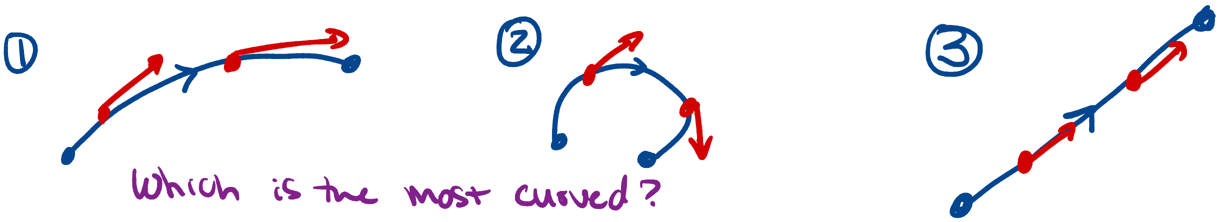
§13.3 & 13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.



Which is the most curved?

Idea: scale 2nd derivative appropriately

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: $r'(s)$
- In terms of any parameter t: $r'(t) / \|r'(t)\|$

(done b/c $\|r'(s)\| = 1$) if $r(s)$ is an arc-length parametrization.

unit vector in the direction of travel (unit velocity vector)

This lets us define the **curvature**, $\boxed{\kappa(s) = \|T'(s)\|}$

What if $r(t)$ isn't an arc-length parametrization?

Example 23. In Example 21 we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$. CHAIN $\frac{d}{ds}\frac{s}{4} = \frac{1}{4}$

$$T(s) = r'(s) = \left\langle 4 \cdot -\frac{1}{4}\sin(s/4),\ 4\cdot\frac{1}{4}\cos(s/4)\right\rangle$$

$T(s) = r'(s)$ if s was an arc-length parameter

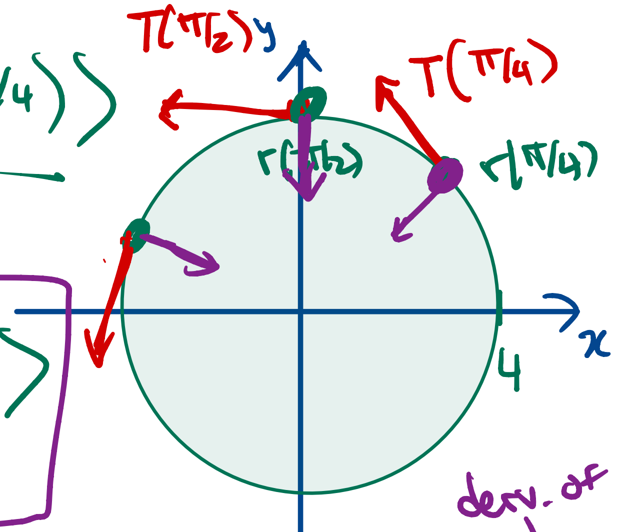
$$\boxed{T(s) = \langle -\sin(s/4),\ \cos(s/4)\rangle}$$

(check $\|r'(s)\| = \sqrt{\sin^2(s/4) + \cos^2(s/4)} = \sqrt{1} = 1$ ✓)

Now
$$\boxed{T'(s) = \left\langle -\frac{1}{4}\cos(s/4),\ -\frac{1}{4}\sin(s/4)\right\rangle}$$

So $\kappa(s) = \|T'(s)\| = \sqrt{\frac{1}{16}\cos^2(s/4) + \frac{1}{16}\sin^2(s/4)} = \sqrt{1/16} = \boxed{\frac{1}{4}}$

$$\frac{T'(s)}{\|T'(s)\|} = \frac{\left\langle -\frac{1}{4}\cos(s/4),\ -\frac{1}{4}\sin(s/4)\right\rangle}{1/4}$$



$$\boxed{\frac{T'(s)}{\|T'(s)\|} = \langle -\cos(s/4),\ -\sin(s/4)\rangle}$$

Question: In which direction is $\mathbf{T}$ changing?

deriv. of unit tangent

This is the direction of the **principal unit normal**, $\mathbf{N}(s) = \dfrac{T'(s)}{\|T'(s)\|}$ its length

Step 0: Find an arc-length parameterization

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) = \mathbf{r}'(t) / \|\vec{r}'(t)\|$
- $\mathbf{N}(t) = \vec{T}'(t) / \|\vec{T}'(t)\|$
- $\kappa(t) = \|T'(t)\| / \|\vec{r}'(t)\|$ or $\dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1\rangle$, $t \in \mathbb{R}$.

