§14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 47. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

https://strawpoll.com/e6Z2AwBYqgN

xbar	42.9
stdev	6.67
%perfect	9%
%within1	22%
Q3	48
Median	45
Q1	39.5
Min	22
%A	52%
%B	23%
%C	12%
%D	6%
%F	7%

Let's investigate graphically.

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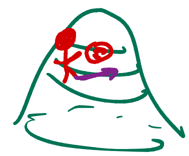


$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

@ $(x,y)=(2,1)$ $h(2,1) = 4 - \frac{1}{4}(2)^2 - \frac{1}{4}(1)^2 = 4 - 1 - \frac{1}{4} = 2 + \frac{3}{4} = 2.75$
$= 11/4$

in y-direction

Idea @ $x=2$ so $(x,y) = (2,y)$ $h(2,y) = 4 - \frac{1}{4}(2)^2 - \frac{1}{4}y^2$
$= 3 - \frac{1}{4}y^2$

Now the y-derivative $\frac{\partial}{\partial y}\left(3 - \frac{1}{4}y^2\right) = -\frac{1}{2}y$

@ $(2,1)$ get $\frac{\partial}{\partial y} h(2,1) = -\frac{1}{2}(1) = \boxed{-\frac{1}{2}}$

So idea @ $(2,1)$ if you travel NORTH (positive y-dir) then down the hill at rate of $-1/2$.

in x-direction

@ $y=1$ $h(x,1) = 4 - \frac{1}{4}x^2 - \frac{1}{4}(1)^2 = 3.75 - \frac{1}{4}x^2$

$\Rightarrow \frac{\partial}{\partial x}\left(3.75 - \frac{1}{4}x^2\right) = -\frac{1}{2}x$

@ $(2,1)$ $\frac{\partial}{\partial x} h(2,1) = -\frac{1}{2}(2) = \boxed{-1.}$

So moving EAST (pos x) then downhill at rate -1.

Let's investigate graphically.

Definition 48. If f is a function of two variables x and y, its partial derivative are the functions f_x and f_y defined by

$\frac{\partial}{\partial x} f(x,y)$ $\qquad$ $\frac{\partial}{\partial y} f(x,y)$

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations:

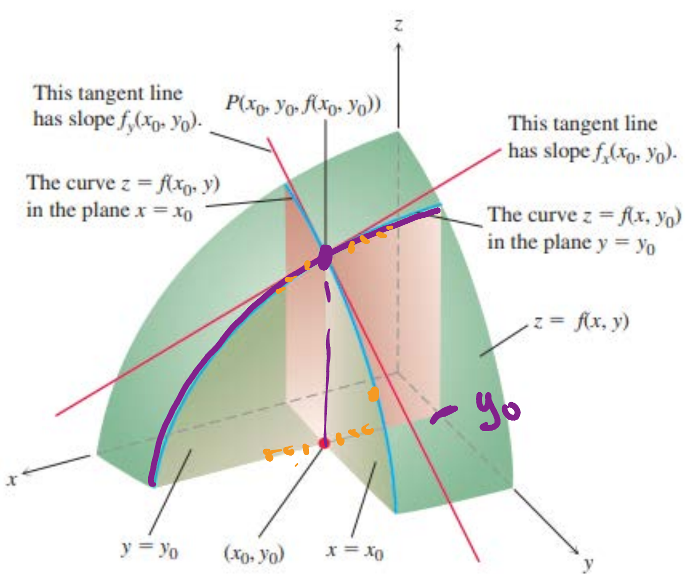
use $f_x = \frac{\partial}{\partial x} f = \frac{\partial f}{\partial x}$ (The x-partial derivative of f)

"f sub x" "del del x of f" "del f del x"

$f_y = \frac{\partial}{\partial y} f = \frac{\partial f}{\partial y}$

Start.

