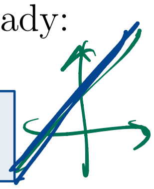
§13.1 Curves in Space & Their Tangents

The description we gave of a line last week generalizes to produce other one-dimensional graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well. We said that a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ produces a straight line when graphed.

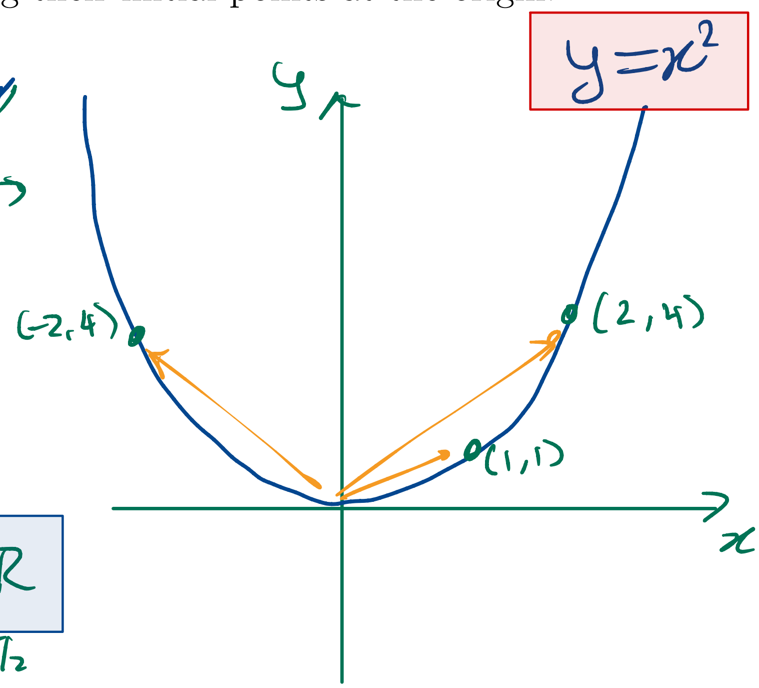
This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

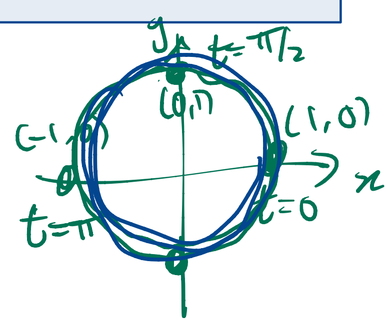
(1) $\boxed{\ell(t) = \vec{OP} + t\vec{v} \;, \; t \in \mathbb{R}}$



(output is a vector)
but we think of it as the points that are pointed by the output vectors.



