EXAM 1 on Thurs @ 2pm in Lecture hall.

Phones must be in backpack. Backpacks must be in front of room

§14.2 Limits & Continuity

COVERED on Exam 1.

Definition 37. What is a limit of a function of two variables?

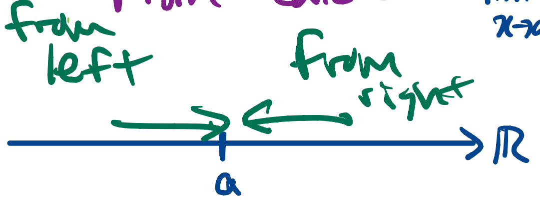
e.g. $f(x,y) = \frac{x+y}{x^2}$ as $(x,y) \to (0,0)$ what happens to $z = f(x,y)$?

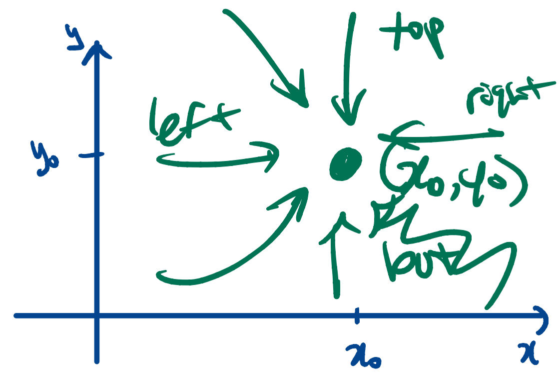
> **DEFINITION** We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write
>
> $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$
>
> if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,
>
> $$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

START on THURS.

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if and only if $f(x,y)$ approaching L regardless of how we approach the point (x_0, y_0).

From Calc I. $\lim_{x\to a} f(x) = L$





Definition 38. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. $f(x_0,y_0) = L$ exist
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist
3. They are equal.

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

e.g. $f(x,y) = x$ cont ✓ $\quad h_1 = x + y$ ✓

$g(x,y) = y$ cont ✓ $\quad h_2 = x(x+y)$ ✓

$h_3 = x^4 + 3xy$ ✓

Example 39. Evaluate $\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Try? evaluate $f(x,y) = \frac{\sqrt{2x-y}-2}{2x-y-4}$ @ $(x,y)=(2,0)$

$$f(2,0) = \frac{\sqrt{4-0}-2}{4-0-4} \stackrel{?}{=} \frac{0}{0} \quad \text{DNE.}$$

(So $f(x,y)$ is not continuous @ $(x,y)=(2,0)$ b/c $f(x,y)$ is NOT DEFINED @ $(2,0)$.)

But can still have a well defined limit.

NOTE: 1-variable techniques include

* L'Hopital's rule (but doesn't seem to work well here)

* algebraic simplification.

Eg. related problem in 1-var calculus just to get warmed up.

has nothing to do w/ 2 var. problem (show technique)

$$\lim_{x\to 2} \frac{\sqrt{2x}-2}{2x-4} = \lim_{x\to 2} \frac{\sqrt{2x}-2}{2x-4} * \frac{\sqrt{2x}+2}{\sqrt{2x}+2} = \lim_{x\to 2} \frac{(\sqrt{2x})^2 - 2^2}{(2x-4)(\sqrt{2x}+2)}$$

$$= \lim_{x\to 2} \frac{\cancel{2x-4}}{\cancel{(2x-4)}(\sqrt{2x}+2)} = \lim_{x\to 2} \frac{1}{\sqrt{2x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

Example 39. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Try? evaluate $f(x,y) = \dfrac{\sqrt{2x-y}-2}{2x-y-4}$ @ $(x,y)=(2,0)$

$$f(2,0) = \frac{\sqrt{4-0}-2}{4-0-4} \stackrel{?}{=} \frac{0}{0} \quad \text{DNE.}$$

(So $f(x,y)$ is not continuous @ $(x,y)=(2,0)$ b/c $f(x,y)$ is NOT DEFINED @ $(2,0)$.)