Step 1: $r'(t) = \langle -2\sin t, 2\cos t, 1\rangle$

and $\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{4+1} = \sqrt{5}$

Q: is $r(t)$ an arc-length parametrization?

A: No, b/c $\|r'(t)\| = 1$ (for all t) if $r(t)$ was an arc-length parametrization

Step 2:

$T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{1}{\sqrt{5}}\langle -2\sin t, 2\cos t, 1\rangle$

$$\boxed{T(t) = \left\langle -\tfrac{2}{\sqrt{5}}\sin t, \tfrac{2}{\sqrt{5}}\cos t, \tfrac{1}{\sqrt{5}}\right\rangle}$$

Step 3:

$N(t) = \dfrac{T'(t)}{\|T'(t)\|}$

$T'(t) = \left\langle -\frac{2}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}}\sin t, 0\right\rangle$

$\|T'(t)\| = 2/\sqrt{5}$ so $N(t) = \dfrac{1}{(2/\sqrt{5})}\left\langle -\frac{2}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}}\sin t, 0\right\rangle$

$$\boxed{N(t) = \langle -\cos t, -\sin t, 0\rangle}$$

Step 4:

$\kappa(t) = \|T'(t)\| / \|r'(t)\| = \dfrac{2/\sqrt{5}}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} \cdot \dfrac{1}{\sqrt{5}} = \boxed{2/5}$

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + \underline{t \sin t})\mathbf{i} + (\sin t - \underline{t \cos t})\mathbf{j} + 3\mathbf{k}, \cancel{t \in \mathbb{R}} \quad t \geq 0$$

PROD PROD

$$r'(t) = \langle -\cancel{\sin t} + t\cos t + \cancel{\sin t}, \cancel{\cos t} - (-t\sin t + \cancel{\cos t}), 0 \rangle$$

$$= \langle t\cos t, t\sin t, 0 \rangle \qquad T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$\|r'(t)\| = \sqrt{t^2\cos^2 t + t^2 \sin^2 t} = \sqrt{t^2} = t \quad (t \geq 0)$$

So $\boxed{T(t) = \langle \cos t, \sin t, 0 \rangle}$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

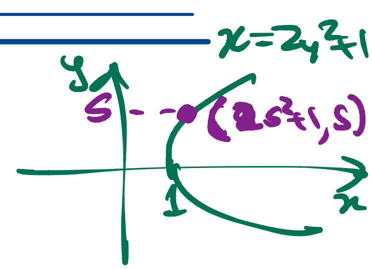
Next, $T'(t) = \langle -\sin t, \cos t, 0 \rangle$

$$\|T'(t)\| = 1 \qquad \kappa = \frac{\|T'(t)\|}{\|r'(t)\|}$$

So $\boxed{N(t) = \langle -\sin t, \cos t, 0 \rangle}$

and $\boxed{\kappa(t) = 1/t}$

IF time: (?)



parametrization ✓

T/F $\boxed{\vec{r}(s) = \langle 2s^2+1, s \rangle, \ s \in \mathbb{R}}$ is an arc-length parametrization of the parabola

$$x = 2y^2 + 1.$$

Check? • $\|r'(s)\| = 1$

$r'(s) = \langle 4s, 1 \rangle$ so $\|r'(s)\| = \sqrt{16s^2+1} \neq 1$.

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}, \ t \in \mathbb{R}.$$

Step 1 Find $r'(t)$ and $\|r'(t)\|$.

$$v = \frac{dr}{dt} = (\cancel{-\sin t} + t\cos t + \cancel{\sin t})\hat{i} + (\cancel{\cos t} - (-t\sin t + \cancel{\cos t}))\hat{j} + 0\hat{k}$$

$$= t\cos t\,\hat{i} + t\sin t\,\hat{j} + 0\hat{k}$$

$|v|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2$ so $|v| = |t|$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$\boxed{T = \frac{v}{|v|} = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}}$$

$\frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}$ & $\left|\frac{dT}{dt}\right| = 1.$

so $\boxed{N = \frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}}$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$\boxed{\kappa = \frac{1}{|t|}}$$

$$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$$

IF time: (?)