(2) $\boxed{\vec{r}(t) = \langle \cos t, \sin t \rangle \quad t \in \mathbb{R}}$



(3) $\boxed{\vec{r}(t) = \langle t, t^2 \rangle, \; t \in \mathbb{R}}$

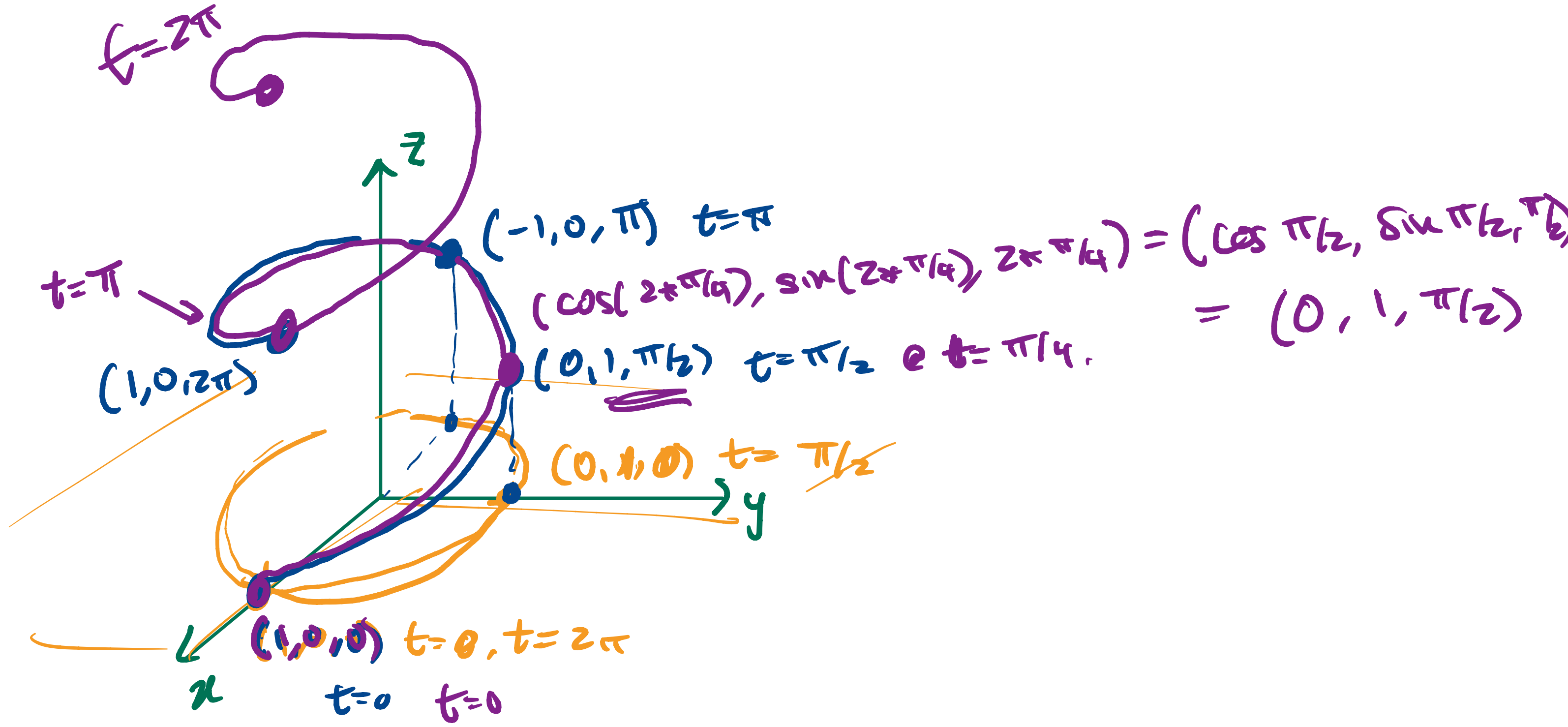
Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called parametrizing the curve C, and $\mathbf{r}$ is called a parametrization of C.

"parameter" aka variable.

$\vec{r}_0(t) = \langle \cos t, \sin t, 0 \rangle$

replace t by $2t$.

Example 10. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $\boxed{[0, 2\pi]}$. What do you think the graph of each looks like? How are they similar and how are they different?



Check your intuition

§13.2: Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits: $r(t) = \langle t^2, 2, \ln t \rangle$, $t > 0$ space curve.

Example 11. Compute $\lim_{t \to e} \langle t^2, 2, \ln(t) \rangle$. near $t = e$ where is $r(t)$?

$$= \left\langle \lim_{t \to e} t^2, \lim_{t \to e} 2, \lim_{t \to e} \ln t \right\rangle$$

$$= \langle e^2, 2, \ln e \rangle = \langle e^2, 2, 1 \rangle$$

[A] credit plz simplify.

And with continuity:

Example 12. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \frac{1}{t^2 - 4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

same. $\updownarrow$ $r(t) = \langle t, -\frac{1}{t^2-4}, \sin t \rangle$

Idea just figure out where each component is continuous & insist that t is in EACH region where EACH component is continuous.

Components:

$x(t) = t$

$y(t) = -\frac{1}{t^2-4}$

$z(t) = \sin t$

$x(t)$ & $z(t)$ are continuous everywhere

So need $y(t)$ to be continuous.

domain of $y(t) = \frac{-1}{t^2-4}$ is

$$\boxed{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}$$

And with derivatives:

Example 13. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

$x(t) = 2t - \frac{1}{2}t^2 + 1$ so $x'(t) = 2 - t$

$y(t) = t - 1$ $\quad y'(t) = 1$ $\quad r'(t)$

So $r'(t) = \langle x'(t), y'(t)\rangle = \boxed{\langle 2-t, 1\rangle}$

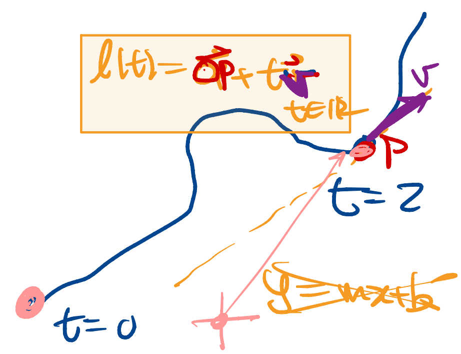
Sanity Check What's $r'(0) = \langle 2, 1\rangle$

$r'(1) = \langle 1, 1\rangle$

What do they mean? "velocity"

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity vector at time t (direction & magnitude)
- $|\mathbf{r}'(t)|$ gives Speed at time t
- $\mathbf{r}''(t)$ gives accelleration at time t



Let's see this graphically

Example 14. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

$t=2$

$\vec{OP} = \vec{r}(2) = \langle 2(2) - \frac{1}{2}(2)^2 + 1, 2 - 1\rangle = \langle 3, 1\rangle$

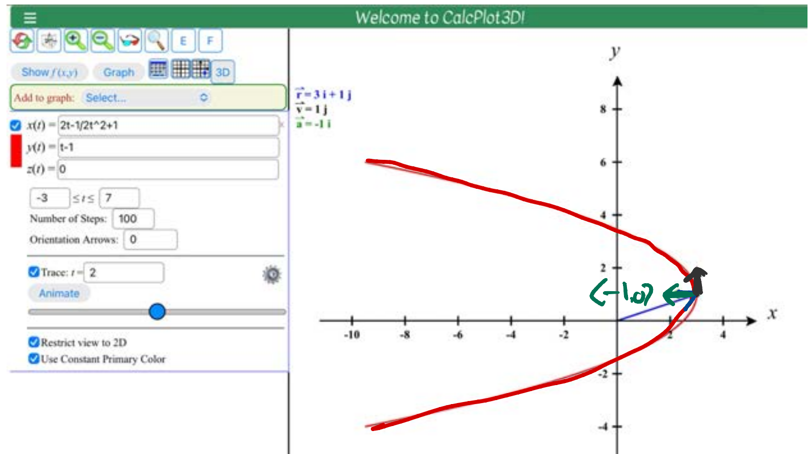
$\vec{v} = \vec{r}'(2) = \langle 2 - 2, 1\rangle = \langle 0, 1\rangle$

So $l(t) = \vec{OP} + t\vec{v}$ becomes

$$\boxed{l(t) = \langle 3, 1\rangle + t\langle 0, 1\rangle, \ t \in \mathbb{R}}$$

Curious $r''(t) = \langle -1, 0\rangle$ for all t.

Example 14. (cont.) Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$ at time $t = 2$.



$r(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle, \ t \in \mathbb{R}$ $\qquad r(t_0)$

so $r'(t) = \langle x'(t), y'(t) \rangle = \langle 2 - t, 1 \rangle$

$r''(t) = \langle -1, 0 \rangle$ for all t.

$l(t) = \langle 3, 1 \rangle + t \langle 0, 1 \rangle, \ t \in \mathbb{R}$

And with integrals: $\vec{r}(t) = \langle t, e^{2t}, \sec^2(t)\rangle$ space curve in $\mathbb{R}^3$ (vs. plane curve in $\mathbb{R}^2$) [N] credit

Example 15. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle\ dt$.

$$= \left\langle \int_0^1 t\,dt,\ \int_0^1 e^{2t}\,dt,\ \int_0^1 \sec^2 t\,dt \right\rangle$$

$$= \left\langle \tfrac{1}{2}t^2,\ \tfrac{1}{2}e^{2t},\ \tan t \right\rangle \Big|_0^1$$

$$= \left\langle \tfrac{1}{2},\ \tfrac{1}{2}e^2,\ \tan(1)\right\rangle - \left\langle 0,\ \tfrac{1}{2}\overset{1}{\cancel{e^0}},\ \overset{0}{\cancel{\tan(0)}}\right\rangle$$

$$= \boxed{\left\langle 1/2,\ \tfrac{1}{2}e^2 - \tfrac{1}{2},\ \tan(1)\right\rangle}$$ ← simplify for [A]-credit.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 16. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle\ m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, use calculus to reconstruct his flight path.