$(a-b)(a+b)=a^2-b^2$

* algebraic simplification. $\quad \times 1$

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} \times \frac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2}$$

$$= \lim_{(x,y)\to(2,0)} \frac{(\sqrt{2x-y})^2-(2)^2}{(2x-y-4)(\sqrt{2x-y}+2)} = \lim_{(x,y)\to(2,0)} \frac{\cancel{2x-y-4}}{\cancel{(2x-y-4)}(\sqrt{2x-y}+2)}$$

$$= \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2} = \frac{1}{\sqrt{4-0}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

I'm evaluating the limit (not dropping)

Example 39. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Try? evaluate $f(x,y) = \dfrac{\sqrt{2x-y}-2}{2x-y-4}$ @ $(x,y)=(2,0)$

$$f(2,0) = \frac{\sqrt{4-0}-2}{4-0-4} \stackrel{?}{=} \frac{0}{0} \quad \text{DNE.}$$

(So $f(x,y)$ is not continuous @ $(x,y)=(2,0)$ b/c $f(x,y)$ is NOT DEFINED @ $(2,0)$.)

$(a-b)(a+b)=a^2-b^2$

* algebraic simplification. $\quad$ $\times 1$

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} \cdot \frac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2}$$

$$= \frac{(\sqrt{2x-y})^2-(2)^2}{(2x-y-4)(\sqrt{2x-y}+2)} = \frac{\cancel{2x-y-4}}{\cancel{(2x-y-4)}(\sqrt{2x-y}+2)}$$

$$= \frac{1}{\sqrt{2x-y}+2} = \frac{1}{\sqrt{4-0}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

Write your limits where appropriate & $\boxed{\text{don't DROP your LIMITS}}$

Example 40. *You try it!* Evaluate $\displaystyle\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$, if it exists.

$f(x,y) = \cos y$ ✓

$g(x,y) = \sin x$ ✓

Example 40. *You try it!* Evaluate $\lim\limits_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$, if it exists.

$$\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x} = \frac{\cos(0) + 1}{0 - \sin(\pi/2)} = \frac{1+1}{0-1} = \frac{2}{-1}$$

$$= \boxed{-2}$$

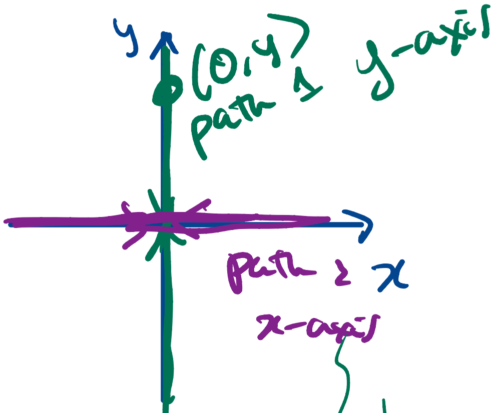
alt soln?

$$f(\pi/2, 0) = \frac{\cos(0)+1}{0-\sin(\pi/2)} = \frac{1+1}{0-1} = \frac{2}{-1} = \boxed{-2}$$ ✓

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 41. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.



$f(x,y) = \frac{x^2}{x^2+y^2}$

limit of $f(x,y)$ as $(x,y)\to(0,0)$ depends on the path you take.

Path 1 along y-axis. @ $(x,y)=(0,y)$ or @ $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0^2}{0^2+y^2}$$

$$= \boxed{\lim_{y\to 0} \frac{0}{y^2} = 0}$$

$\frac{0}{0}$??

← along the path $x=0$ the limit value is $L=0$.

Path 2 along x-axis @ $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}} = 1$$

along the second path $y=0$ the limit value is 1.

$\boxed{\text{DNE by TWO PATH TEST}}$

This idea is called the **two-path test:**

If we can find two different paths approach ~~to~~ (x_0, y_0) along which the limit value takes on two different values, then conclude the limit is DNE.

