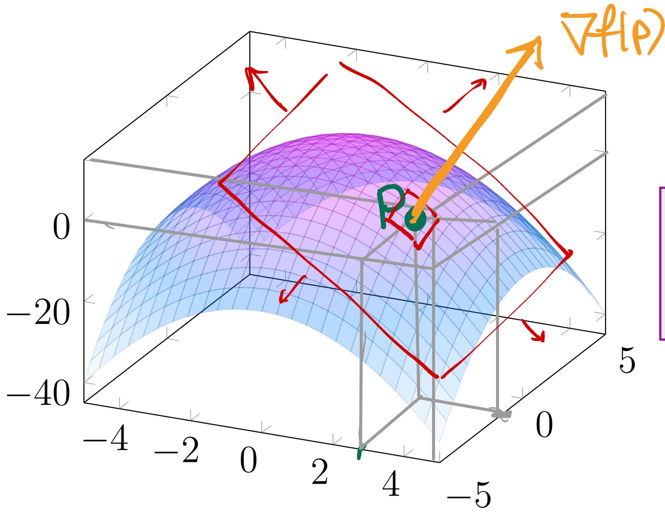
§14.6 Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

$\nabla f(p)$ is orthogonal to the level set $F(x_0, y_0, z_0) = k$.

So $\boxed{\vec{n} = \nabla f(p) \text{ normal vector}}$

$$\boxed{\begin{gathered} a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \\ \text{where } \vec{n} = \langle a,b,c \rangle. \end{gathered}}$$

plane EQN.

So ① identify F function which our surface is a level set of

② compute ∇F & plug in point P.

③ fill out the plane eqn

$F(x,y,z) = x^2+y^2+z$

∇F is $(DF(p))^T = \nabla F(p)$

$DF(p) = \begin{bmatrix} -2 & 6 & 1 \end{bmatrix}$

$$\nabla F = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} = \langle 2x, 2y, 1 \rangle \quad @ P(-1,3,0) \text{ get } \quad \nabla F(-1,3,0) = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$

So plane eqn for tangent plane is

$$\boxed{-2(x+1)+6(y-3)+1(z-0)=0}$$

Example 67. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Surface is graph of
$z = f(x,y) = 4 - x^2 - y$

① Identify F: $F(x,y,z) = 4 - x^2 - y - z = 0$

② Find $\vec{n}$: $\vec{n} = \nabla F(P)$ so $\nabla F = \begin{bmatrix} -2x \\ -1 \\ -1 \end{bmatrix}$ @ $(-2,1,-1)$

③ Plane eq

$\nabla F(-2,1,-1) = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

So plane eqn is.

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ $P(-2,1,-1)$, $\vec{n} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

get $\boxed{4(x+2) - (y-1) - (z+1) = 0}$

let's rearrange & solve for z.

$4(x+2) - (y-1) - z - 1 = 0$

$4x + 8 - y + 1 - z - 1 = 0$

$\Rightarrow \boxed{z = 4x - y + 8.}$ or $\boxed{z = 4(x+2) - (y-1) - 1}$

$f_x(-2,1)$ $f_y(-2,1)$ $f(-2,1)$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This should look familiar: it's the linearization

Example 68. *You try it!* Consider the surface in $\mathbb{R}^3$ containing the point P and defined by

$$x^2 + 2xy - y^2 + z^2 = 7, \; P(1, -1, 3).$$

Identity the function $F(x, y, z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

Example 68. *You try it!* Consider the surface in $\mathbb{R}^3$ containing the point P and defined by

$$x^2 + 2xy - y^2 + z^2 = 7, \; P(1, -1, 3).$$

Q: can we use (Chew-ization way? $f(x,y) = z$?

Identity the function $F(x, y, z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

(a)

① $F(x,y,z) = x^2 + 2xy - y^2 + z^2 = k$, $k = 7$

② $\nabla F = \begin{bmatrix} 2x + 2y \\ 2x - 2y \\ 2z \end{bmatrix}$ $\nabla F(1,-1,3) = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$

③ Plane eqn becomes

$$\boxed{0(x-1) + 4(y+1) + 6(z-3) = 0}$$

(b) $l(t) = \vec{OP} + t\vec{v}$, $t \in \mathbb{R}$

$\vec{v} = \nabla F(P) = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$ $\vec{OP} = \langle 1, -1, 3 \rangle$

So $\boxed{l(t) = \langle 1, -1, 3 \rangle + t\langle 0, 4, 6 \rangle, \; t \in \mathbb{R}}$

