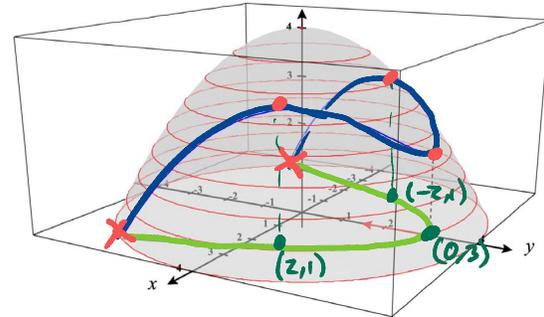


§14.8 Constrained Optimization, Lagrange Multipliers

Goal: Maximize or minimize $f(x, y)$ or $f(x, y, z)$ subject to a *constraint*, $g(x, y) = c$.

Example 77. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy -plane. What is the highest point on the hill on this path?



Objective function: what we want to max/minimize

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

Constraint equation:

restriction on the inputs into the objective function.

$$y = -0.5x^2 + 3$$

$$g(x, y) = y + 0.5x^2 = 3$$

constraint eqn written as a level set.

IDEA Solve $\begin{cases} \nabla h = \lambda \nabla g \\ g(x, y) = k \end{cases}$
system of eqns.

$$\nabla h = \begin{bmatrix} -1/2 x \\ -1/2 y \end{bmatrix} \quad \nabla g = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Solve the Lagrange eqns

$$\begin{bmatrix} -1/2 x \\ -1/2 y \end{bmatrix} = \lambda \begin{bmatrix} x \\ 1 \end{bmatrix} \Rightarrow \begin{cases} \textcircled{1} -\frac{1}{2}x = \lambda x \\ \textcircled{2} -\frac{1}{2}y = \lambda \\ \textcircled{3} y + \frac{1}{2}x^2 = 3 \end{cases}$$

Note $\textcircled{1}$ is true if $x=0$. then $\textcircled{3}$ becomes $y=3$.

So $x=0, y=3$ solves $\textcircled{1}, \textcircled{3}$ (set $\lambda = -3/2$ $\textcircled{2}$ ✓)