Example 42. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

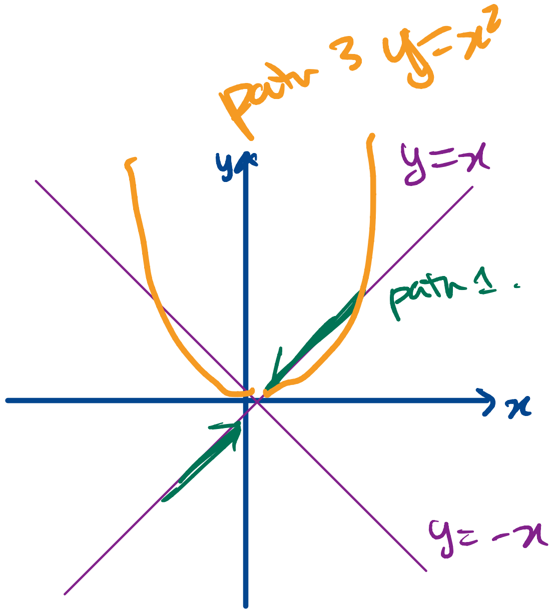
$$\lim_{x\to 0} \frac{1}{x} = \text{DNE}$$

Path 1

Try path along $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2x}{x^4+x^2}$$

$$= \lim_{x\to 0} \frac{x^3}{x^4+x^2} = \lim_{x\to 0} \frac{x}{x^2+1} = \frac{0}{0^2+1} = \boxed{0}$$



Try path $y=-x$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,-x)\to(0,0)} \frac{x^2(-x)}{x^4+(-x)^2}$$

$$= \lim_{x\to 0} \frac{-x^3}{x^4+x^2} = \lim_{x\to 0} \frac{-x}{x^2+1} = 0$$

Path 3

Try $y=x^2$ path.

along @ $y=x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,x^2)\to(0,0)} \frac{x^2x^2}{x^4+(x^2)^2}$$

$$= \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \boxed{\frac{1}{2}}$$

Since path 1 & path 3 have have different limit values the limit is $\boxed{\text{DNE by the TWO PATH TEST}}$

Example 42. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2x}$$

add

$$\lim_{x\to 0} \frac{1}{x} = \text{DNE}$$

does not exist.

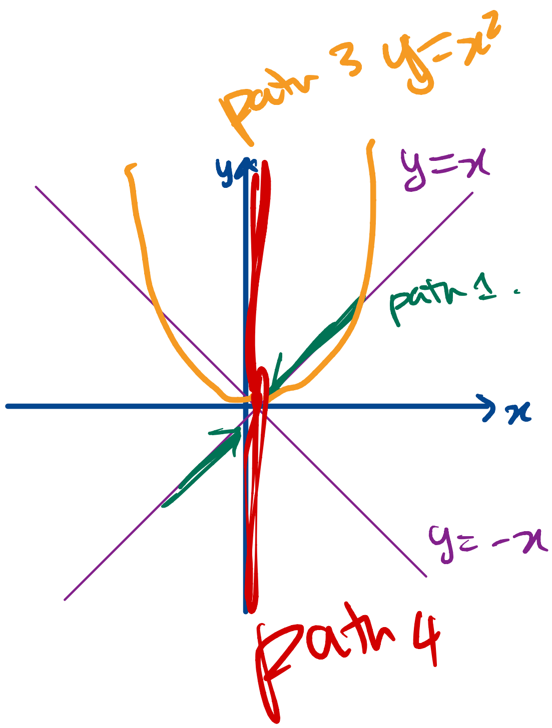
Path 1

Try path along $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2x}{x^4+x^2}$$

$$= \lim_{x\to 0} \frac{x^3}{x^4+x^2} = \lim_{x\to 0} \frac{x}{x^2+1}$$

$$= \frac{0}{0^2+1} = \boxed{0}$$



Path 4 $x=0$

Watch out for

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2x} = \lim_{(0,y)\to(0,0)} \frac{0}{0}$$

Note $\lim_{(x,y)\to(1,1)} f(x,y)$ then the paths MUST INCLUDE (1,1)!!

Example 43. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test. Hint: try two parabolas.

Path 1 @ $y=x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{(x,x^2)\to(0,0)} \frac{x^4}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$$

Path 2 @ $x=y^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{(y^2,y)\to(0,0)} \frac{(y^2)^4}{(y^2)^4+y^2} = \lim_{y\to 0} \frac{y^8}{y^8+y^2} = \cancel{1}$$

$$= \lim_{y\to 0} \frac{y^6}{y^6+1} = 0$$

The limit is $\boxed{\text{DNE by the TWO PATH TEST}}$

other paths $y=2x^2$ $L=1/5$

$\boxed{x=0, y=0 \quad L=0 \quad L=1}$??