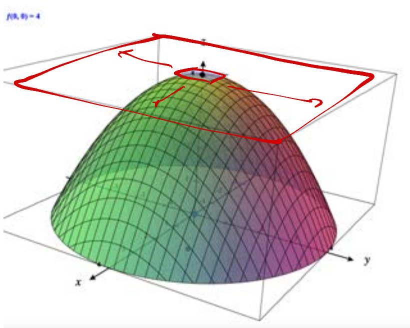
§14.7 Optimization: Local & Global

Gradient: If $f(x,y)$ is a function of two variables, we said $\nabla f(a,b)$ points in the direction of greatest change of f.

Back to the hill $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$.

What should we expect to get if we compute $\nabla h(0,0)$? Why? What does the tangent plane to $z = h(x,y)$ at $(0,0,4)$ look like?

know plane eqn. (linearization)



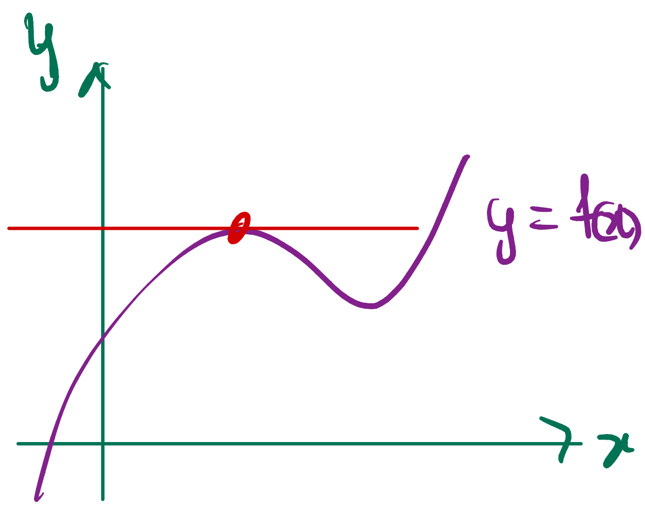
$$\boxed{z = h(0,0) + h_x(0,0)(x-0) + h_y(0,0)(y-0)}$$

$$\nabla h = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x \\ -\frac{1}{2}y \end{bmatrix} \text{ @ } (0,0) \quad \nabla h(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

simplify

$$z = 4 + 0(x) + 0(y) = 4$$

$\boxed{z = 4}$ horizontal plane (parallel to floor).



horizontal tangent line.

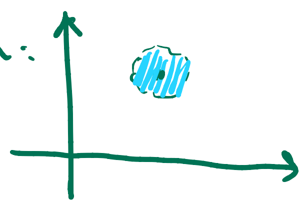
$\Longrightarrow$ location of local max/min.

Definition 68. Let $f(x,y)$ be defined on a region containing the point (a,b). We say

- $f(a,b)$ is a <u>local maximum</u> value of f if $f(a,b)$ <u>$\geq$</u> $f(x,y)$ for all domain points (x,y) in a disk centered at (a,b)
- $f(a,b)$ is a <u>local minimum</u> value of f if $f(a,b)$ <u>$\leq$</u> $f(x,y)$ for all domain points (x,y) in a disk centered at (a,b)

Graph:

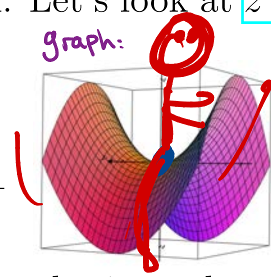
Domain:



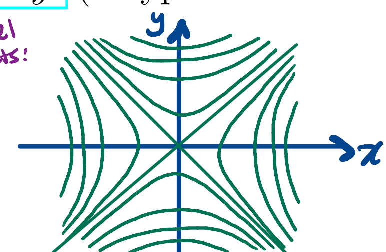
local min.

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $\boxed{z = \underbrace{x^2 - y^2}_{f(x,y)}}$ (a hyperbolic paraboloid!) near $(0,0)$.

graph:



level sets!



This is called a <u>Saddle point</u>

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

$\nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$ @ $(0,0)$ $\nabla f(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

One more nuance.