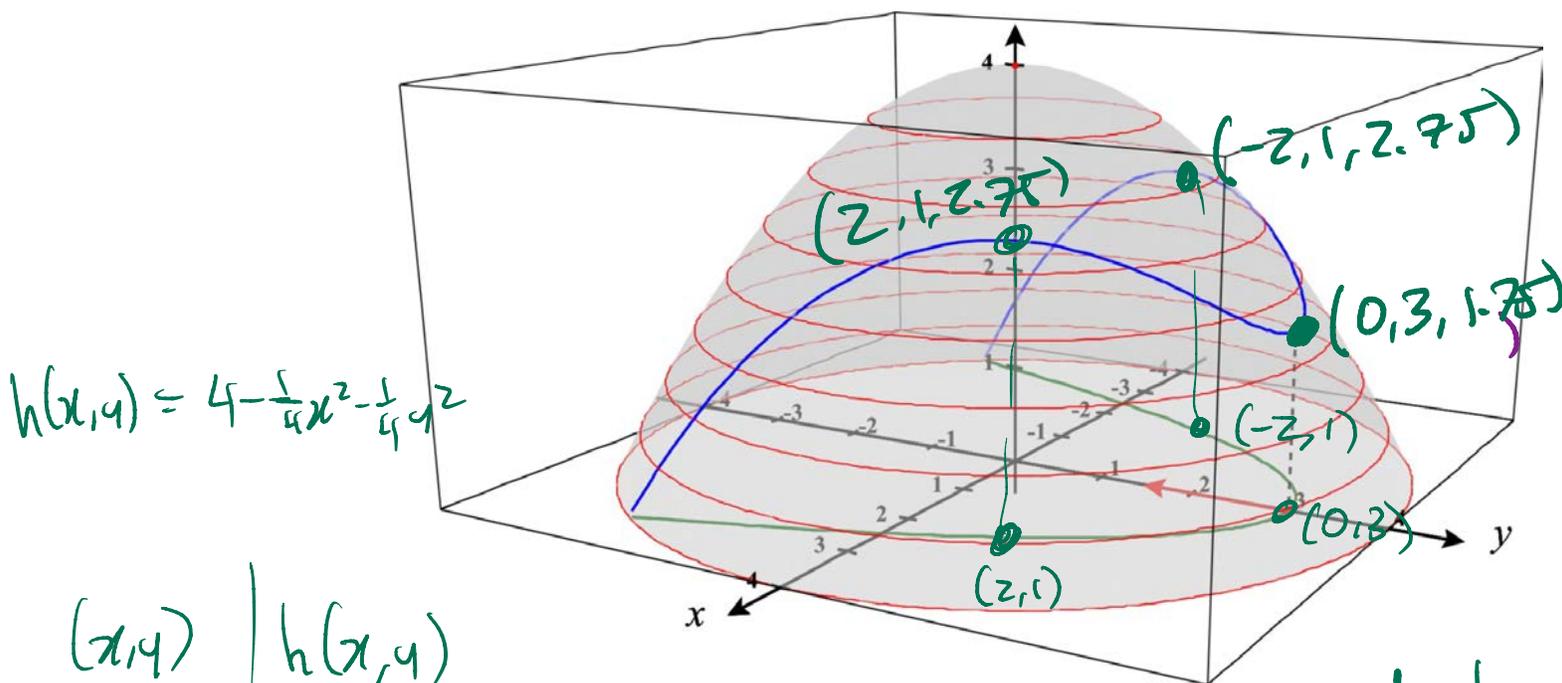
$(x, y) \neq (0, 3)$ works for $\textcircled{1}, \textcircled{2}, \textcircled{3}$.

If $x \neq 0$. $\textcircled{1} -\frac{1}{2}x - \lambda x = 0 \Rightarrow x(-\frac{1}{2} - \lambda) = 0 \Rightarrow \lambda = -1/2$

then $\textcircled{2} -\frac{1}{2}y = -\frac{1}{2} \Rightarrow y=1$, plug $y=1$ into $\textcircled{3}$ get $1 + \frac{1}{2}x^2 = 3 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
set pts. $(2, 1), (-2, 1)$

Example 77. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy -plane. What is the highest point on the hill on this path?

(Cont.)



$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

(x, y)

$h(x, y)$

$(2, 1)$

$$4 - \frac{1}{4}(2)^2 - \frac{1}{4}(1)^2 = 2.75$$

$(-2, 1)$

$$4 - \frac{1}{4}(-2)^2 - \frac{1}{4}(1)^2 = 2.75$$

$(0, 3)$

$$4 - \frac{1}{4}(0)^2 - \frac{1}{4}(3)^2 = 4 - \frac{9}{4} = 1.75$$

MAX value of h
occurs @ $(2, 1)$
and $(-2, 1)$

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

don't remember seeing this on sample exams
 If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Objective function: distance function

$$d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Constraint: be on the surface

$$g(x, y, z) = xy + 4 - z^2 = 0$$

$$g(x, y, z) = xy + 4 - z^2 = 0$$

defines the surface as a level set of g .



IDEA FROM CALC 1
 minimize instead

$$d^2 = f(x, y, z) = x^2 + y^2 + z^2$$

(no square root, easier w/ same answer)
 use this objective function instead.