T/F

$\vec{r}(s) = \langle 2s^2+1, s\rangle$, $s \in \mathbb{R}$ is an arc-length parametrization of the parabola $x = 2y^2 + 1$.

Multivariable calculus

So far $r(t) = \langle x(t), y(t), z(t)\rangle$

single input variable

multiple output variables.

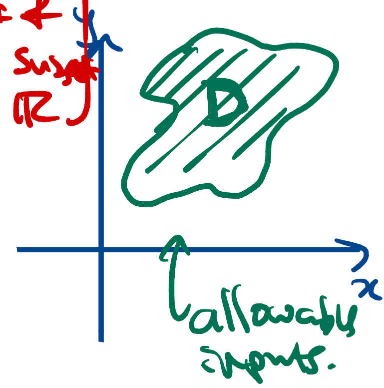
§14.1 Functions of Multiple Variables

Definition 26. A Function of two variables is a rule that assigns to each 2-tuple of real numbers (x, y) in a set D a uniquely determined output denoted by $f(x, y)$.

codomain of f.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

[range of f is some subset of $\mathbb{R}$]



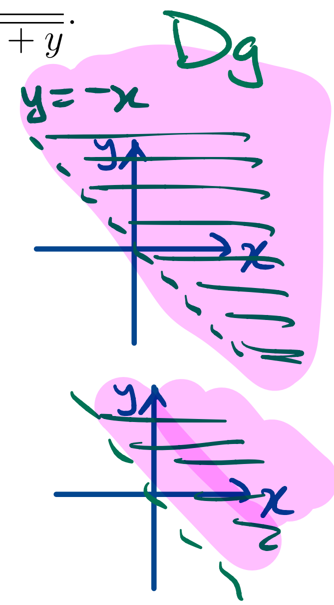
D some subset of $\mathbb{R}^2$ consisting of all points you're allowed to plug into the function f. (The domain of f)

Example 27. Three examples are

$(x,y,z) = (x,y,f(x,y))$

$$z = f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \frac{1}{\sqrt{x + y}}.$$

Example 28. Find the largest possible domains of f, g, and h.