Interpretations:



MAIN IDEA: treat the other variable as a constant

Example 49. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

$(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

a) $f(x, y) = \sqrt{5x - y}$

$f_x(x,y) = \frac{\partial}{\partial x}\left((5x-y)^{1/2}\right)$

$= \frac{1}{2}(5x-y)^{-1/2} * \frac{\partial}{\partial x}(5x-y)$

$= \frac{1}{2}(5x-y)^{-1/2} * 5 = \frac{5}{2\sqrt{5x-y}}$ @ $(1,2)$ $\frac{\partial}{\partial x}f(1,2) = \frac{5}{2\sqrt{5(1)-2}} = \boxed{\frac{5}{2\sqrt{3}}}$

$f_y(x,y) = \frac{\partial}{\partial y}\sqrt{5x-y} = \frac{1}{2\sqrt{5x-y}} * \frac{\partial}{\partial y}(5x-y) = \frac{-1}{2\sqrt{5x-y}}$ @ $(1,2)$ $\frac{\partial}{\partial y}f(1,2) = \frac{-1}{2\sqrt{5(1)-2}}$

$= \boxed{\frac{-1}{2\sqrt{3}}}$

b) $f(x, y) = \tan(xy)$

$(\tan x)' = \sec^2 x$

$f_x = y\sec^2(xy)$ (from CHAIN rule)

$f_y = x\sec^2(xy)$

So @ $(1,2)$ $f_x(1,2) = \boxed{2\sec^2(2)}$

$f_y(1,2) = 1\sec^2(2) = \boxed{\sec^2 2}$

What about $f'' = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2} f$ (2nd derivative?)

Question: How would you define the second partial derivatives?

① $f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$

② $f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$

③ $f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$

④ $f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$

Example 50. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function below.

a) $f(x, y) = \sqrt{5x - y}$ prev slide $f_x = \frac{5}{2\sqrt{5x-y}}$ $f_y = \frac{-1}{2\sqrt{5x-y}}$

$f_x = \frac{5}{2}(5x-y)^{-1/2}$ $f_y = -\frac{1}{2}(5x-y)^{-1/2}$

① $f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2} * 5 = \boxed{\frac{-25}{4}(5x-y)^{-3/2}}$

② $f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2} * (-1) = \boxed{\frac{5}{4}(5x-y)^{-3/2}}$

SAME ?

③ $f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2} * 5 = \boxed{\frac{5}{4}(5x-y)^{-3/2}}$

④ $f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2}(-1) = \boxed{\frac{-1}{4}(5x-y)^{-3/2}}$

don't need to show work for ② + ③

just pick one & <u>STATE</u> $f_{xy} = f_{yx}$ ✓

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 51 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

Then: $\boxed{f_{xy} = f_{yx}}$

Immediate consequence

$$f_{xyx} = f_{xxy} = f_{yxx}$$

Example 52. *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 51 (Clairaut's Theorem). *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

$$\boxed{f_{xy} = f_{yx}}$$

corollary: $f_{xyx} = f_{xxy} = f_{yxx}$ ~~Same~~?? why?

also $f_{yxy} = f_{yxy} = f_{xyy}$

Example 52. *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

$$f_x = 2yz - 0 = \boxed{2yz}$$

$$f_y = \boxed{2xz - z^2}$$

$$f_z = \boxed{2xy - 2zy}$$

$f: \mathbb{R}^2 \to \mathbb{R}^3$

Example 53. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

Idea. $f(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

$\to x(s,t) = s^2+t$

$\to y(s,t) = 2s-t$

$\to z(s,t) = st$

Idea. we know how to deal with each component (scalar valued function)

For first component

$\frac{\partial}{\partial s} x(s,t) = 2s$

$\frac{\partial}{\partial t} x(s,t) = 1$

For 2nd component

$\frac{\partial}{\partial s} y(s,t) = 2$

$\frac{\partial}{\partial t} y(s,t) = -1$

For 3rd component

$\frac{\partial}{\partial s} z(s,t) = t$

$\frac{\partial}{\partial t} z(s,t) = s$

take all these and put them in a rectangular array.

$$\begin{array}{c|cc} & \partial/\partial s & \partial/\partial t \\ \hline x & 2s & 1 \\ y & 2 & -1 \\ z & t & s \end{array} = Df$$

"big D f"

Called the total derivative of f.

