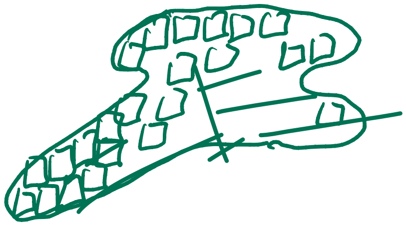
# §15.2 Double Integrals on General Regions

R maybe isn't a rectangle.

**Question:** What if the region $R$ we wish to integrate over is not a rectangle?



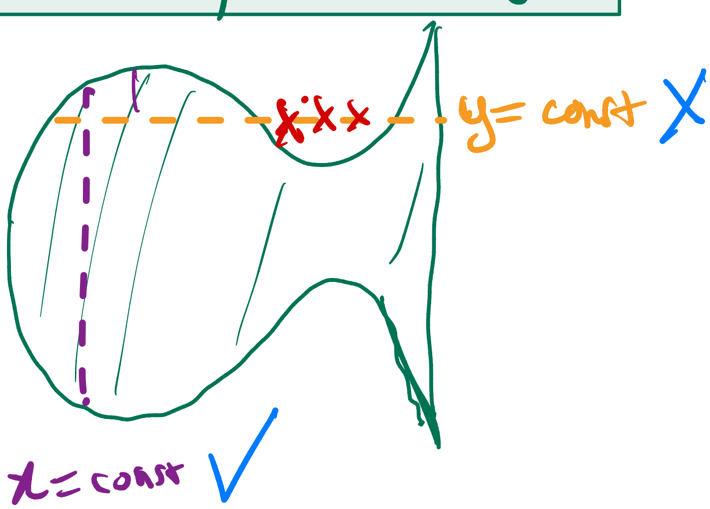
$\iint_R f(x,y)\, dA$

Same idea: use rectangles & lots of to approximate the volume Vol $= \int\dots$

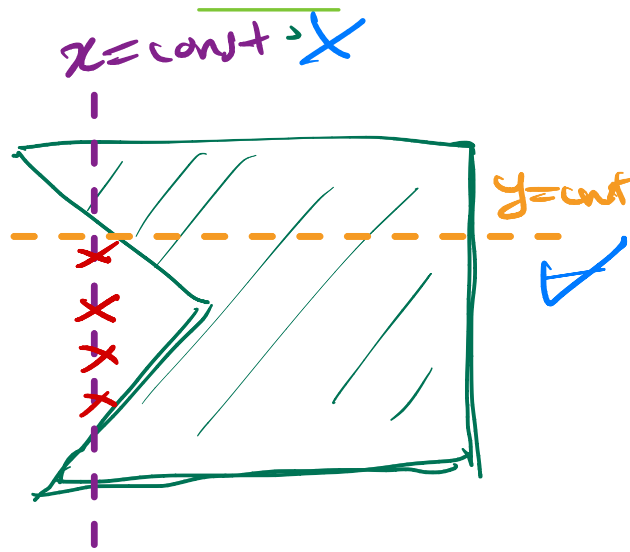
**Answer:** Repeat same procedure - it will work if the boundary of $R$ is smooth and $f$ is continuous.

IDEA when choosing dxdy vs. dydx it helps to look at R the region.

vertically simple region



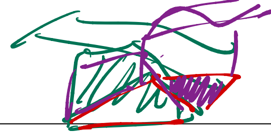

vs.




$\iint f(x,y)\, dA$

$dA = dy\,dx$ inner integral w/ y (x const)

$dA = dx\,dy$ inner integral w/ x (y - const)



**Example 84.** Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the $xy$-plane and whose top is $z=2-x-y$.

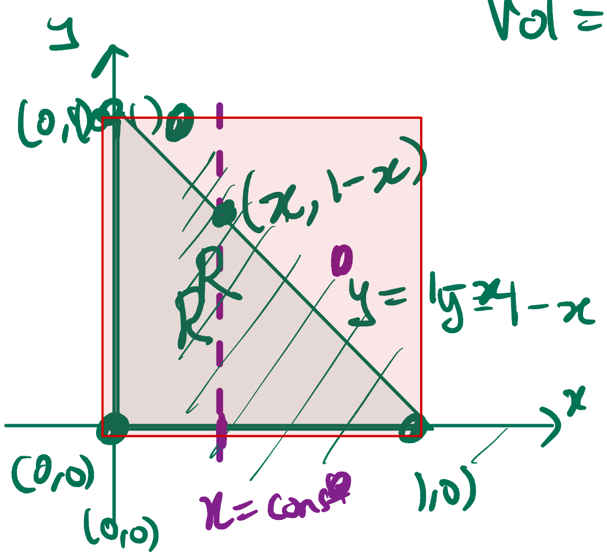
y-values range from 0 to $1-x$

$f(x,y)=2-x-y.$

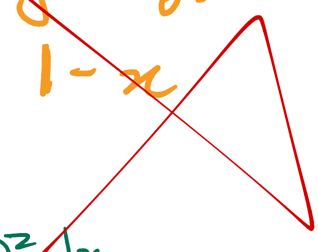
**Vertically simple:**

$$\text{Vol}=\int_0^1\int_0^{1-x} 2-x-y \, dy \, dx$$

← $x$ treated as constant in inner integral.