D_f can be $\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$

D_g need $x + y > 0$

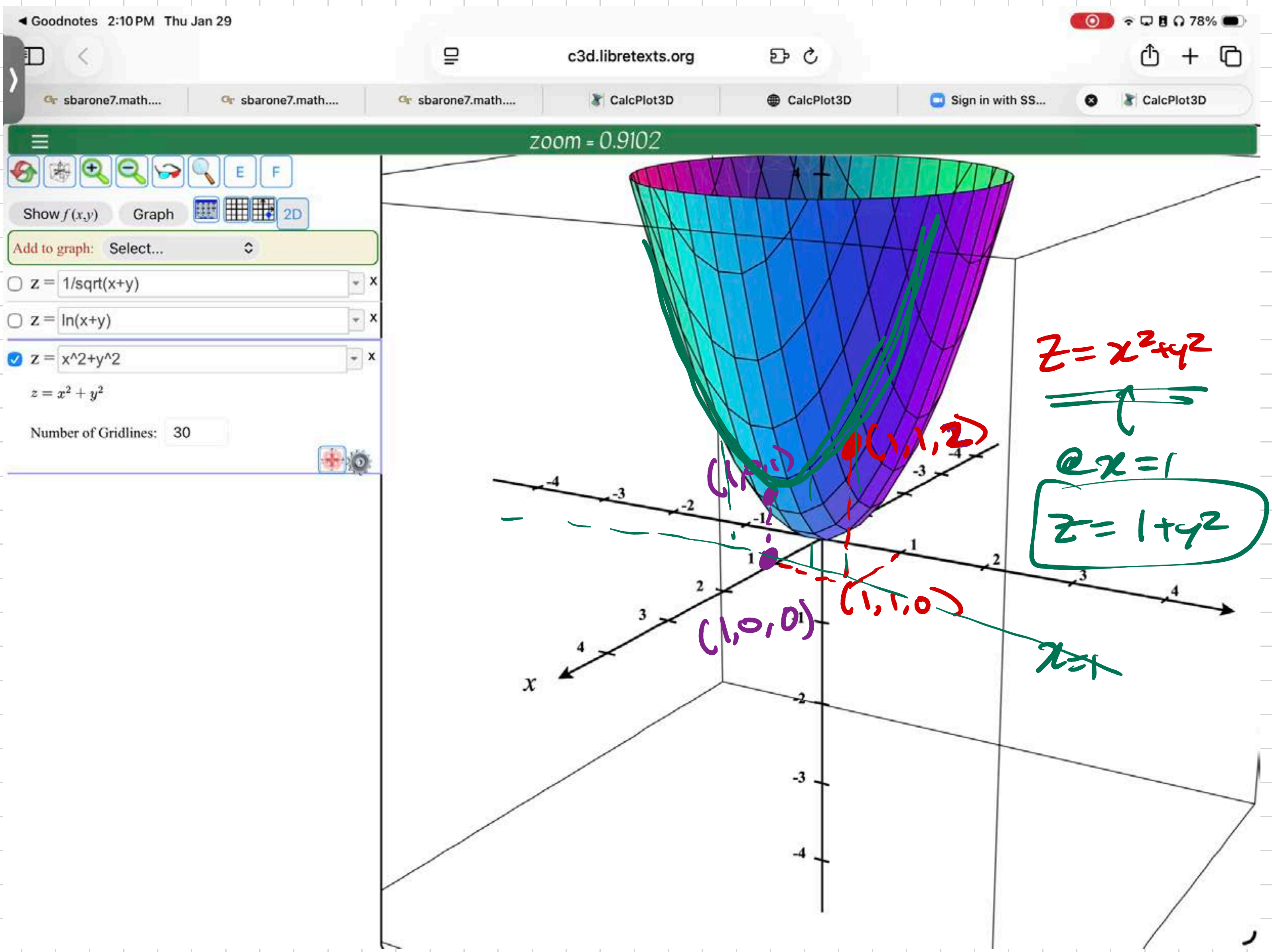
so $D_g = \{(x,y) \mid x+y>0\}$

$= \{(x,y) \mid y > -x\}$

D_h same need $x+y>0$ $D_h = D_g$

Definition 29. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

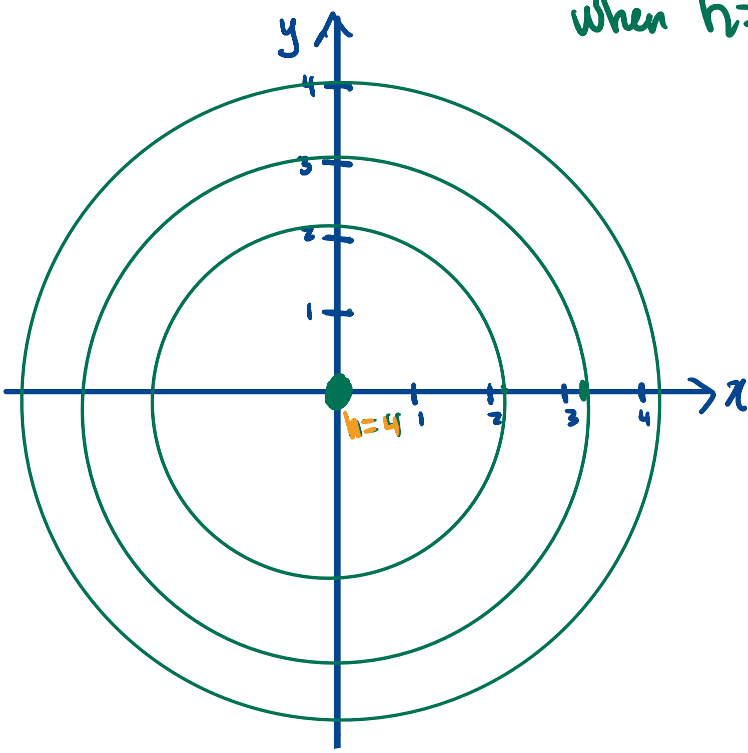

c3d.libretexts.org
CalcPlot3D
zoom = 0.9102
Show f(x,y)
Graph
2D
Add to graph: Select...
z = 1/sqrt(x+y)
z = ln(x+y)
z = x^2+y^2
$z = x^2 + y^2$
Number of Gridlines: 30
(1,1,2)
(1,0,1)
(1,1,0)
(1,0,0)
x=1
$z = x^2+y^2$
@ $x = 1$
$\boxed{z = 1+y^2}$

height function (z-value)