Lagrange eqns. $\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y, z) = c \end{cases}$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \quad \nabla g = \begin{bmatrix} y \\ x \\ -2z \end{bmatrix}$$

So we set L-G eqns

Solve $\begin{cases} ① \ 2x = \lambda y \\ ② \ 2y = \lambda x \\ ③ \ 2z = -2\lambda z \\ ④ \ xy + 4 - z^2 = 0 \end{cases}$

Case $z=0$.

$$\begin{cases} ① \ 2x = \lambda y \\ ② \ 2y = \lambda x \\ ③ \ xy + 4 = 0 \end{cases}$$

For ③ to be true $x \neq 0$ and $y \neq 0$.

$$\begin{cases} ① \ \frac{2x}{y} = \lambda \\ ② \ \frac{2y}{x} = \lambda \end{cases}$$

$$\Rightarrow \frac{2x}{y} = \frac{2y}{x} \Rightarrow 2x^2 = 2y^2$$

$$\begin{cases} ③ \ xy = -4 \\ \text{if } x=y \text{ is impossible} \\ \text{so } x = -y \text{ and } x^2 = 4 \Rightarrow x = \pm 2 \end{cases}$$

$$(2, -2, 0), (-2, 2, 0)$$

So $x = \pm y$.

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

(Cont.)

Case $d = -1$

L-G eqns become

Solve

- ① $2x = \lambda y$
- ② $2y = \lambda x$
- ③ $2z = -2\lambda z$
- ④ $xy + 4 - z^2 = 0$

- ① $2x = -y$
- ② $2y = -x$
- ③ $2z = 2z$
- ④ $xy + 4 - z^2 = 0$

① becomes $y = -2x$

plug into ② $z(-2x) = -x$

$\Rightarrow -4x + x = 0 \Rightarrow -3x = 0$

so $x = 0$.

Sub into ① get $y = 0$.

Sub into ④ $0 + 4 - z^2 = 0$

$\Rightarrow z^2 = 4$

$\Rightarrow z = \pm 2$

get $(0, 0, 2), (0, 0, -2)$

(x, y, z)	$f(x, y, z) = x^2 + y^2 + z^2$
$(2, -2, 0)$	$4 + 4 = 8$
$(-2, 2, 0)$	$4 + 4 = 8$
$(0, 0, 2)$	4
$(0, 0, -2)$	4

↑ these two points are closest to origin w/ distance equal to 2.

$$d = \sqrt{x^2 + y^2 + z^2} \quad \nabla d = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$dx = \frac{2x}{z\sqrt{x^2 + y^2 + z^2}}$$

Example 79. *You try it!* Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy -plane that are nearest to and farthest from the origin.

Example 79. *You try it!* Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy -plane that are nearest to and farthest from the origin.

Set up: $\nabla f = \lambda \nabla g$ $d(x,y) = \sqrt{x^2+y^2}$ but do instead $f(x,y) = x^2+y^2$
 $\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x+y \\ 2y+x \end{bmatrix}$ $g(x,y) = x^2+xy+y^2$

So Lagrange equations to solve are

- ① $2x = \lambda(2x+y)$
- ② $2y = \lambda(2y+x)$
- ③ $x^2 + xy + y^2 = 1$

Case 1: If $x=0$ then ③ $y^2=1 \Rightarrow y = \pm 1$.
 get $(0, 1)$ & $(0, -1)$.

Case 2: If $y=0$ Then ③ $x^2=1 \Rightarrow x = \pm 1$ get $(1, 0), (-1, 0)$.

Case 3: $x \neq 0$ and $y \neq 0$. Then ① & ② become $\lambda = \frac{2x}{2x+y} = \frac{2y}{2y+x}$

$$\Rightarrow 2x(2y+x) = 2y(2x+y)$$

$$\Rightarrow 4xy + 2x^2 = 4xy + 2y^2$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow \overset{(a)}{x=y} \text{ or } \overset{(b)}{x=-y}$$

(x, y)	$f(x, y)$
$(0, 1)$	1
$(0, -1)$	1
$(-1, 0)$	1
$(1, 0)$	1
$(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$	$\frac{2}{3}$
$(1, -1)$	2
$(-1, 1)$	2