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(\underbrace{x_1, \ldots, x_n}_{n \text{ inputs}}) = \left.\begin{bmatrix} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_m(x_1,\ldots,x_n) \end{bmatrix}\right\} m \text{ output}$,

we have __n__ inputs, __m__ output, and __m×n__ partial derivatives, which we can use to form the **total derivative**. Df

This is a __linear__ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an __m×n matrix__, with one column per input and one row per output.

It has the formula $Df_{ij} = \frac{\partial}{\partial x_j} f_i(x_1, \ldots, x_n)$ $\quad 1 \le j \le n$ (# inputs) $\quad 1 \le i \le m$ (# outputs)

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2 y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

Df is a MATRIX

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

$f: \mathbb{R} \to \mathbb{R}$

Df has size 1x1

$[2x]$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$r: \mathbb{R} \to \mathbb{R}^3$

Df has size 3x1

$r_1(t) = \cos t$

$r_2(t) = \sin t$

$r_3(t) = t$

$\frac{\partial}{\partial t}$ or $\frac{d}{dt}$

$$Df = \begin{bmatrix} r_1'(t) \\ r_2'(t) \\ r_3'(t) \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix} & \begin{array}{l} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{array} \end{array}$$

c) $f(x, y) = \sqrt{5x - y}$

$f: \mathbb{R}^2 \to \mathbb{R}$

Df is 1x2

$\partial/\partial x$ $\partial/\partial y$

$$Df = [f_x \ \ f_y] = \left[\frac{5}{2\sqrt{5x-y}} \quad \frac{-1}{2\sqrt{5x-y}} \right] \leftarrow \text{"}x\text{"}$$

d) $f(x, y, z) = 2xyz - z^2y$

$f: \mathbb{R}^3 \to \mathbb{R}$

Df is 1x3

$$Df = [f_x \ f_y \ f_z] = [2yz \quad 2xz - z^2 \quad 2xy - 2yz]$$

e) $\mathbf{f}(s, t) = \langle \underbrace{s^2 + t}_{x}, \underbrace{2s - t}_{y}, \underbrace{st}_{z} \rangle$

$F: \mathbb{R}^2 \to \mathbb{R}^3$

Df size 3x2

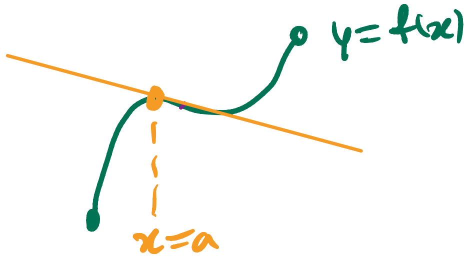
$$Df = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

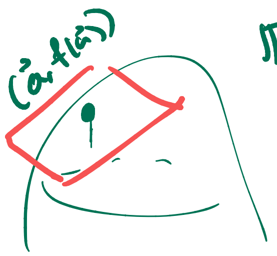
Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable:
$f(x) \approx f(a) + f'(a)(x - a) = L(x)$



Definition 55. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is



$\mathbb{R}^{m\times n}$

$$L(\mathbf{x}) = \underbrace{f(\vec a)}_{\text{vector}} + \underbrace{Df(\vec a)}_{\text{total derivative}}\underbrace{(\vec x - \vec a)}_{\text{vector}}$$

$L(\vec x) \approx f(\vec x)$ near $\vec x \approx \vec a$.

plug in $(1.1, 1.1)$ get ≈ 2.1

Example 56. Find the linearization of the function $\boxed{f(x, y) = \sqrt{5x - y}}$ at the point $\underbrace{(1, 1)}_{\text{nice point}}$. Use it to approximate $f(1.1, 1.1)$.