Example 30. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

draw all points in the input space w/ a given height.

solve for x, y



when $h=0$

$0 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\Rightarrow \frac{1}{4}x^2 + \frac{1}{4}y^2 = 4$

$\Rightarrow x^2 + y^2 = 16$

circle w/ center $(0,0)$ & $r = 4$

when $h=1$

$1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\Rightarrow \frac{1}{4}x^2 + \frac{1}{4}y^2 = 3$

$\Rightarrow x^2 + y^2 = 12$

circle w/ center $(0,0)$ & $r = \sqrt{12} = 2\sqrt{3}$

when $h=4$

$4 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\Rightarrow x^2 + y^2 = 0$

only happens at $(x, y) = (0, 0)$

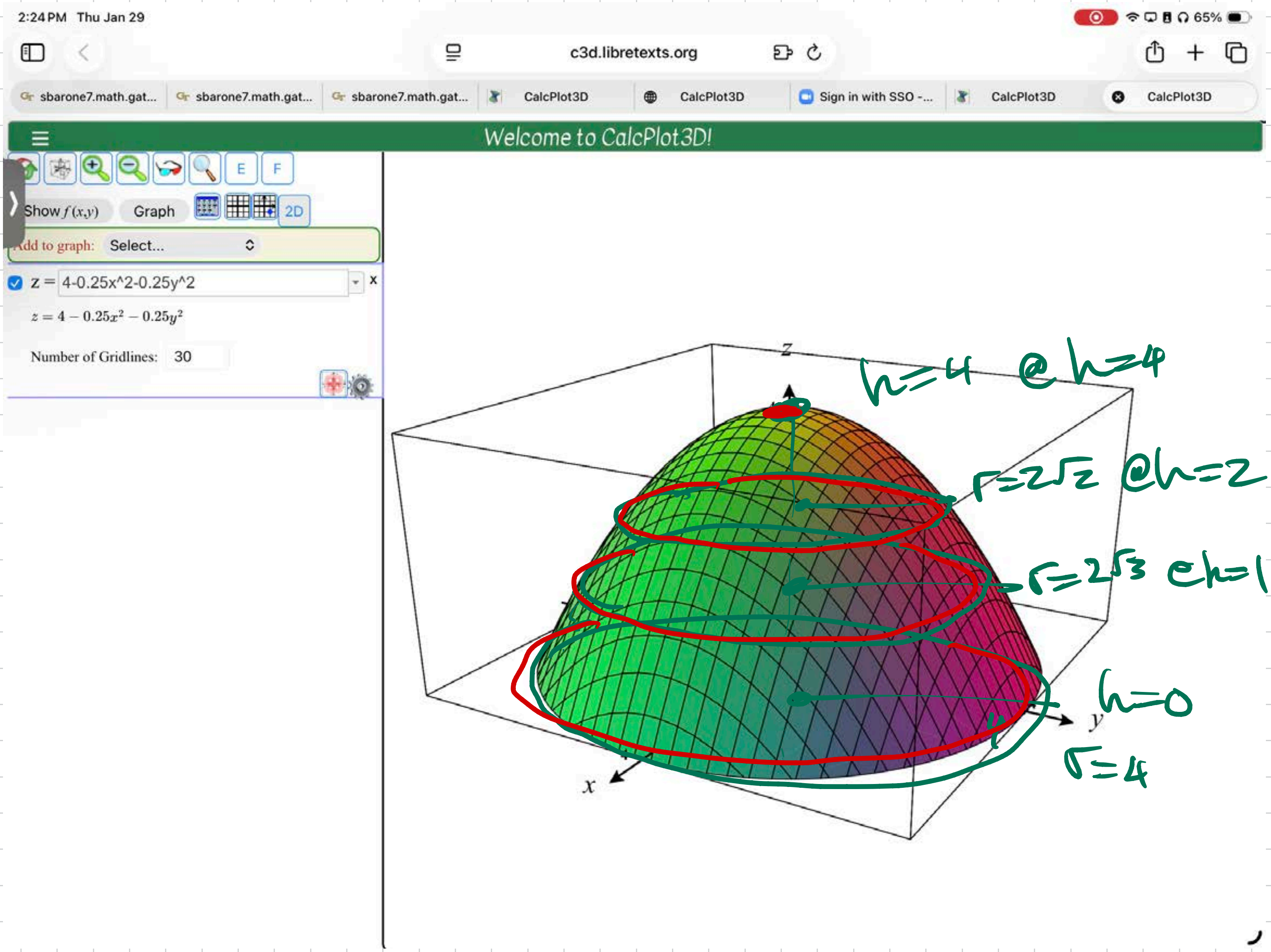
when $h=2$

$2 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\Rightarrow x^2 + y^2 = 8$

circle w/ center $(0,0)$ $r = 2\sqrt{2}$

In 3D, it looks like this.