Example 43. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test. Hint: try two parabolas.

$f(x,y) = \frac{x^4}{x^4+y^2}$ Let $y = mx^2$

Then @ $y=mx^2$ $\lim_{(x,mx^2)\to(0,0)} f(x,mx^2) = \lim_{x\to 0} \frac{x^4}{x^4+(mx^2)^2}$

$= \lim_{x\to 0} \frac{x^4}{x^4+m^2x^4} = \lim_{x\to 0} \frac{x^4}{x^4} \cdot \frac{1}{1+m^2} = \frac{1}{1+m^2}$

which takes different values depending on m.

So, by $\boxed{\text{Two-path test the limit is DNE.}}$

Example 44. [Challenge:] Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4+y^2}$$

does exist using the Squeeze Theorem.

Theorem 45 (Squeeze Theorem)**.** *If* $f(x,y) = g(x,y)h(x,y)$, *where* $\lim_{(x,y)\to(a,b)} g(x,y) = 0$ *and* $|h(x,y)| \le C$ *for some constant* C *near* (a,b), *then* $\lim_{(x,y)\to(a,b)} f(x,y) = 0$.

IDEA. Show that $\dfrac{x^4 y}{x^4+y^2} = \underbrace{y}_{\text{goes to zero}} \times \underbrace{\dfrac{x^4}{x^4+y^2}}_{\text{bounded}}$

Note $\dfrac{x^4}{x^4+y^2}$ is bounded. ie. $\left|\dfrac{x^4}{x^4+y^2}\right| \le C$ for all $(x,y) \in \mathbb{R}^2$ for some C

whole expression is larger

add something non-neg to numerator

In particular

$$0 \le \frac{x^4}{x^4+y^2} \le \frac{x^4+y^2}{x^4+y^2} = 1.$$

x^4, y^2 both nonneg. So $\left|\dfrac{x^4}{x^4+y^2}\right| \le 1$.

and $y \to 0$ as $(x,y)\to(0,0)$.

So By SQUEEZE THM the $\boxed{\lim_{(x,y)\to(0,0)} \dfrac{x^4}{x^4+y^2} = 0}$

Math 2551 Worksheet 8 - Review for Exam 1

1. Set up the integral to find the arc length of the curve $y = e^x$ from the point $(0, 1)$ to the point $(1, e)$. Focus on finding a parameterization, and on what values of t give these two points. Is this an integral you would want to compute? Why or why not?

2. Parameterize the line tangent to the curve

$$\mathbf{r}(t) = \langle \cos^2(t), \sin(t)\cos(t), \cos(t) \rangle$$

at the point where $t = \pi/2$.

3. Compute the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ to the circle

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle.$$

Before checking, should the normal vector be pointing into or out of the circle? Why?

4. We have seen that the curvature of a circle with radius a is $1/a$. Thinking about the geometry of a helix with radius a, do you think its curvature will be greater than or less than $1/a$? Why? Compute the curvature using the parameterization

$$\mathbf{r}(t) = \langle a\cos(t), t, a\sin(t) \rangle$$

to confirm or challenge your intuition.

5. The function $\ell(t)$ below describes a line. There is a particular plane that $\ell(t)$ is normal to at the point $t = 0$. Find an equation of this plane.

$$\ell(t) = \langle 3 - 3t, 2 + t, -2t \rangle.$$

Where does this line intersect the different plane $3x - y + 2z = -7$?

6. Find and sketch the domain of each of the following functions of two variables:

   (a) $\sqrt{9 - x^2} + \sqrt{y^2 - 4}$

   (b) $\arcsin(x^2 + y^2 - 2)$

   (c) $\sqrt{16 - x^2 - 4y^2}$

7. Solve the differential equation below, together with its given initial conditions. Remember that this means finding all functions $\mathbf{r}(t)$ which satisfy the given equations.

$$\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}, \quad \mathbf{r}'(1) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(1) = \mathbf{i} + \mathbf{j}$$

8. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?