<u>In words</u>: as $x$ ranges from 0 to 1, then $y$ ranges from 0 to $1-x$



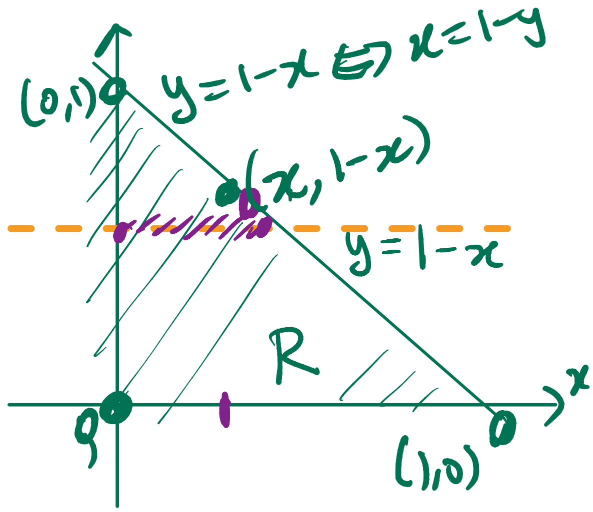
$$\text{Vol}=\int_0^1 2y-xy-\tfrac{1}{2}y^2\Big|_0^{1-x} dx=\int_0^1 2(1-x)-x(1-x)-\tfrac{1}{2}(1-x)^2\,dx$$

$$=\int_0^1 2-2x-x+x^2-\tfrac{1}{2}(1-2x+x^2)\,dx=\int_0^1 2-3x+x^2-\tfrac{1}{2}+x-\tfrac{1}{2}x^2\,dx$$

$$=\int_0^1 \tfrac{1}{2}x^2-2x+\tfrac{3}{2}\,dx=\tfrac{1}{6}x^3-x^2+\tfrac{3}{2}x\Big|_0^1$$

$$=\left(\tfrac{1}{6}-1+\tfrac{3}{2}\right)-(0)=\boxed{2/3}$$

**Horizontally simple:**




Now choose $y$ in $[0,1]$ first then $x$ goes between 0 and $1-y$

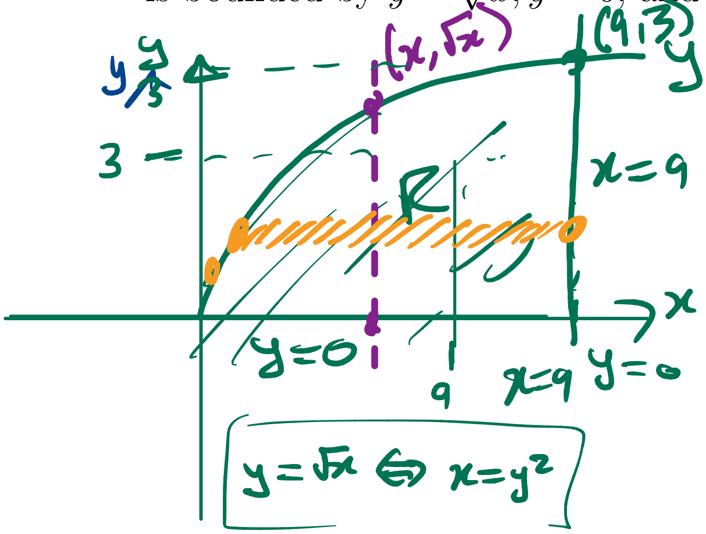
$$\text{Vol}=\int_0^1\int_0^{1-y} 2-x-y\,dx\,dy$$

$$=\cdots=\boxed{2/3}$$

① $\iint f(x,y)\,dy\,dx$

② $\iint f(x,y)\,dx\,dy$

**Example 85.** Write the two iterated integrals for $\iint_R 1\, dA$ for the region $R$ which is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.



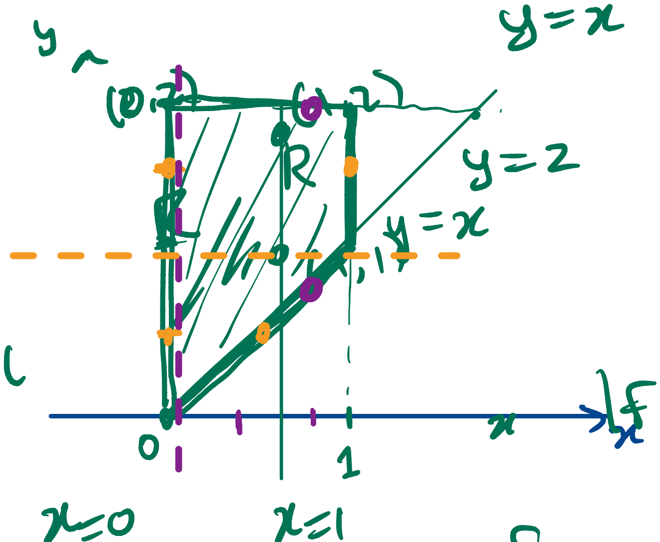

First $[0,9]$ then $y\in[0,\sqrt{x}]$

① $\int_0^9 1\; dy\, dx$

First $\in[0,3]$ then $x\in[y^2,9]$

② $\int^3 1\; dx\, dy$

**Example 86.** Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\, dA$, where $R$ is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.




Easier Choice of order is $\iint f(x,y)\,dy\,dx$

If $x\in[0,1]$, then $y\in[x,2]$

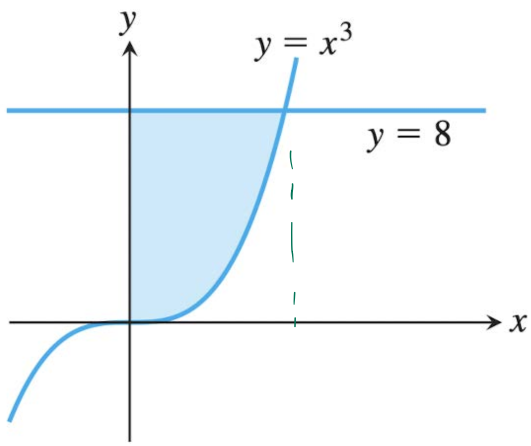
So double integral is

$$\int_0^1 \int_x^2 6x^2 y\; dy\; dx$$

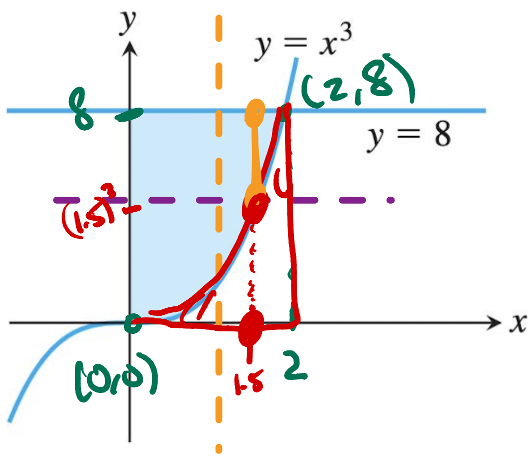
Then y-values in range $[x,2]$.

x values in $[0,1]$

**Example 87.** *You try it!* Write the two iterated integrals for $\iint_R 1 \; dA$ for the region $R$ which is bounded by $x = 0, y = 8$, and $y = x^3$.