$f(1,1) = \sqrt{5-1} = \sqrt{4} = 2$

$Df = \left[\frac{5}{2\sqrt{5x-y}} \quad \frac{-1}{2\sqrt{5x-y}}\right]$ @ $\vec a(1,1)$ $Df(1,1) = \left[\frac{5}{2\sqrt4} \quad \frac{-1}{2\sqrt4}\right]$

$= \left[\frac{5}{4} \quad \frac{-1}{4}\right]$

plug into $L(\vec x)$ formula

$L(\vec x) = 2 + \left[\frac{5}{4} \quad -\frac{1}{4}\right]\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

@ $\underbrace{L(1.1, 1.1)}_{\text{ugly point}} = 2 + \left[\frac{5}{4} \quad -\frac{1}{4}\right]\left(\begin{bmatrix} 1.1 \\ 1.1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2 + [1.25 \quad -.25]\begin{bmatrix} .1 \\ .1 \end{bmatrix}$

Question: What do you notice about the equation of the linearization?

$= 2 + (0.125 - 0.025)$

$= 2 + .1 = \boxed{2.1}$

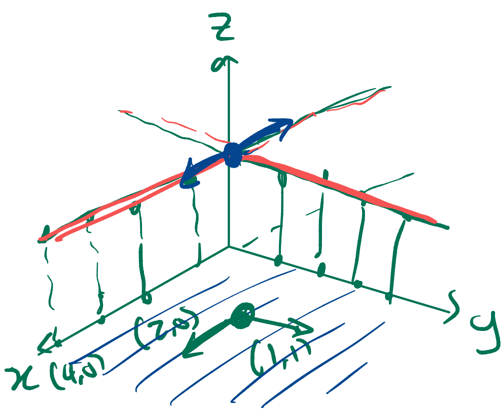
f_x & f_y can both be defined but still have some problem w/ f.

We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$.

$$\boxed{\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|} = 0.}$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) its graph has a unique tangent plane at $(a,b,f(a,b))$.

Example 57. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.



The partial derivatives of f wrt

D.

$f_x(0,0) = 0$

$f_y(0,0) = 0$

$f_x(1,1) = 0$

$f_y(1,1) = 0$

$Df = [f_x \;\; f_y]$
$= [0 \;\; 0]$

Problem is that

@(0,0) $L(\vec{x}) = f(0,0) + Df(0,0)(\vec{x} - a)$

$L(\vec{x}) \equiv 0$

§14.4 The Chain Rule

Recall the Chain Rule from single variable calculus:

$$\frac{d}{dx}(f(g(x)) = f'(g(x)) * g'(x)$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

compare

$$\boxed{\underbrace{D(f(g(\mathbf{x})))}_{\text{matrix}}^{m\times n} = \underbrace{Df(g(\mathbf{x}))}_{\text{matrix}}^{m\times p}\underbrace{Dg(\mathbf{x})}_{\text{matrix}}^{p\times n}.}$$

(matrix multiplication)

$f\circ g(\vec{x}) = f(g(\vec{x}))$

$f : \mathbb{R}^p \to \mathbb{R}^m$

$g : \mathbb{R}^n \to \mathbb{R}^p$

$f\circ g : \mathbb{R}^n \to \mathbb{R}^m$

$\circ$ \circ (composition symbol)

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

Example 58. Suppose we are walking on our hill with height $h(\cancel{x}^{t+1}, \cancel{y}^{2-t^2}) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2\rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

$$h(r(t)) = h(t+1, 2-t^2) \overset{?}{=} \underbrace{4 - \tfrac{1}{4}(t-1)^2 - \tfrac{1}{4}(2-t^2)^2}_{\text{simplify \& take } d/dt}$$

$$Dh(t) = Dh(r(t)) * Dr(t)$$

$$Dh = [h_x \ \ h_y] = [-\tfrac{1}{2}x \ \ -\tfrac{1}{2}y] \quad @ \ t=1$$

$$r(1) = (2, 1) = (x,y)$$

$$Dh(2,1) = [-1 \ \ -\tfrac{1}{2}]$$

$$Dr = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (t+1)' \\ (2-t^2)' \end{bmatrix} = \begin{bmatrix} 1 \\ -2t \end{bmatrix} \qquad Dr(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$Dh(1) = [-1 \ \ -\tfrac{1}{2}]\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1+1 = \boxed{0}$$