$f(x,y) = \sqrt{x^2+y^2} = z$

Cone

$Df = \left[\frac{x}{\sqrt{x^2+y^2}} \;\; \frac{y}{\sqrt{x^2+y^2}}\right]$ @ $(0,0)$ $Df(0,0) = [0 \;\; 0]$

Definition 69. If $f(x,y)$ is a function of two variables, a point (a,b) in the domain of f with $Df(a,b) =$ <u>$[0 \;\; 0]$</u> or where $Df(a,b)$ <u>is DNE</u> is called a <u>critical point</u> of f.

Example 70. Find the critical points of the function

$$f(x,y) = x^3 + y^3 - 3xy.$$

Idea:

Solve $\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for x & y.

alt notation

$Df = [3x^2-3y \;\; 3y^2-3x] = [0 \;\; 0]$

$\boxed{\text{ad hoc}}$

step ① $\nabla f = \begin{bmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Step ② Solve $\begin{cases} (1) & 3x^2 - 3y = 0 \\ (2) & 3y^2 - 3x = 0 \end{cases}$

so (1) becomes $3x^2 = 3y \Rightarrow \boxed{y = x^2 \;(1)}$

sub into (2) and get $3(x^2)^2 - 3x = 0$

now $3x^4 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0$

$\Rightarrow$ So $3x = 0$ or $x^3 - 1 = 0$

$\Rightarrow x = 0$ or $x = 1$.

plug back into (1) to figure out y.

$y = 0$ at $x = 0$, $y = 1$ at $x = 1$.

two crit pts. $\boxed{(0,0),\ (1,1)}$

$\boxed{\text{why not } (0,1)\text{?}}$

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0,0)$.

a) $f(x,y) = 3x + y^3 + 2y^2$

b) $g(x,y) = \cos(x) + \sin(x)$

c) $h(x,y) = \dfrac{4}{x^2 + y^2}$

d) $k(x,y) = x^2 + y^2$

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0,0)$.

a) $f(x,y) = 3x + y^3 + 2y^2$

$Df = [3 \quad 3y^2 + 4y]$

NO CRIT POINTS since $Df \neq [0 \quad 0]$ for any $(x,y) \in \mathbb{R}^2$, so $\boxed{\text{NO}}$

b) $g(x,y) = \cos(x) + \sin(x)$

$Dg = [g_x \quad g_y]$

$Dg = [-\sin x + \cos x \quad 0] \stackrel{?}{=} [0 \quad 0]$

need to check $-\sin(0) + \cos(0) \stackrel{?}{=} 0$?

$0 + 1 \neq 0$.

So $(0,0)$ is NOT crit pt. $\boxed{\text{NO}}$

Note: if $x = \frac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$, $y \in \mathbb{R}$.
but $Df(0,0) \neq [0 \quad 0]$

c) $h(x,y) = \dfrac{4}{x^2 + y^2}$

$Dh = \left[\dfrac{-8x}{(x^2+y^2)^2} \quad \dfrac{-8y}{(x^2+y^2)^2}\right]$ is <u>DNE</u> @ $(0,0)$

<u>BUT</u> $(0,0)$ not in the DOMAIN OF h!! So $\boxed{\text{NO}}$

Even though $Dh(0,0)$ is DNE, we Don't call $(0,0)$ a crit point of h. Why not? b/c $h(0,0)$ is DNE.

d) $k(x,y) = x^2 + y^2$

$Dk = [2x \quad 2y]$ and

$Dk(0,0) = [0 \quad 0]$ so $\boxed{\text{yes}}$

Recall from calc 1.

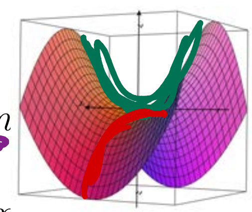
MAX $f'(a)=0$, $f''(a)<0$

MIN $f'(a)=0$, $f''(a)>0$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x,y)$ at (a,b) is

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix} = D^2 f(a,b)$$

Theorem 72 (2nd Derivative Test). *Suppose (a,b) is a critical point of $f(x,y)$ and f has continuous second partial derivatives. Then we have:*



- *If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) > 0$, $f(a,b)$ is a local minimum*
- *If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) < 0$, $f(a,b)$ is a local maximum*
- *If $\det(Hf(a,b)) < 0$, f has a saddle point at (a,b)*
- *If $\det(Hf(a,b)) = 0$, the test is inconclusive.*

Ex. $f(x,y) = x^2 - y^2$

$Df = [2x \;\; -2y]$

$D^2 f = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

A is 2x2. $\det A = \lambda_1 \lambda_2$ product of eigenvalues

Example 73. Classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$ from Example 70.