**Example 87.** *You try it!* Write the two iterated integrals for $\iint_R 1\ dA$ for the region $R$ which is bounded by $x = 0, y = 8$, and $y = x^3$.




✓ (x first then y)

① $\iint f(x,y)\, dy\, dx$ (not right but what does it give?)

$$= \int_0^2 \int_{\cancel{x^3}\ 0}^{\cancel{8}\ x^3} 1\ dy\, dx$$

✓ (y first then x)

② $\iint f(x,y)\, dx\, dy$

$$= \int_0^8 \int_0^{\sqrt[3]{y}} 1\ dx\, dy$$

$y = x^3$

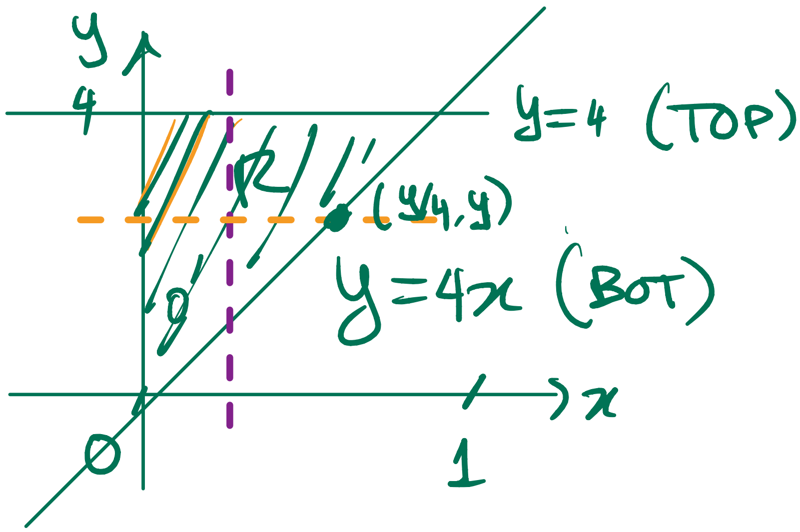
$\Leftrightarrow\ x = \sqrt[3]{y}.$

**Example 88.** Sketch the region of integration for the integral

$$V = \int_0^1 \int_{4x}^4 f(x,y)\, dy\, dx.$$

Then write an equivalent iterated integral in the order $dx\, dy$.

Q: how to get R from the iterated integral?




$$② \int_0^4 \int_0^{y/4} f(x,y)\, dx\, dy$$

One idea

$y = 4x \iff x = y/4$

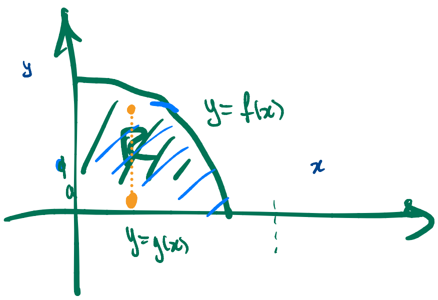
and $y \in [0,4]$ so $x \in [0, y/4]$

# §15.3 Area & Average Value

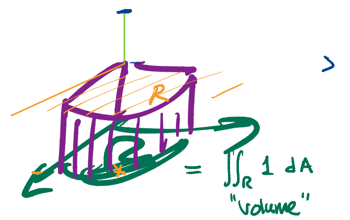
Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain. also Area(R)

**Area:** If $R$ is a region bounded by smooth curves, then

$$\text{Area}(R) = \iint_R 1\, dA$$

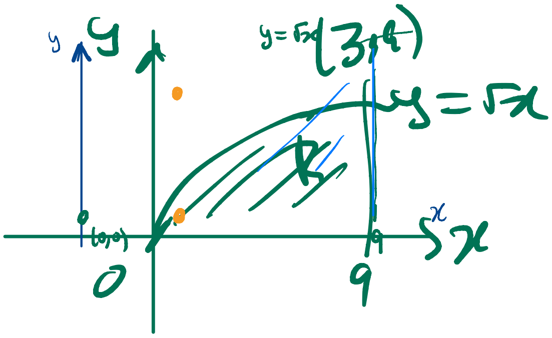



$$\iint_R 1\, dA \stackrel{?}{=} \text{Volume?}$$




Volume of this solid is $\iint_R 1\, dA$

**Example 89.** Find the area of the region $R$ bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.




$$\text{Area} = \iint_R 1\, dA = \int_0^9 \int_0^{\sqrt{x}} 1\, dy\, dx$$

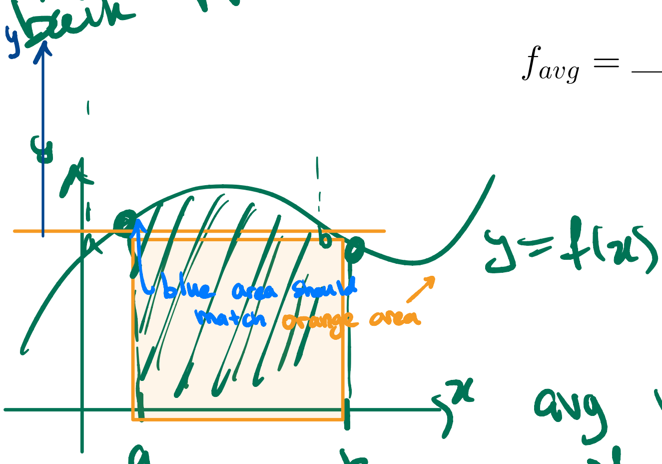
$$= \int_0^9 y \Big|_0^{\sqrt{x}} dx = \int_0^9 \sqrt{x}\, dx$$