$g(s,t) = \langle u(s,t), v(s,t) \rangle$

Example 59. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$u(1,0) = 2$	$v(1,0) = 3$
$u_s(1,0) = -2$	$v_s(1,0) = 5$
$u_t(1,0) = 6$	$v_t(1,0) = 4$
$F_u(2,3) = -1$	$F_v(2,3) = 10$

$F : \mathbb{R}^2 \to \mathbb{R}^1$

$DF = [F_u \ F_v]$

1×2 matrix.

Idea:

$DW = [W_s \ W_t]$

$g : \mathbb{R}^2 \to \mathbb{R}^2$

$Dg = \begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix}$

2×2 matrix

$\boxed{\text{Find } W_s(1,0) \text{ and } W_t(1,0).}$

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

Sanity check

$W : \mathbb{R}^2 \to \mathbb{R}$ so DW is a 1×2 matrix.

$DW = DF * Dg$

@ $(s,t) = (1,0)$

then $u(1,0) = 2$
$v(1,0) = 3$

$$DF\Big|_{(u,v)=(2,3)} = \begin{bmatrix} -1 & 10 \end{bmatrix}$$

$$Dg\Big|_{(s,t)=(1,0)} = \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$

$$\text{So } DW = \underbrace{\begin{bmatrix} -1 & 10 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}}_{2\times 2} = \underbrace{\begin{bmatrix} 2+50 & -6+40 \end{bmatrix}}_{1\times 2}$$

$$= \begin{bmatrix} 52 & 34 \end{bmatrix}$$

$\uparrow$ $W_s(1,0)$ $\quad$ $\uparrow$ $W_t(1,0)$

$\boxed{W_s(1,0) = 52 \text{ and } W_t(1,0) = 34}$

if z is one of the variables for an implicitly defined surface

Application to Implicit Differentiation: If $\boxed{F(x, y, z) = c}$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

Idea: $F(x,y,z)=c$ then $DF = [F_x \; F_y \; F_z]$

Formula
$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

Example 60. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $\boxed{x^2 + y^2 + z^2 = 4.}$

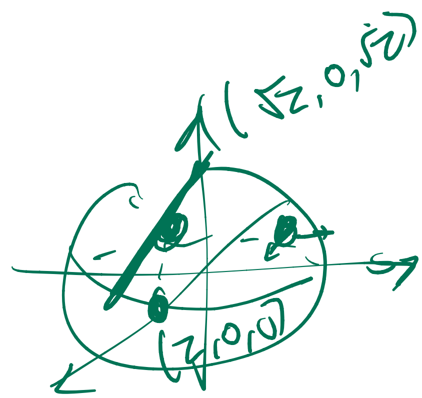
Compare top-half

$z = \sqrt{4-x^2-y^2} = f(x,y)$ (height z is function of x & y)

Step 1: Identify the function $F(x,y,z)$ s.t. the surface is a level set of F.

$F(x,y,z) = x^2+y^2+z^2$ so that surface level set at $c=4$.

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 2z$$



$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = \frac{-2x}{2z} = \frac{-x}{z} \\ \frac{\partial z}{\partial y} &= \frac{-F_y}{F_z} = \frac{-2y}{2z} = \frac{-y}{z} \end{aligned}}$$

Aside

$$\boxed{\underbrace{DF(g(\vec{x}))}_{} = DF * Dg}$$

Textbook. has all partial derivative info.

Gives a formula for each partial derivative

$h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \qquad r(t) = \langle t+1, 2-t^2 \rangle$

Then $\boxed{\dfrac{dh}{dt} = \dfrac{\partial h}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial h}{\partial y}\dfrac{dy}{dt}}$ another way to compute a chain rule.