In case $y=x$ then

$$\textcircled{3} \quad x^2 + x^2 + x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{3}$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

get $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$

In case $y=-x$ then

$$\textcircled{3} \quad x^2 - x^2 + x^2 = 1$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

get $(1, -1)$ and $(-1, 1)$

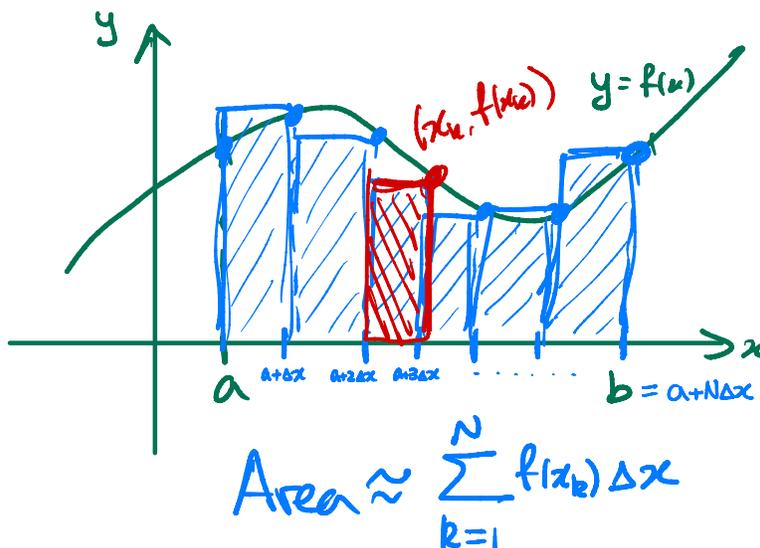
So MIN value $\sqrt{\frac{2}{3}}$
 @ $(1, -1)$ & $(-1, 1)$
 MAX value $\frac{2}{3}$
 @ $(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$

§15.1 Double Integrals, Iterated Integrals, Change of Order

Recall: Riemann sum and the definite integral from single-variable calculus.

$$\text{Area} = \int_a^b f(x) \, dx = \lim_{N \rightarrow \infty} \sum_{k=1}^N \underbrace{f(x_k) \Delta x}_{\text{differential.}}$$

where $x_k = a + \Delta x k$
 $\Delta x = \frac{b-a}{N}$



Main idea:

in the limit as $N \rightarrow \infty$
 you get the exact area under
 the curve.

(as you have more rectangles the width of each rectangle
 went to zero in the limit)

Also, same area w/ other choices of how to
 pick height so long as $x_k^* \in [a+k\Delta x, a+(k+1)\Delta x]$

- such as :
- * Right endpoint
 - * left endpoint
 - * midpoint
 - even other → * average height (trapezoidal rule)
 - ways eg. → * Simpson's rule (more complicated) etc. etc.

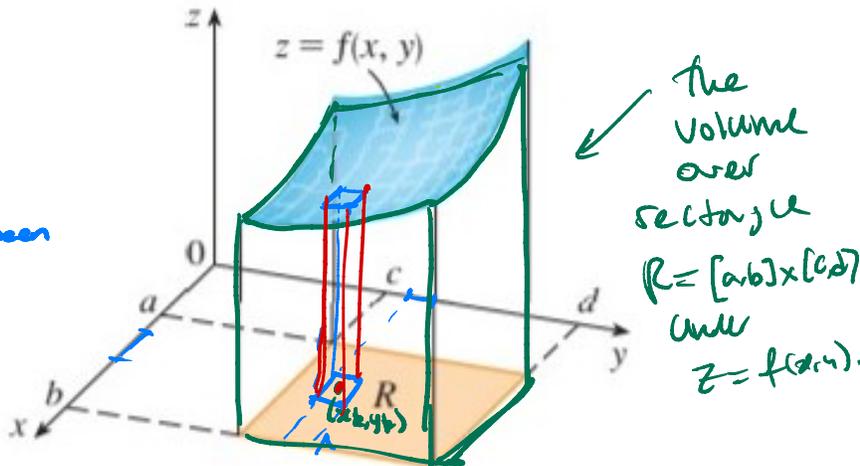
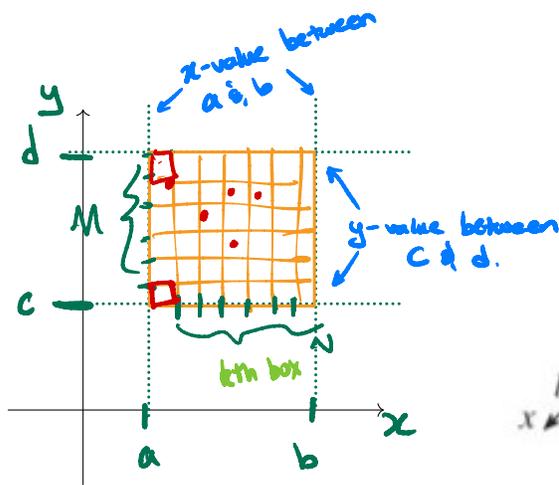
Double Integrals