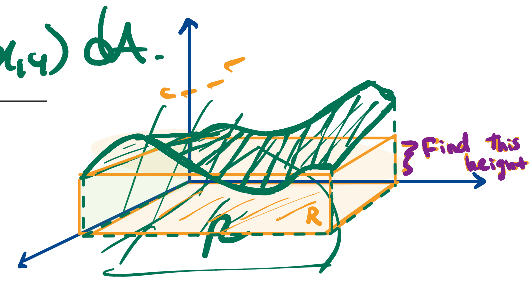
$$= \frac{2}{3} x^{3/2} \Big|_0^9 = \frac{2}{3}(27) - 0 = \boxed{18}$$

**Average Value:** The average value of $f(x, y)$ on a region $R$ contained in $\mathbb{R}^2$ is

$$f_{avg} = \frac{1}{\cancel{A}\ (R)} \iint_R f(x,y)\, dA.$$

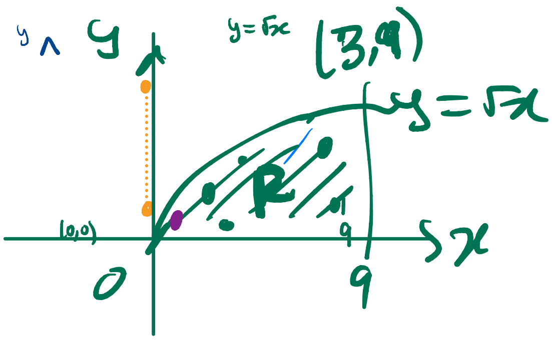
From Calc I Calc II.




avg value of $f$ over $[a,b]$ $= \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\, dx$

**Example 90.** Find the average temperature on the region $R$ in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.




$\text{Area}(R) = 18$

Idea: Compute ① $\iint_R T(x,y)\, dA$

& divide by Area(R).

① $$\int_0^9 \int_0^{\sqrt{x}} 4xy^2 \, dy \, dx = \int_0^9 \frac{4}{3} x y^3 \Big|_0^{\sqrt{x}} dx$$

$$= \int_0^9 \frac{4}{3} x (\sqrt{x})^3 - \frac{4}{3}(0)\, dx = \int_0^9 \frac{4}{3} x^{5/2}\, dx$$

$$= \frac{4}{3} * \frac{2}{7} x^{7/2} \Big|_0^9 = \frac{8}{21} (9)^{7/2} = \frac{8}{21} 3^7 = \frac{8 * 3^6}{7}$$

② $$\text{Avg} f = \frac{1}{18} \frac{8 * 3^6}{7} = \frac{4 * 3^4}{7} = \frac{4 * 81}{7}$$

$$= \boxed{324/7} \approx 46.28°$$

**Example 91.** *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

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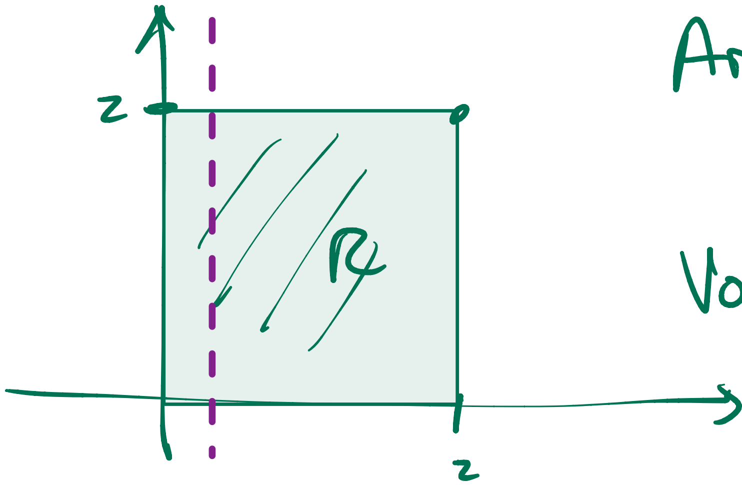



Area $R = 4$ (no need to do $\iint_R 1 \, dA$)

$$\text{Vol} = \iint_R x^2 + y^2 \, dA$$

$$= \int_0^2 \int_0^2 x^2 + y^2 \, dy \, dx$$

$$= \int_0^2 x^2 y + \tfrac{1}{3} y^3 \Big|_0^2 \, dx$$

$$= \int_0^2 2x^2 + 8/3 \, dx$$

$$= \tfrac{2}{3} x^3 + \tfrac{8}{3} x \Big|_0^2$$

$$= \tfrac{2}{3} * 8 + \tfrac{2}{3}(8) = \boxed{\cancel{32/3}}$$

↑ volume!!

Step 2 divide volume of solid by area of base

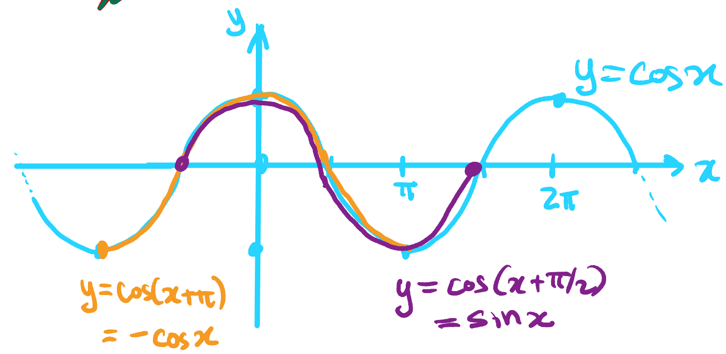
$$\text{Avg } f = \frac{1}{\text{Area } R} \iint_R f \, dA = \frac{1}{4} * \frac{32}{3} = \boxed{8/3}$$

REMINDER: ① Exam 2 in ONE WEEK, Thur March 12
② Quiz 3 due Friday on Gradescope.