EX. $h(x,y)$, $r(s,t) = \langle x,y \rangle$

$$\boxed{\begin{aligned} \frac{\partial h}{\partial s} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial h}{\partial t} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$

$$Dh = \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix}$$

$$Dr = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

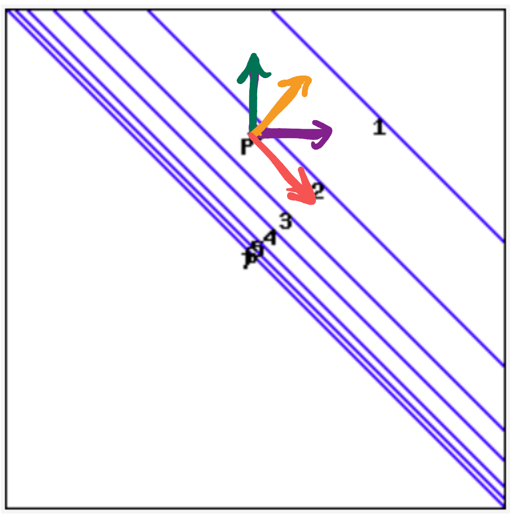
$$Dh(r(s,t)) = \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial h}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t} \end{bmatrix}$$

§14.5 Directional Derivatives & Gradient Vectors

Example 61. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?

Level sets for $z=k$.
Shows all (x,y) pairs s.t. $f(x,y)=k$



$f_x(P) < 0$

$f_y(P) < 0$

$f_{\langle 1,1\rangle}(P) < 0$ (more negative)

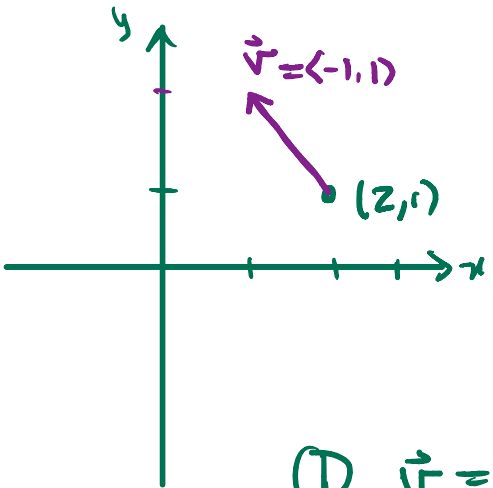
$f_{\langle 1,-1\rangle}(P) = 0$

In general $f_{\vec{u}}$ measures the change in height (output value) as we move in the $\vec{u}$-direction.

$D_{\vec{u}} f(P)$ the directional derivative of f in the direction of $\vec{u}$ at the point P.

Let's go back to our hill example again, $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2,1)$ if we move in the direction $\langle -1, 1\rangle$?

How do we compute in this case?



First what's the current height @ $(2,1)$

$$h(2,1) = 4 - \tfrac{1}{4}(2)^2 - \tfrac{1}{4}(1)^2 = 4 - 1 - \tfrac{1}{4} = 2.75$$

Idea: normalize $\vec{v}$ to a unit vector & do difference quotient.

① $\vec{v} = \langle -1, 1\rangle$ normalize to $\vec{u} = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$

② $$\lim_{t\to 0} \frac{h(\vec{p} + t\vec{u}) - h(\vec{p})}{t} = \lim_{t\to 0} \frac{h(\langle 2,1\rangle + t\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle) - h(\langle 2,1\rangle)}{t}$$

$$= \lim_{t\to 0} \frac{h(2 - t/\sqrt{2}, 1 + t/\sqrt{2}) - 2.75}{t} = \lim_{t\to 0} \frac{\cancel{t}/2\sqrt{2} - t^{\cancel{2}}/4}{\cancel{t}}$$

$$h(2 - t/\sqrt{2}, 1 + t/\sqrt{2}) - 2.75 = 4 - \tfrac{1}{4}(2 - t/\sqrt{2})^2 - \tfrac{1}{4}(1 + t/\sqrt{2})^2 - 2.75 = \frac{t}{2\sqrt{2}} - \frac{t^2}{4}$$

