered about the z-axis, oriented away from the z-axis. [AJN]

   (a) Find a parametrization $r(u,v)$ of the surface S. *Hint: cylindrical coordinates.*

   (b) Set up and **evaluate** a surface integral which computes the flux of $\mathbf{F}$ through S. $\iint_S \mathbf{F}\cdot\vec{n}\, d\sigma$



(a) $r(u,v) = \langle 3\cos u, 3\sin u, v \rangle$
$u \in [0, 2\pi)$ $v \in [0, 2]$

(b) 36π

$$r(u,v) = \langle 3\cos(u), 3\sin(u), v \rangle, \quad u \in [0, 2\pi), \quad v \in [0, 2]$$

$$r_u = \langle -3\sin u, 3\cos u, 0 \rangle \qquad r_v = \langle 0, 0, 1 \rangle$$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 3\cos u, -(-3\sin u), 0 \rangle$$

$$= \langle 3\cos u, 3\sin u, 0 \rangle$$

$$\text{Flux} = \iint_S \mathbf{F}\cdot n\, d\sigma = \iint_R \mathbf{F}(r(u,v)) \cdot r_u \times r_v \, dA$$

$$= \int_0^{2\pi}\int_0^2 \langle 3\cos u, 3\sin u, v^2 \rangle \cdot \langle 3\cos u, 3\sin u, 0 \rangle \, dv\, du$$

$$= \int_0^{2\pi}\int_0^2 9\cos^2 u + 9\sin^2 u + 0 \, dv\, du = \int_0^{2\pi}\int_0^2 9 \, dv\, du$$

$$= \int_0^{2\pi} 9v \Big|_0^2 du = \int_0^{2\pi} 18\, du = 18u \Big|_0^{2\pi} = \boxed{36\pi}$$

7. (10 points) Consider the vector field $\mathbf{F} = \langle 2y, 3x \rangle$ and the curve C which is the circle $x^2 + y^2 = 4$ oriented counterclockwise with outward pointing normal vector. [AJN]

(a) Compute the flow of $\mathbf{F}$ around C, which is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, **with or without** using Green's Theorem. *Use either method, but you can use the other integral to check your answer.*

(b) Compute the flux of F around C, which is $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$, **using** Green's Theorem.

(c) Is F conservative? Explain in at least one or two complete sentences how you know.

(a) $\boxed{8\pi}$

(b) $\boxed{0}$

(a) C: $r(t) = \langle 2\cos t, 2\sin t \rangle,\ t \in [0, 2\pi]$

$r'(t) = \langle -2\sin t, 2\cos t \rangle$

$$\text{Flow} = \int_C F \cdot T\, ds = \int_0^{2\pi} \langle -2\sin t, 2\cos t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\, dt$$

$$= \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t\, dt = \int_0^{2\pi} 4\, dt = 4 * 2\pi = 8\pi$$

using G's T

$$\text{Flow} = \int_C F \cdot T\, ds = \iint_R \text{curl} F \cdot \vec{k}\, dA = \iint_R Q_x - P_y\, dA$$

$$= \iint_R 1 - (-1)\, dA = 2 * \iint_R 1\, dA = 2 * \text{Area}(R)$$

$$= 2 * \pi 2^2$$

$$= 8\pi \checkmark$$

(b) $$\text{Flux} = \int_C F \cdot n\, ds = \iint_R \text{div} F\, dA = \iint_R P_x + Q_y\, dA = \iint_R 0\, dA$$

$$= 0$$

(c) no, b/c $\text{curl} F \neq \vec{0}$ and/or circulation around C is non-zero.

FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$ or $\dfrac{\int_C f(x,y,z)\ ds}{\text{length of } C}$,
  Mass: $M = \iiint_D \delta\ dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Scalar line integral: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- Surface Area$=\iint_S 1\ d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_D \nabla\cdot\mathbf{F}\ dV.$$

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