**Example 92.** Find the average value of the function $f(x, y) = \sin(x + y)$ on (a) the region $R_1 = [0, \pi] \times [0, \pi]$, and (b) the region $R_2 = [0, \pi] \times [0, \pi/2]$.
*Hint: choose your order of integration carefully!*

x-values    y-values

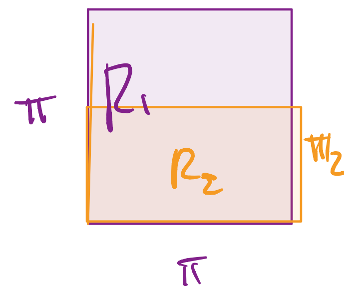



(a) For $R_1$

$$V = \int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx \quad ①$$

$$= \int_0^\pi \int_0^\pi \sin(x+y)\,dx\,dy \quad ②$$

lets try option 2.

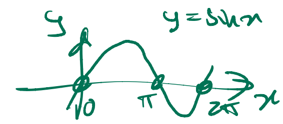



(b) For $R_2$

$$V = \int_0^\pi \int_0^{\pi/2} \sin(x+y)\,dy\,dx \quad ①$$

$$= \int_0^{\pi/2} \int_0^\pi \sin(x+y)\,dx\,dy \quad ②$$

same




For (a)

$$V = \int_0^\pi \underbrace{-\cos(x+y)\Big|_0^\pi}_{\text{same}} dy = \int_0^\pi \underbrace{-\cos(\pi+y) - (-\cos(0+y))}_{\text{Same}}\,dy$$

$$= \int_0^\pi -\cos(y+\pi) + \cos(y)\,dy$$

$$= -\sin(y+\pi) + \sin(y)\Big|_0^\pi = \left(-\cancel{\sin(2\pi)}^{0} + \cancel{\sin(\pi)}^{0}\right) - \left(-\cancel{\sin(\pi)}^{0} + \cancel{\sin(0)}^{0}\right)$$

$$= 0$$

(a) avg value is $\boxed{0}$ b/c $0/\pi^2 = 0$

For (b)

$$V = \int_0^{\pi/2} -\cos(y+\pi) + \cos(y)\,dy = -\sin(y+\pi) + \sin(y)\Big|_0^{\pi/2}$$

$$= \left(-\sin(\underbrace{\pi/2+\pi}_{3\pi/2}) + \sin(\pi/2)\right) - \left(\cancel{\sin \pi}^{0} - \cancel{\sin 0}^{0}\right) = -(-1)+1 = 2$$

(b) avg value is

$$2/(\pi^2/2) = \boxed{\frac{4}{\pi^2}}$$

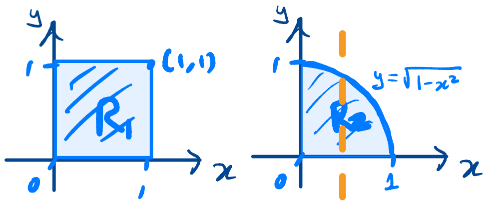
Formula: $\text{avg } f = \dfrac{1}{\text{Area}R} \times \text{Vol}$

$$= \frac{1}{\text{Area}R} \iint_R f\,dA$$

① Compute avg value of $f$ over $R_1$ & $R_2$

② guess which will be bigger?

**Example 93.** *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of $f$ over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of $f$ over $R_2$ the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.




$\text{Avg}_{R_1}(f)$ or $\boxed{\text{Avg}_{R_2}(f)}$?

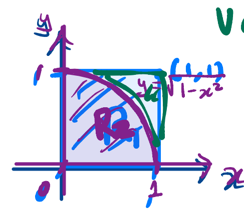
guess b/c ÷ smaller area, so larger total?

$$\text{Vol}_{R_1} = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \tfrac{1}{2} x y^2 \Big|_0^1 dx = \int_0^1 \tfrac{1}{2} x \, dx$$

$$= \tfrac{1}{4} x^2 \Big|_0^1 = \tfrac{1}{4}.$$

$\text{Area}(R_1) = 1.$ So $\text{Avg}_{R_1} f = \boxed{1/4}$

$$\text{Vol}_{R_2} = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx$$




Values in this region are generally higher than average

$$= \int_0^1 \tfrac{1}{2} x y^2 \Big|_0^{\sqrt{1-x^2}} dx$$

$\text{Avg}_{R_2} f = ?$ (bigger than 1/4 or smaller than 1/4 ...)

$$= \int_0^1 \tfrac{1}{2} x (1 - x^2) \, dx = \int_0^1 \tfrac{1}{2} x - \tfrac{1}{2} x^3 \, dx = \tfrac{1}{4} x^2 - \tfrac{1}{8} x^4 \Big|_0^1$$

$$= \tfrac{1}{4} - \tfrac{1}{8} = 1/8.$$

So $\text{Area } R_2 = \tfrac{1}{4} \pi = \pi/4$

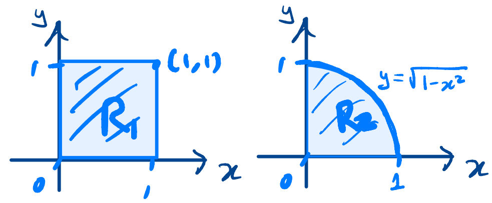
$$\text{Avg}_{R_2} f = \frac{1}{\pi/4} \cdot \frac{1}{8} = \frac{4}{\pi \cdot 8} = \boxed{\frac{1}{2\pi}} \approx \frac{1}{6}$$

Smaller than $\text{Avg}_{R_1} f$.

8:50 15 min 9:05



**Example 93.** *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of $f$ over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of $f$ over $R_2$ the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.