Definition 62. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{t\to 0} \frac{f(\vec{p} + t\vec{u}) - f(\vec{p})}{h}$$

$$= \lim_{t\to 0} \frac{1}{2\sqrt{2}} - \frac{t}{4} = \boxed{\frac{1}{2\sqrt{2}}}$$

if this limit exists. (Hard)

E.g. for our hill example above we have:

(easy)

$$D_u f(p) = [Df(p)]^T \cdot \vec{u}$$

Note that $D_{\mathbf{i}}f =$ $D_{\mathbf{j}}f =$ $D_{\mathbf{k}}f =$

Definition 63. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

$$\nabla f(\mathbf{p}) = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = (Df)^T$$

"gradient"

∇ \nabla.

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = (Df(p))^T \cdot \vec{u} = \nabla f(p) \cdot \vec{u}$$

Ex. $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ @ $(2,1)$ in direction $\vec{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$Dh = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}$ $Dh(2,1) = \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix}$

$\vec{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$

So $(Dh(2,1))^T \cdot \vec{u}$

$$= \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{2}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$$

Example 64. *You try it!* Find the gradient vector and the directional derivative of each function at the given point **p** in the direction of the given vector **u**.

a) $f(x, y) = \ln(x^2 + y^2)$, $\mathbf{p} = \langle -1, 1 \rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = \langle 1, 2, 1 \rangle$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Example 64. *You try it!* Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x,y) = \ln(x^2+y^2)$, $\mathbf{p} = \langle -1, 1 \rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 2x/(x^2+y^2) \\ 2y/(x^2+y^2) \end{bmatrix} \quad @ \ \vec{p} = \langle -1, 1 \rangle \quad \nabla f(\vec{p}) = \begin{bmatrix} -2/2 \\ 2/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and $$D_{\vec{u}} f(\vec{p}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = -\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \boxed{1/\sqrt{5}}$$

b) $g(x,y,z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = \langle 1, 2, 1 \rangle$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

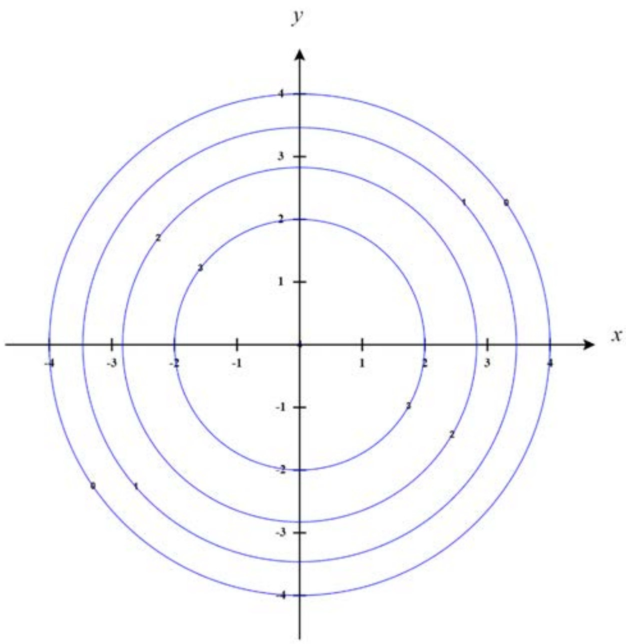
$$\nabla g = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} 2x+4y^2 \\ 8xy \\ 2z \end{bmatrix} \quad @ \ \vec{p} = \langle 1,2,1 \rangle \quad \nabla g(\vec{p}) = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix}$$

$$v = \langle 1, 2, -1 \rangle \Rightarrow u = \frac{1}{\|v\|} v = \langle 1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6} \rangle$$