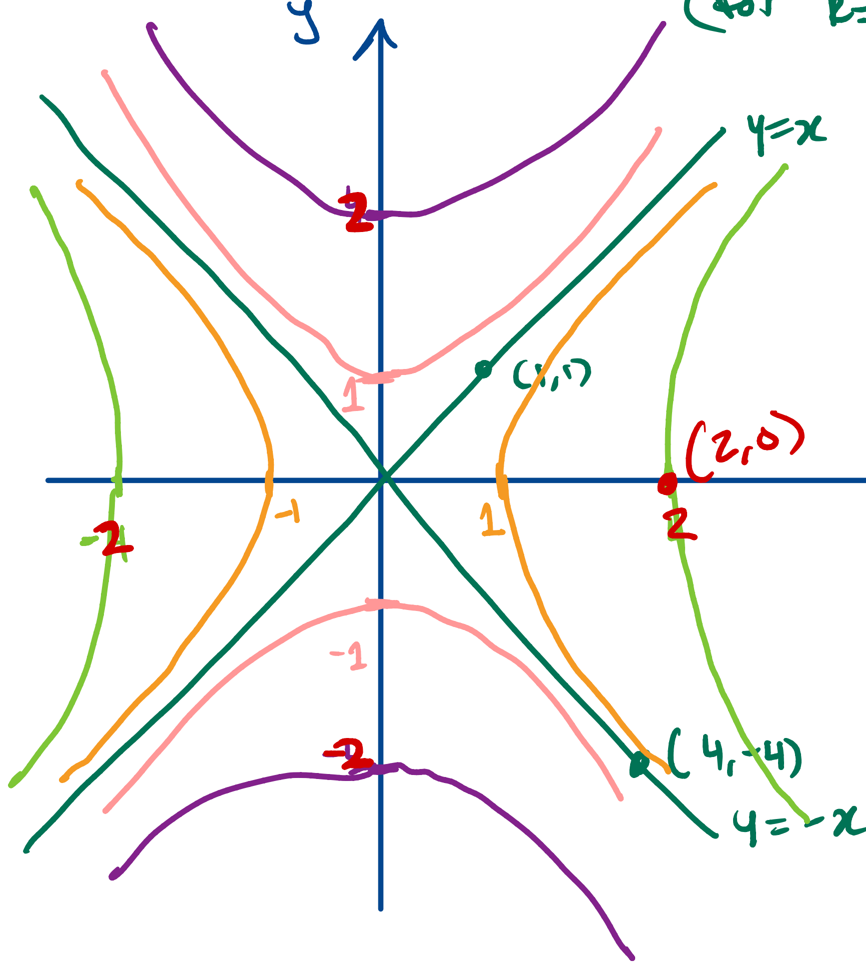

2:24 PM Thu Jan 29
65%
c3d.libretexts.org
sbarone7.math.gat...
CalcPlot3D
Sign in with SSO -...
Welcome to CalcPlot3D!
E
F
Show f(x,y)
Graph
2D
Add to graph:
Select...
z = 4-0.25x^2-0.25y^2
z = 4 − 0.25x² − 0.25y²
Number of Gridlines: 30
z
x
y
h=4 @ h=4
r=2√2 @h=2
r=2√3 @h=1
h=0
r=4

Definition 31. The level sets (also called contours) of a function f of two variables are the curves with equations $k = f(x,y)$, where k is a constant (in the range of f). A plot of contours for various values of z is a contour plot (or level set map).

Some common examples of these are:

- Maps topography (mountain maps)
- potential lines (electromagnetic potentials)
- depth charts.
- air pressure map.

Example 32. Create a contour diagram of $\boxed{f(x,y) = x^2 - y^2}$ Try $k = 0, 1, 4$.