initial value problem

Idea: Given $\vec{r}\,'(t) = \vec{v}(t)$ & initial value $\vec{r}(0)$, reconstruct $\vec{r}(t)$. [IVP]

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left\langle -200\sin 2t,\ 200\cos t,\ 400 - \frac{400}{1+t}\right\rangle dt$$

$$= \left\langle +\frac{\overset{100}{\cancel{200}}}{\cancel{2}}\cos 2t + \overset{?}{C_1},\ 200\sin t + \overset{?}{C_2},\ 400t - 400\ln|1+t| + \overset{?}{C_3}\right\rangle$$

To get C_1, C_2, C_3 figured out we use $r(0) = \langle 0,0,0\rangle$ the initial value.

So $\vec{r}(0) = \langle 0,0,0\rangle = \langle 100\overset{1}{\cancel{\cos(0)}} + C_1,\ 200\overset{0}{\cancel{\sin(0)}} + C_2,\ 400(0) - 400\overset{0}{\cancel{\ln|1|}} + C_3\rangle$

$$= \langle 100 + C_1, C_2, C_3\rangle = \langle 0,0,0\rangle \quad \text{So } C_1 = -100,\ C_2 = C_3 = 0$$

$$\boxed{\vec{r}(t) = \langle 100\cos 2t - 100,\ 200\sin t,\ 400t - 400\ln|1+t|\rangle} \quad \boxed{t \in [0,\infty)}$$

From §13.2 workbook on Sal's website

You try it!

#7. Suppose $r(t) = \langle te^{t^2}, e^{-t}, 1 \rangle$. Integrate $\int_0^1 r(t)\,dt$

#13. Solve the IVP $\frac{dr}{dt} = \langle (t+1)^{1/2}, e^{-t}, \frac{1}{t+1} \rangle$ $r(0) = \langle 0, 0, 1 \rangle$
(initial value problem)

§13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

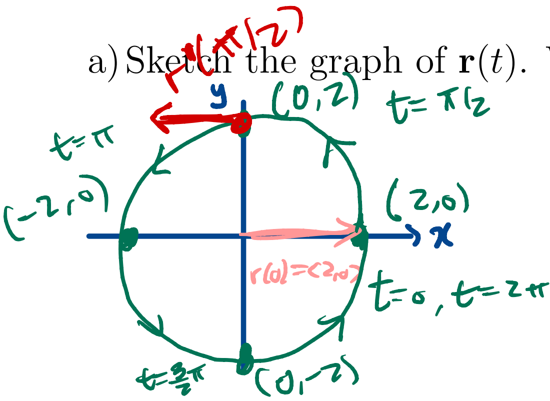
Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\boxed{\vec{\mathbf{r}}(t) = \langle 2\cos(t), 2\sin(t) \rangle,}$$

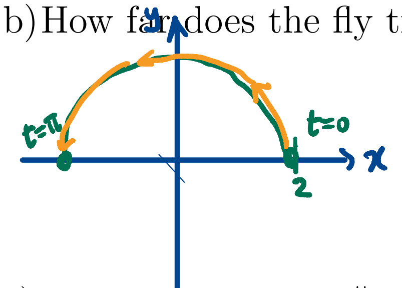
where $0 \leq t \leq 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?



Circle, centered @ (0,0) w/ radius 2.

b) How far does the fly travel between $t = 0$ and $t = \pi$?



$C = 2\pi r$ if $r = 2$

$C = 4\pi$ half circle so total distance is $\boxed{2\pi.}$

c) What is the speed $\|\mathbf{v}(t)\|$ of the fly at time t?