So $$D_{\vec{u}} g(\vec{p}) = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}}(18+32-2)$$

$$= \boxed{\frac{48}{\sqrt{6}}}$$

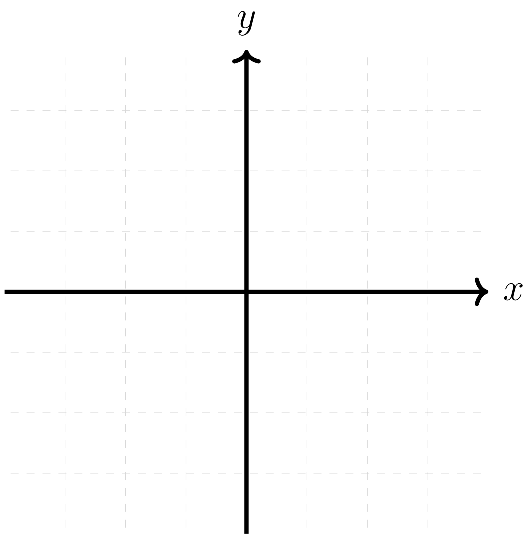
Example 65. If $h(x, y) = 4 - \dfrac{1}{4}x^2 - \dfrac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2, 0), (0, 4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2, 0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.



Note that the gradient vector ∇f is ________________ to the level curves of the function ____________.

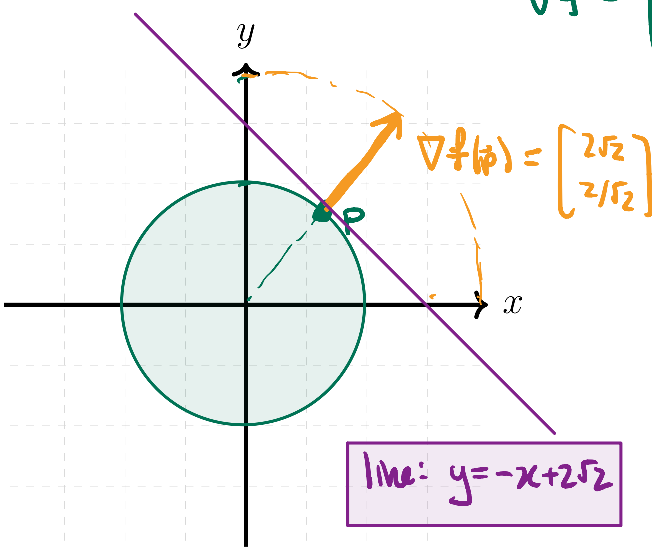
Similarly, for $f(x, y, z)$, $\nabla f(a, b, c)$ is ____

Example 66. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f\,|_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.



Example 66. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f \mid_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.

$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ @ $P(\sqrt{2},\sqrt{2})$ $\quad \nabla f(\vec{P}) = \begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$



$8 + 8 = 16 = 4^2$

$\langle -1, 1 \rangle$ orth. to $\nabla f|(\vec{p})$, so.

$\ell(t) = \langle \sqrt{2}, \sqrt{2} \rangle + t\langle 1, -1 \rangle$

(so $m = -1$)

$y = -x + b$ and passes thru $(\sqrt{2}, \sqrt{2})$

So $\sqrt{2} = -\sqrt{2} + b$

$\Rightarrow 2\sqrt{2} = b$

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what's the latex symbol \nabla

The LaTeX symbol `\nabla` renders as:

$$\nabla$$

It is called **"nabla"** or the **del operator**, and it is used primarily in vector calculus. It represents the vector differential operator:

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right]$$

Common Uses:

- **Gradient** of a scalar function f:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \right]$$

- **Divergence** of a vector field $\vec{F}$:

$$\nabla \cdot \vec{F}$$

- **Curl** of a vector field $\vec{F}$:

$$\nabla \times \vec{F}$$

- **Laplacian** of a scalar field f:

$$\nabla^2 f = \nabla \cdot \nabla f$$

So in summary, `\nabla` is a compact and powerful symbol in multivariable calculus, especially when working with fields and differential operators.

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