$\text{Avg}_{R_1}(f)$ or $\boxed{\text{Avg}_{R_2}(f)}$ ?

guess b/c $\div$ smaller area, so larger total?

$$V_{R_1} = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^1 = \int_0^1 \tfrac{1}{2}x(1-0)\,dx = \int_0^1 \tfrac{1}{2}x \, dx$$

$$= \tfrac{1}{4}x^2 \Big|_0^1 = 1/4 \qquad \text{So } \text{Avg}_{R_1}(f) = \frac{1}{\text{Area } R_1} V_{R_1} = \frac{1}{1} * \frac{1}{4} = \frac{1}{4}$$

$$V_{R_2} = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \tfrac{1}{2}x(1-x^2)\,dx$$

$$= \int_0^1 \tfrac{1}{2}(x - x^3)\,dx = \tfrac{1}{2}\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1 = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2} * \tfrac{1}{4} = 1/8$$

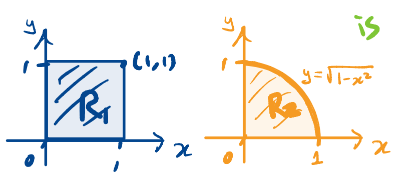
$$\text{So } \text{Avg}_{R_2}(f) = \frac{1}{\text{Area } R_2} V_{R_2} = \frac{1}{\pi/4} * \frac{1}{8} = \frac{1}{2\pi} \approx \frac{1}{6}$$

Well I'll be...

!!

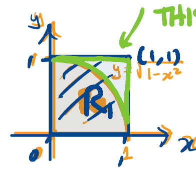
how's that work...

In retrospect, what seems to be happening




is that in the outer region of $R_1$, (the part not contained in $R_2$)

THIS BIT HERE

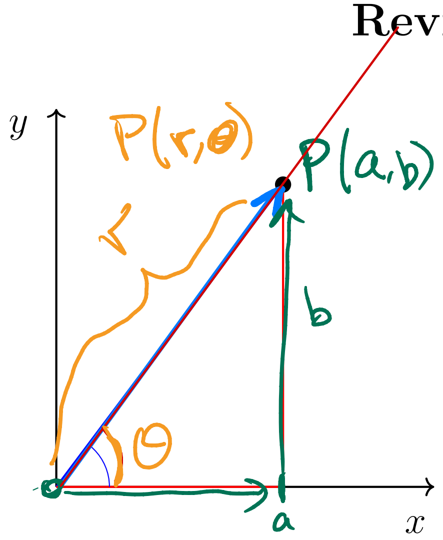



in the part of $R_1$ not contained in $R_2$ the value of $f(x,y) = xy$ is "particularly large" since $xy$ is small when $x$ or $y$ is small. And apparently this more than compensates for $\div$ by larger area.

who woulda think it.

# §15.4 Double Integrals in Polar Coordinates

## Review of Polar Coordinates




aka rectangular coordinates.

**Cartesian coordinates:** Give the distances in $x$ and $y$ directions from $(0,0)$

**Polar coordinates:**

- $r$ = distance from $(0,0)$ to $P(a,b)$ $\boxed{r \geq 0}$
- $\theta$ = angle between the ray $\overrightarrow{OP}$ and the positive $x$-axis

$\boxed{\theta \in [0, 2\pi) \text{ or } \theta \in [-\pi, \pi]}$

We can use trigonometry to go back and forth.

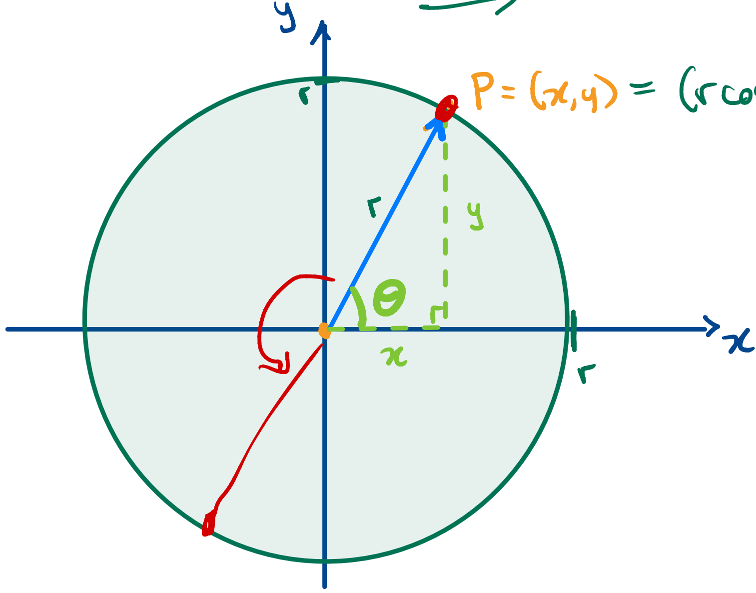
https://www.geogebra.org/classic/thaxxzzp

**Polar to Cartesian:**

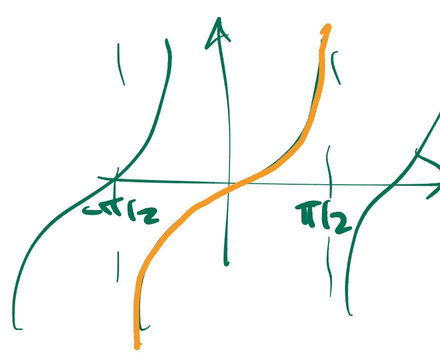
have $(r,\theta)$ want $(x,y)$ Then use $\rightarrow$ $\quad x = r\cos(\theta) \qquad y = r\sin(\theta)$

**Cartesian to Polar:**

have $(x,y)$ want $(r,\theta)$ $\rightarrow$ $\quad r^2 = x^2 + y^2 \qquad \tan(\theta) = \dfrac{y}{x}$ So $\theta = \underline{\arctan(y/x)}$.