(for k=0) $0 = x^2 - y^2 \Rightarrow x^2 = y^2$

$|y| = |x|$

$x = y$ or $x = -y$

(For k=1) $\boxed{1 = x^2 - y^2}$ hyperbola opening sideways

(k=4) $4 = x^2 - y^2$

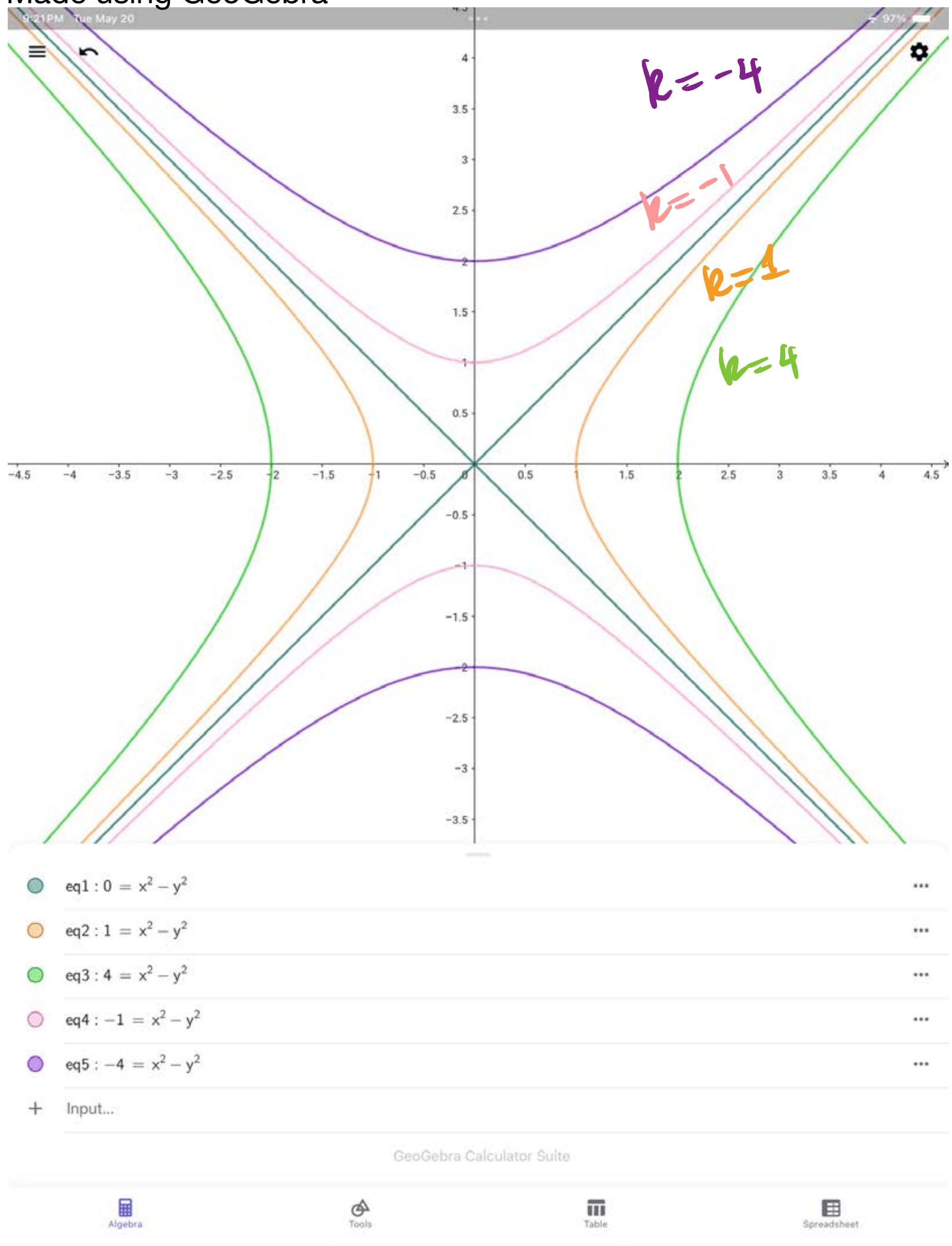
(k=-1) $-1 = x^2 - y^2$

$\Rightarrow \boxed{1 = -x^2 + y^2}$

(k=-4) $-4 = x^2 - y^2$

$\Rightarrow 4 = -x^2 + y^2$

Made using GeoGebra



(Contours were intersections w/ planes parallel to xy-plane)