$\int_a^b f(x) dx$ the area over interval $[a, b]$ below $y = f(x)$

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x, y)$ be a function defined on R such that $f(x, y) \geq 0$. Let S be the solid that lies above R and under the graph f .



add up these $f(x_i, y_j) * \Delta x \Delta y$

Question: How can we estimate the volume of S ?

$$\text{Volume} \approx \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) \Delta y \Delta x$$

Volume =

Definition 79. The _____ of $f(x, y)$ over a rectangle R is

$$\iint_R \overbrace{f(x, y)}^{\text{integrand}} dA = \lim_{|P| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

if this limit exists.

region
of integration

P is the "mesh size", largest size among all
of the bases of the tall boxes.

•

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Question: How can we compute a double integral?

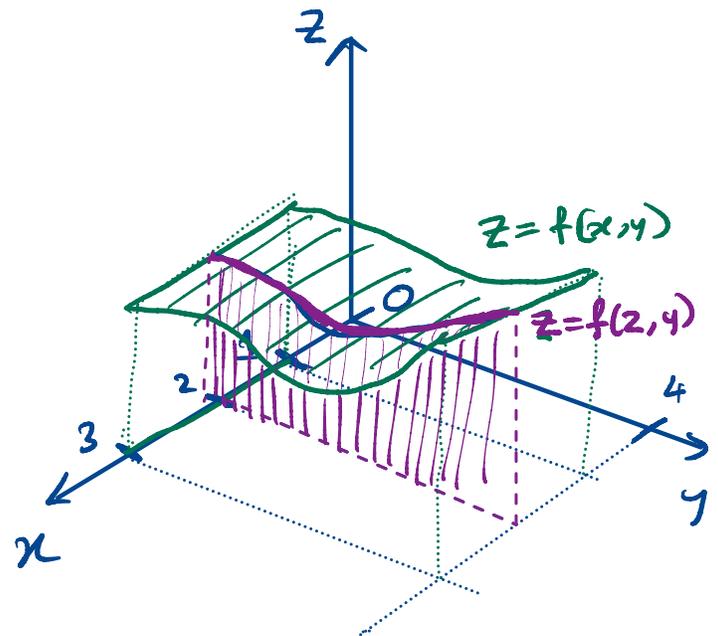
Answer:

Let $f(x, y) = 2xy$ and let's integrate over the rectangle $R = [1, 3] \times [0, 4]$.

We want to compute $\int_1^3 \int_0^4 f(x, y) \, dy \, dx$, but let's consider the slice at $x = 2$.

What does $\int_0^4 f(2, y) \, dy$ represent here?

Purple slice has area



Vol =

In general, if $f(x, y)$ is integrable over $R = [a, b] \times [c, d]$, then $\int_c^d f(x, y) dy$ represents:

What about $\int_c^d f(x, y) dy$?