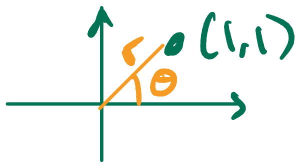



$P = (x,y) = (r\cos\theta, \underbrace{r\sin\theta}_{y})$




Warning: many possible $(r,\theta)$ for one particular $(x,y)$.

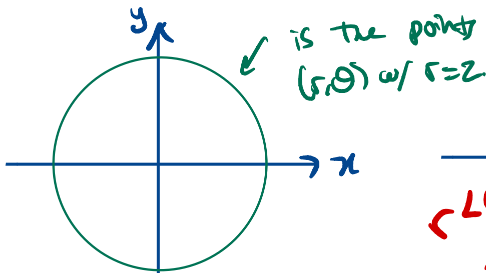
**Example 94.** a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

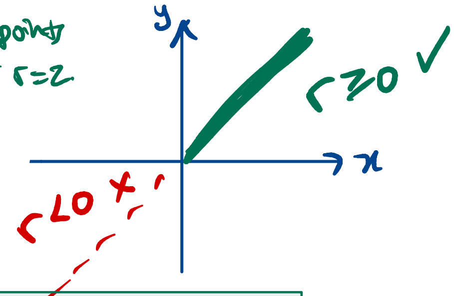



$r^2 = (1)^2 + (1)^2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$.

$\theta = \arctan(1/1) = \pi/4$.

b) Graph the set of points $(x, y)$ that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the** $xy$**-plane.**




c) Write the function $\boxed{f(x, y) = \sqrt{x^2 + y^2}}$ in polar coordinates. $\boxed{f(r,\theta) = r}$

Idea: substitute
$x = r\cos\theta$
$y = r\sin\theta$

$f(r,\theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = |r| = r$ ($r \geq 0$)

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$. rewrite $\boxed{r = 2\sin\theta}$ using $x$'s & $y$'s.

$r\sin\theta = y$
$r^2 = x^2+y^2$

$\Rightarrow r^2 = 2r\sin\theta$

$\Rightarrow \boxed{x^2+y^2 = 2y}$

$\Rightarrow x^2 + y^2 - 2y + 1 = 1$

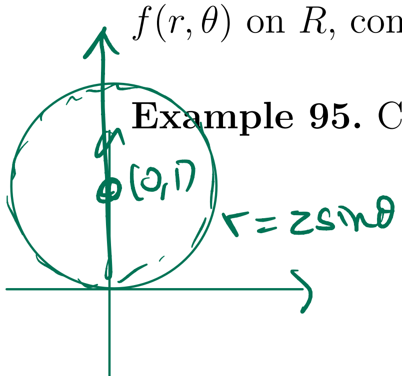
$\Rightarrow x^2 + (y-1)^2 = 1$

① $r = \sqrt{x^2+y^2}$

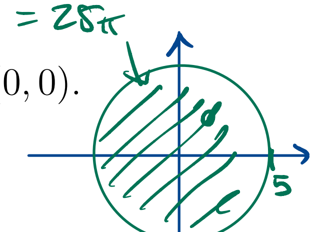
② $y = r\sin\theta \Rightarrow \sin\theta = y/r$

$r\sqrt{x^2+y^2} = 2\dfrac{y}{\cancel{r}} \Rightarrow \boxed{x^2+y^2 = 2y}$

**Goal:** Given a region $R$ in the $xy$-plane described in polar coordinates and a function $f(r, \theta)$ on $R$, compute $\iint_R f(r, \theta)\, dA$.




$A = \pi R^2 = 25\pi$ $R=5$




**Example 95.** Compute the area of the disk of radius 5 centered at $(0, 0)$.

Naive approach

$A \overset{??}{=} \iint_R 1\, dA = \int_0^{2\pi}\int_0^5 1\, dr\, d\theta = \int_0^{2\pi} r\Big|_0^5 d\theta = \int_0^{2\pi} 5\, d\theta = 5\theta\Big|_0^{2\pi} = \boxed{10\pi}$

$r$ values in $[0,5]$
$\theta$ values in $[0,2\pi]$

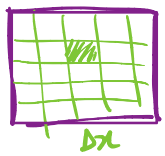
**Remember:** In polar coordinates, the area form $dA =$ ? ________

$dA \neq dr\, d\theta$

**Goal:** Given a region $R$ in the $xy$-plane described in polar coordinates and a function $f(r,\theta)$ on $R$, compute $\iint_R f(r,\theta)\; dA$.

**Example 96.** Compute the area of the disk of radius 5 centered at $(0,0)$.
*Cont.*

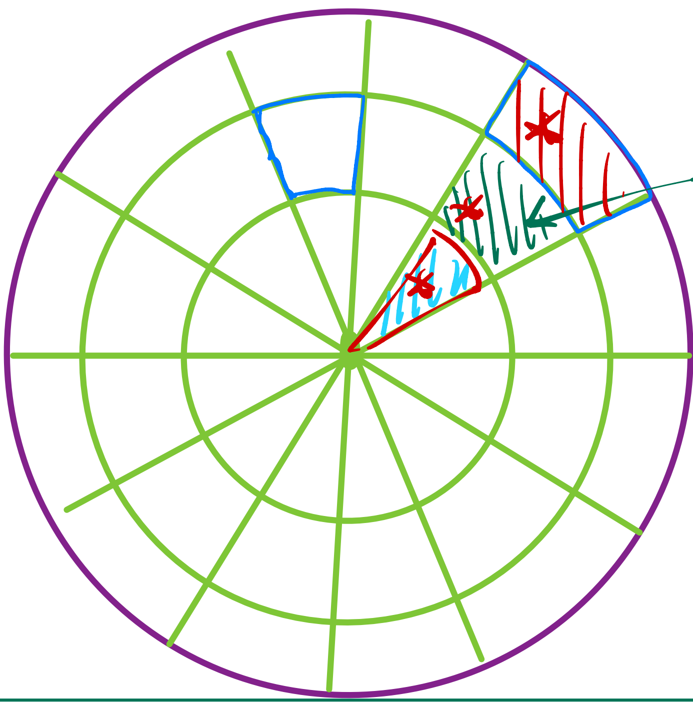
recall.