Definition 32. The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. x-z plane or y-z plane

Example 33. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

$\Longrightarrow (x \geq 0, y \geq 0, z \geq 0)$ (x, y, z)

contours are $z=k$ level sets.

at $z=0$ then $0 = 4-2x-y^2$

$\Rightarrow x = 2 - \frac{1}{2}y^2$

at $z=k>0$ then $k = 4-2x-y^2$

$\Rightarrow x = \frac{4-k}{2} - \frac{1}{2}y^2$

traces w/ $y=k$

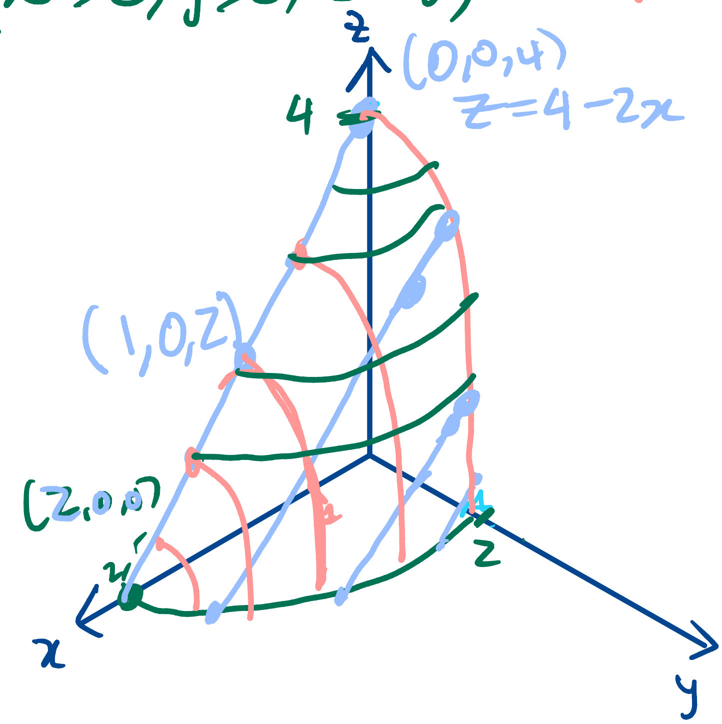
If $y=0$ $z = 4-2x-0^2$

If $y=1$, $z = 4-2x-1^2$

$\Rightarrow z = 3-2x$

If $y=k>0$, $z = 4-2x-k^2$

$\Rightarrow z = (4-k^2)-2x$



$z = 4-2x-y^2$ (2,0,0) ✓

(1,0,2) ✓

(0,0,4) ✓

traces w/ $x=k$

If $x=0$ then $z = 4-2(0)-y^2 = 4-y^2$

If $x=1$, then $z = 4-2(1)-y^2 = 2-y^2$

If $x=k$, then $z = (4-2k)-y^2$

$(k>0)$

Let's check our work: https://tinyurl.com/math2551-2var-graph

Definition 34. A Function of 3-variables is a rule that assigns to each 3-tuple of real numbers (x, y, z) in a set D a unique output denoted by $f(x, y, z)$. ← The function evaluated at (x,y,z).

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 35. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Need $4 - x^2 - y^2 - z^2 \neq 0$

so avoid $4 - x^2 - y^2 - z^2 = 0$

$\Rightarrow x^2 + y^2 + z^2 = 4$ (eqn. of a sphere centered $(0,0,0)$ w/ $r = 2$.)

Domain of f

$D = \{ (x,y,z) \mid x^2 + y^2 + z^2 \neq 4 \}$

Example 36. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

(output variable is constant)

set $g(x,y,z) = k$ (constant) & describe resulting set of points (x,y,z)

e.g. $k = 0$ $\quad 0 = 2x^2 + y^2 + z^2$ only true if $(x,y,z) = (0,0,0)$

$k = 1$ $\quad 1 = 2x^2 + y^2 + z^2$ implicitly defined surface in $\mathbb{R}^3$.

$k = 2$ $\quad 2 = 2x^2 + y^2 + z^2$

larger ellipsoid.

ellipsoid