Crit points are $(0,0)$, and $(1,1)$

lets classify them using $Hf = D^2 f$ the Hessian matrix.

$$Df = \begin{bmatrix} f_x & f_y \end{bmatrix} \quad \text{and} \quad Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

$$= \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix} \qquad = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

Now plug in crit points one at a time to classify them.

$$@(0,0) \quad Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \quad \det Hf(0,0) = 0 - 9 = -9 < 0$$

So $\boxed{(0,0) \text{ is a SADDLE pt.}}$

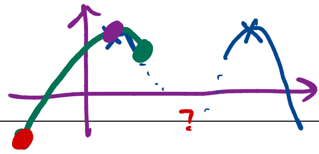
$$@(1,1) \quad Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \quad \det Hf(1,1) = 36 - 9$$

$$= 27 > 0$$

So $(1,1)$ is either MAX/MIN & need to look at f_{xx} or f_{yy} to decide.

$f_{xx}(1,1) = 6 > 0$ (concavity positive) $\Rightarrow$ $\boxed{(1,1) \text{ is a MIN}}$

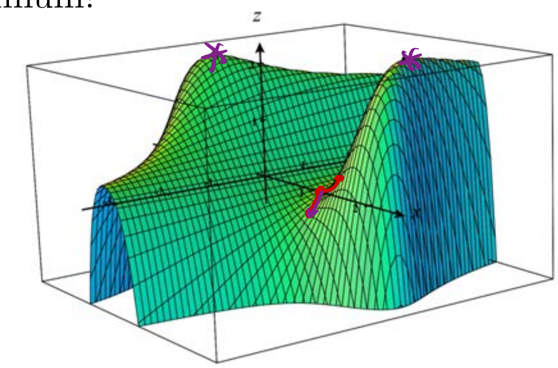
Recall:



Plausibility: between any two local maxs there must be a local min.

Two Local Maxima, No Local Minimum: The function $g(x, y) = -(x^2 - 1)^2 - (x^2 y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!

In one variable calculus, the situation (⟶) can't happen.



A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 74. *On a $\boxed{\textit{closed \& bounded}}$ domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

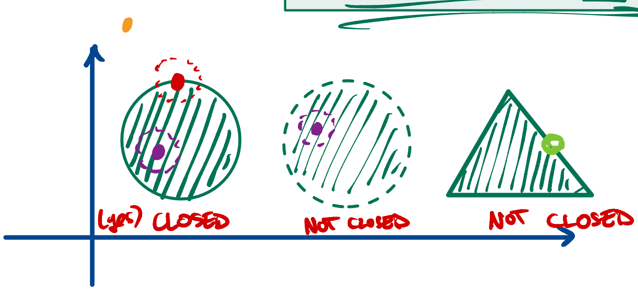
A region in $\mathbb{R}^2$ is closed if it

Closed: contains its boundary points

14.1 Functions of Several Variables **795**

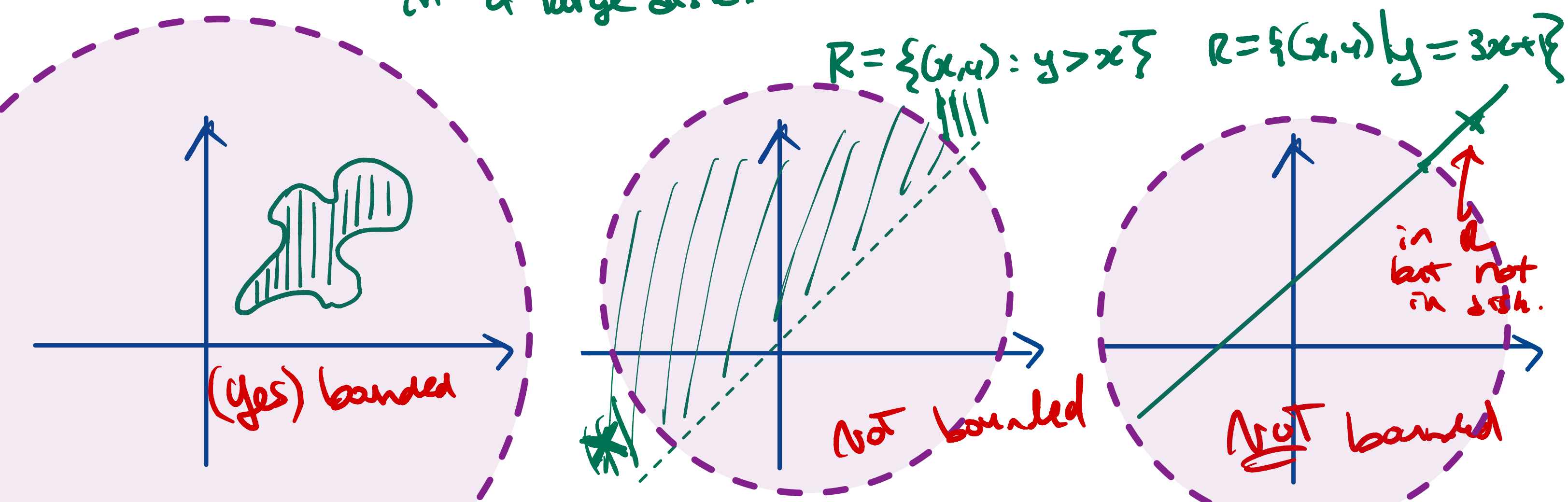
DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Bounded: set is contained in a large disk.



Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

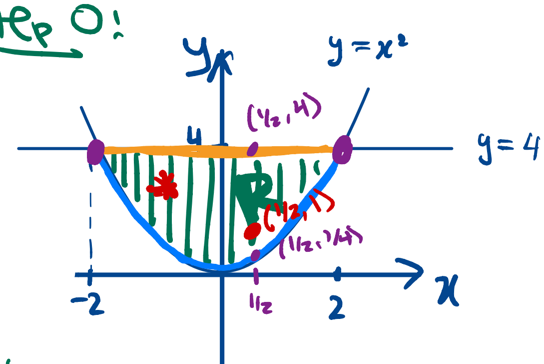
✓ continuous on $\mathbb{R}^2$

Example 75. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

(*)

Step 1: find crit points in the interior of R.

Step 0!



Solve.

$$Df = [f_x \;\; f_y] = [8x - 4y \;\; -4x + 2] \overset{\downarrow}{=} [0 \;\; 0]$$

ie solve $\begin{cases} 8x - 4y = 0 \\ -4x + 2 = 0 \end{cases}$ From 2nd eqn. $-4x + 2 = 0 \Rightarrow x = 1/2$.

Sub into 1st eqn.

$8(\frac{1}{2}) - 4y = 0 \Rightarrow 4 - 4y = 0$ so $y = 1$.

Only one crit pt @ $(1/2, 1)$. [Should we include $(1/2, 1)$? is it in R??]

Step 2: Find crit pts on bdry.

@ $y = 4$ sub into $f(x,4)$ @ $y = 4$ $f(x, 4) = 4x^2 - 4x(4) + 2(4)$

$= 4x^2 - 16x + 8.$

on $x \in [-2, 2]$

@ $y = x^2$ $f(x, x^2) = 4x^2 - 4x\,x^2 + 2x^2$

$= 6x^2 - 4x^3$

on $x \in [-2, 2]$

@ two endpts $(2, 4), (-2, 4)$

Example 76. Find the global minimum and maximum of $\boxed{f(x,y) = 4x^2 - 4xy + 2y}$ on the closed region R bounded by $y = x^2$ and $y = 4$.

(Cont.)

Step 2: Find crit pts on bdry.

@ $y=4$ sub into $f(x,y)$ @ $y=4$

$f(x,4) = 4x^2 - 4x(4) + 2(4) = 4x^2 - 16x + 8$ on $x \in [-2,2]$

$\frac{d}{dx} f(x,4) = 8x - 16 = 0 \Rightarrow x = 2$. So crit point is @ $(2,4)$

@ $y = x^2$ $f(x,x^2) = 4x^2 - 4x\,x^2 + 2x^2 = 6x^2 - 4x^3$ on $x \in [-2,2]$

$\frac{d}{dx} f(x,x^2) = 12x - 12x^2 = 0 \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

So $x = 1$ or $x = 0$ (& $y = x^2$) so crit pts are $(0,0)$ & $(1,1)$

@ two endpts $(2,4)$, $(-2,4)$

Step 3: Eval

(x,y)	$f(x,y)$
$(1/2, 1)$	$f(1/2,1) = 4(\frac{1}{2})^2 - 4(\frac{1}{2})(1) + 2(1) = 1$
$(2,4)$	-8 MIN value @ $(2,4)$
$(0,0)$	0
$(1,1)$	2
$(-2,4)$	56 MAX value @ $(-2,4)$
$(2,4)$	-8