Let $A(x) = \int_c^d f(x, y) dy$. Then,

$$\text{Volume} = \int_a^b A(x) dx =$$

This is called an _____.

Example 80. Evaluate $\int_1^2 \int_3^4 6x^2y dy dx$.

do inside first

Theorem 81 (Fubini's Theorem). *If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then*

"order doesn't matter"

More generally, this is true if we assume that f is bounded on R , f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 82. *You try it!* Integrate:

a) $\int_0^2 \int_{-1}^1 x - y \, dy \, dx$ **easy**

b) $\int_0^1 \int_0^1 \frac{y}{1 + xy} \, dx \, dy$ **medium**

c) $\int_1^4 \int_1^e \frac{\ln x}{xy} \, dx \, dy$ **HARD!**

Example 82. *You try it!* Integrate:

a) $\int_0^2 \int_{-1}^1 x - y \, dy \, dx$ **easy**

$$= \int_0^2 \left. xy - \frac{1}{2}y^2 \right|_{-1}^1 dx$$

$$= \int_0^2 x - (-x) - 0 \, dx$$

$$= \int_0^2 2x \, dx = \left. x^2 \right|_0^2 = \boxed{4}$$

b) $\int_0^1 \int_0^1 \frac{y}{1+xy} \, dx \, dy$ **medium**

$$\int_0^1 \int_0^1 \frac{y}{1+xy} dx dy = \int_0^1 \ln(1+xy) \Big|_0^1 dy = \int_0^1 \ln(1+y) - y \ln(1) dy$$

u-sub

$$\begin{aligned} u &= 1+xy \\ du &= y \, dx \end{aligned}$$

IBP $\int u \, dv = uv - \int v \, du$

$$\begin{aligned} u &= \ln(t) & dv &= 1 \, dt \\ du &= \frac{1}{t} dt & v &= t \end{aligned}$$

$$\begin{aligned} \text{So } \int \ln t \, dt &= t \ln t - \int dt \\ &= t \ln t - t + C \end{aligned}$$

$$= (1+y) \ln(1+y) - (1+y) \Big|_0^1$$

$$= [2 \ln 2 - 2] - [1 \ln(1) - 1] = \boxed{2 \ln 2 - 1}$$

c) $\int_1^4 \int_1^e \frac{\ln x}{xy} \, dx \, dy$ **HARD!**

$$= \int_1^4 \left[\frac{1}{2} (\ln x)^2 + \frac{1}{y} \right] \Big|_1^e dy = \int_1^4 \left[\frac{1}{2} (\ln e)^2 - \frac{1}{2} (\ln 1)^2 \right] + \frac{1}{y} dy$$

$$= \int_1^4 \frac{1}{2} + \frac{1}{y} dy = \frac{1}{2} \ln(y) \Big|_1^4$$

$$= \frac{1}{2} \ln 4 - \frac{1}{2} \ln 1 = \boxed{\ln 2}$$

IBP $\int u \, dv = uv - \int v \, du$

$$\begin{aligned} u &= \ln x & dv &= \frac{1}{x} dx \\ du &= \frac{1}{x} dx & v &= \ln x \end{aligned}$$

$$a \ln b = \ln(b^a)$$

$$\begin{aligned} \text{so } \frac{1}{2} \ln 4 &= \ln(\sqrt{4}) \\ &= \ln 2 \end{aligned}$$

$$\begin{aligned} * &= \int \frac{\ln x}{x} dx = (\ln x)^2 - \int \ln x \cdot \frac{1}{x} dx \\ \text{So } 2* &= (\ln x)^2 + C \\ * &= \frac{1}{2} (\ln x)^2 + C \checkmark \end{aligned}$$

Example 83. Compute $\iint_R x e^{e^y} dA$, where R is the rectangle $[-1, 1] \times [0, 4]$.

Hint: Fubini's Theorem.

IDEA? $\int_{-1}^1 \int_0^4 x e^{e^y} dy dx =$

Better
Idea?