(velocity) $v = r'(t) = \langle -2\sin t, 2\cos t \rangle$

(speed) so $\|v\| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4(\underbrace{\cancel{\sin^2 t + \cos^2 t}}_{1})} = \sqrt{4} = 2$

(doesn't depend on t)

d) Compute the integral $\boxed{\int_0^\pi \|\mathbf{v}(t)\| \, dt}$. What do you notice?

$$\int_0^\pi 2 \, dt = 2t \Big|_0^\pi = 2\pi - 0 = \boxed{2\pi}$$

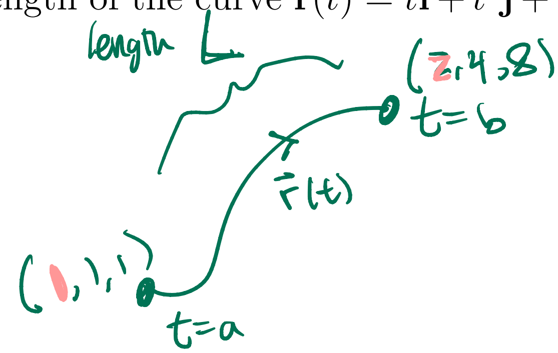
same.

integrate speed to get distance travelled.

Definition 17. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $\underline{t=a}$ to $\underline{t=b}$ that is traced out exactly once is

$$L = \underline{\int_a^b \|\vec{r}\,'(t)\|\,dt}$$

Example 18. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.



$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

$\vec{r}(t) = \langle t, t^2, t^3\rangle$

so $r'(t) = \langle 1, 2t, 3t^2\rangle$

how about a? b?

want $\vec{r}(a) = \langle 1,1,1\rangle$ $a=1$

$= \langle a, a^2, a^3\rangle$

$\vec{r}(b) = \langle 2,4,8\rangle$ $b=2$

$= \langle b, b^2, b^3\rangle$

So

$$L = \int_1^2 \|\langle 1, 2t, 3t^2\rangle\|\,dt$$

$$= \int_1^2 \sqrt{1^2 + (2t)^2 + (3t^2)^2}\,dt$$

$$= \boxed{\int_1^2 \sqrt{1 + 4t^2 + 9t^4}\,dt}$$

??

Example 19. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = \langle 6\sin(2t), 6\cos(2t), 5t\rangle,\ 0 \le t \le 2\pi$.

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \le t \le 8$.

Check your intuition

Example 19. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = \langle 6\sin(2t), 6\cos(2t), 5t\rangle,\ 0 \le t \le 2\pi$.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle 12\cos 2t, -12\sin 2t, 5\rangle$$

$$\Rightarrow \quad |\vec{v}(t)|^2 = 144(\cos^2 2t + \sin^2 2t) + 25 = 169$$

$$\Rightarrow \quad |\vec{v}(t)| = 13$$

So,

$$L = \int_0^{2\pi} |\vec{v}(t)|\, dt = \int_0^{2\pi} 13\, dt = 13t\Big|_0^{2\pi} = \boxed{26\pi}$$

Check your intuition

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \leq t \leq 8.$

Check your intuition

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k}$, $0 \le t \le 8$.

$$L = \int_a^b \|\vec{v}(t)\| \, dt$$

$x(t) = t$ $a=0$
$y(t) = 0$ $b=8$
$z(t) = \frac{2}{3}t^{3/2}$

$$\vec{v}(t) = \vec{r}'(t) = 1 * \hat{\imath} + \cancel{\tfrac{2}{3} * \tfrac{3}{2}}^{\,1} t^{1/2} \hat{k}$$

So $\|\vec{v}(t)\| = \sqrt{1^2 + (t^{1/2})^2} = \sqrt{1+t}$

and
$$L = \int_0^8 \sqrt{1+t} \, dt$$

u-sub Box
$$\boxed{\begin{aligned} u &= 1+t \\ du &= dt \end{aligned}}$$

$t=0 \Rightarrow u=1$
$t=8 \Rightarrow u=9$

$$= \int_1^9 \sqrt{u} \, du = \frac{2}{3} u^{3/2} \Big|_1^9$$

$$= \frac{2}{3} 9^{3/2} - \frac{2}{3} 1^{3/2}$$

$$= \frac{2}{3} 27 - \frac{2}{3} = \frac{2}{3} * 26 = \boxed{52/3}$$

Check your intuition

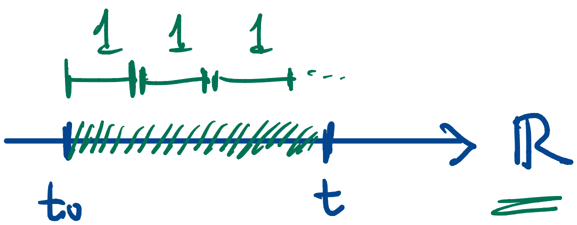
 $\int x\,dx = x^2 + C \qquad \int y\,dy = y^2 + C$

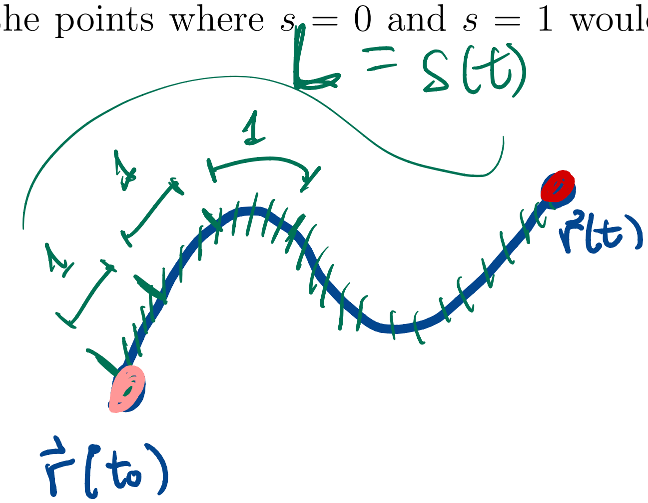
Arc length parametrization

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \int_{t_0}^{t} \|\vec{v}(\tau)\|\, d\tau$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.





Idea: why should we be measuring out units in the parameter space $\mathbb{R}$?

More natural to measure units in the codomain of $\vec{r}(t)$, $\vec{r} : \mathbb{R} \to \mathbb{R}^n$.

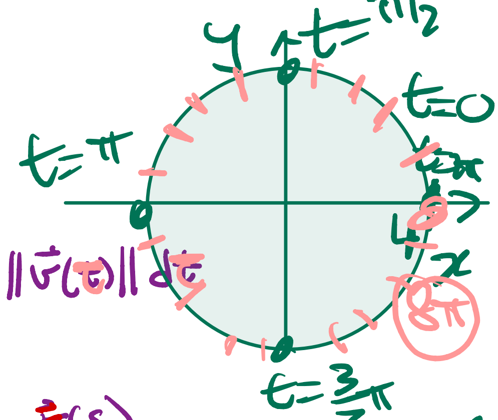
$\boxed{\text{FACT. If } \vec{r}_2(s) \text{ is an arc-length parametrization of } C\text{, then } \|\vec{r}_2{}'(s)\| = 1 \text{ for all } s.}$

(constant speed of 1)

give me a way to locate the point at "length s" along the curve.

Example 21. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.

$t_0 = 0$



doesn't depend on t.

META

① Compute arc length function $s(t) = \int_{t_0}^{t} \|\vec{v}(\tau)\|\, d\tau$

② Solve $s = s(t)$ for $t = f(s)$

③ Substitute back into $\vec{r}(t)$ to obtain $\vec{r}(s)$.

① Note: $r'(t) = \langle -4\sin t, 4\cos t\rangle$

$$\Rightarrow \|r'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t} = \sqrt{16} = 4$$

Find $s(t) = \int_0^t \|r'(\tau)\|\, d\tau = \int_0^t 4\, d\tau$

$$\Rightarrow s(t) = 4\tau \Big|_0^t = 4t - 0 \Rightarrow s(t) = 4t$$

② Solve $s = 4t$ for t, $t = s/4$.

③ Substitute into $\vec{r}(t)$ to get $\vec{r}_2(s)$

where $\vec{r}_2(s) = \vec{r}(s/4)$

$0 \le t \le 2\pi \quad t = s/4$

sub $\boxed{t = s/4}$ into $r(t) = r(s/4) = \boxed{\langle 4\cos(s/4), 4\sin(s/4)\rangle = \vec{r}_2(s), \quad 0 \le s \le 8\pi}$

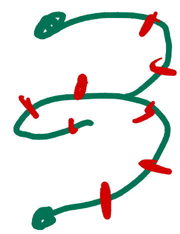
Example 22. *You try it!* Find (a) an arc length parameterization $s(t)$ of the curve $\mathcal{C}$, the portion of the helix of radius 4 in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$, $0 \le t \le \pi/2$, and (b) use $s(t)$ to find L the length of $\mathcal{C}$

$\vec{r}_2(s)$ arc-length parametrization.

(a) Find $\vec{r}_2(s)$

META

1. Compute arc length function $S(t) = \int_{t_0}^{t} \|\vec{v}(\tau)\|\, d\tau$
2. Solve $S = s(t)$ for $t = f(s)$
3. Substitute back into $\vec{r}(t)$ to obtain $\vec{r}_2(s)$.



① $r'(t) = \langle -4\sin t, 4\cos t, 3\rangle$

$\|r'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = \sqrt{16+9} = \sqrt{25} = 5$

$S(t) = \int_0^t \|r'(\tau)\|\, d\tau = \int_0^t 5\, d\tau = 5\tau \Big|_0^t = 5t - 0 = 5t$

$\boxed{s = 5t}$

So $t = s/5$ and

$$\boxed{\begin{gathered} r_2(s) = \left\langle 4\cos(s/5), 4\sin(s/5), 3\tfrac{s}{5}\right\rangle \\ 0 \le s \le 5\pi/2 \end{gathered}}$$

what $s=$? when $t = \pi/2$ using $t = s/5$

(b) Need to compute

$L = \int_0^{5\pi/2} \|r_2'(s)\|\, ds$

$r_2'(s) = \left\langle -\frac{4}{5}\sin(s/5), \frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right\rangle$

$\|r_2'(s)\| = \sqrt{\frac{16}{25}\sin^2(s/5) + \frac{16}{25}\cos^2\left(\frac{s}{5}\right) + \frac{9}{25}}$

$= \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{25/25} = \boxed{1}$

$= \int_0^{5\pi/2} 1\, ds = s\Big|_0^{5\pi/2}$

$= \boxed{5\pi/2}$

Example 22. *You try it!* (a) Find an arc length parameterization of the portion $\mathcal{C}$ of the helix of radius 4 in $\mathbb{R}^3$ parametrized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle, 0 \le t \le \pi/2$.

(b) use s(t) to find L the length of $\mathcal{C}$.

① Find s(t):

$$\vec{v} = \frac{d\vec{r}}{dt} = \langle -4\sin t, 4\cos t, 3\rangle \qquad |\vec{v}|^2 = 16+9=25$$

so $|\vec{v}| = 5$

So,

$$s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t 5\, d\tau = 5\tau \Big|_0^t$$

and $s = 5t$ ///

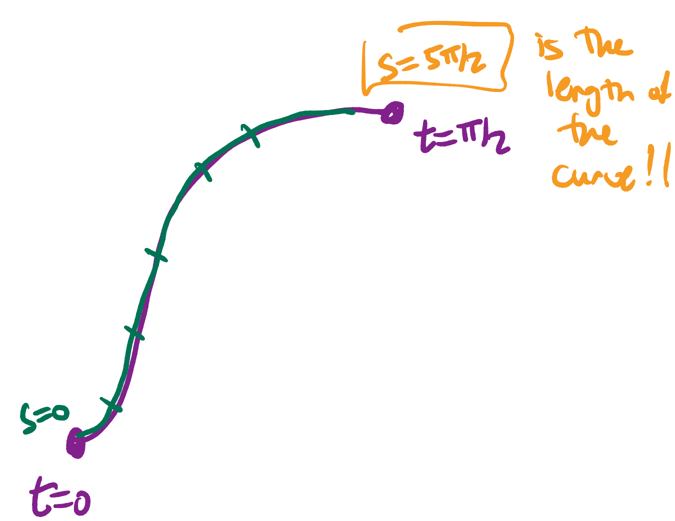
② Solve for t:

$$s = s(t) = 5t \quad \text{so} \quad t = f(s) = s/5$$

Sub into $\vec{r}$: $\vec{r}_2(s) = r(f(s))$

③ $$\boxed{\vec{r}_2(s) = \langle -4\sin(s/5), 4\cos(s/5), 3\rangle, \quad 0 \le s \le 5\pi/2}$$

$$\begin{cases} t=0 \\ t=\pi/2 \end{cases} \Rightarrow \begin{matrix} s=0 \\ s=5\pi/2 \end{matrix}$$



(b)
$$L = \int_0^{\pi/2} |\vec{v}|\, d\tau = s\Big|_{t=\pi/2} = \boxed{\frac{5}{2}\pi}$$