$dA_k = \Delta x \Delta y$

So $\iint_R 1\, dA = \int_a^b \int_c^d 1\, dx\, dy$

but now.



current radius

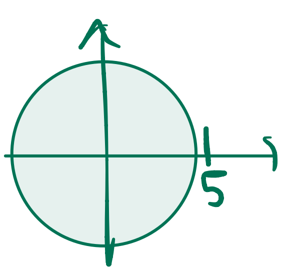
$A_k = r_1 \nabla\theta \nabla r$

$\boxed{dA = r\, d\theta\, dr}$

So $\iint_R 1\, dA = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} 1 \; \ldots \; ?$

Let's try it out!

Disk area $\iint_R 1\, dA = \int_0^{2\pi} \int_0^5 1 * \underbrace{\boxed{r\;\; dr\;\; d\theta}}_{dA}$

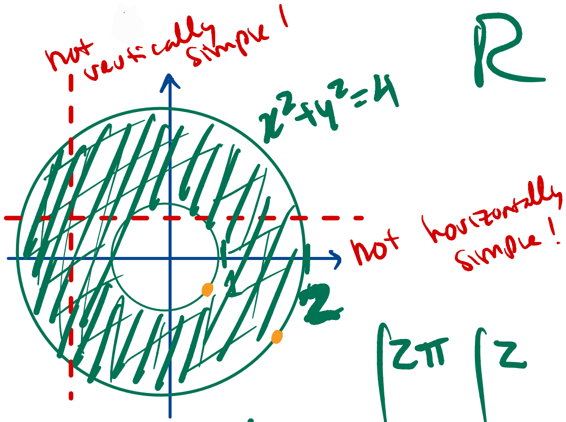



$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^5 \, d\theta = \int_0^{2\pi} \frac{25}{2}\, d\theta$

$= \frac{25}{2}\theta \Big|_0^{2\pi} = \frac{25}{2} * 2\pi = \boxed{25\pi}$

**Remember:** In polar coordinates, the area form $dA = \boxed{r\, dr\, d\theta}$

**Example 96.** Compute $\iint_D \overbrace{e^{-(x^2+y^2)}} \, dA$ on the washer-shaped region $1 \le x^2 + y^2 \le 4$.

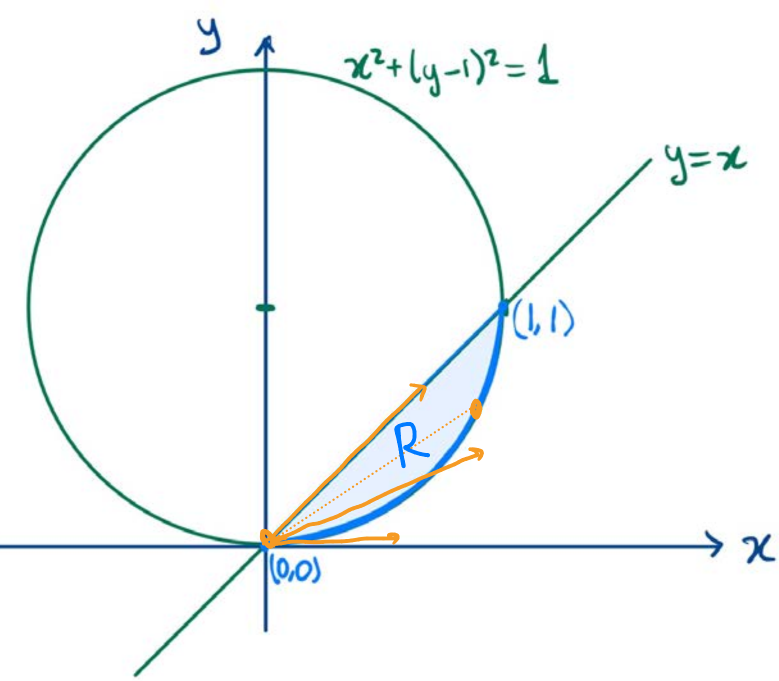



$R = \{ (r,\theta) \mid 0 \le \theta \le 2\pi, \ 1 \le r \le 2 \}$

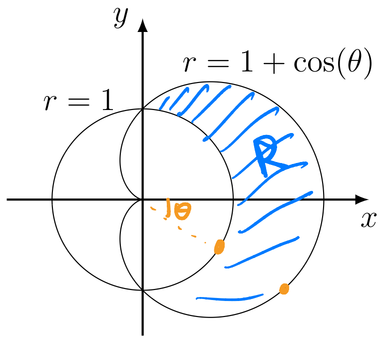
"polar rectangle" $R = ([0,2\pi] \times [1,2])_P$

$$V = \int_0^{2\pi} \int_1^2 \underbrace{e^{-r^2}} \, r \, dr \, d\theta$$

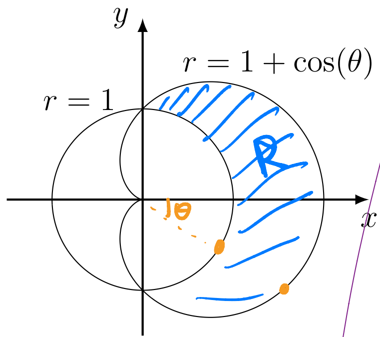
**Example 97.** Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.



**Example 98.** *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



**Example 98.** *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.




$\theta \in [-\pi/2, \pi/2]$

$r \in [1, 1+\cos\theta]$

So $f(x,y) = x$

$f(r,\theta) = r\cos\theta$

don't forget r!!

So $\displaystyle\iint_R f(x,y)\, dA = \boxed{\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\cos\theta * r\, dr\, d\theta}$

Stop here ☺

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \cos\theta \Big|_1^{1+\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\left[(1+\cos\theta)^3\cos\theta - \cos\theta\right] d\theta$$

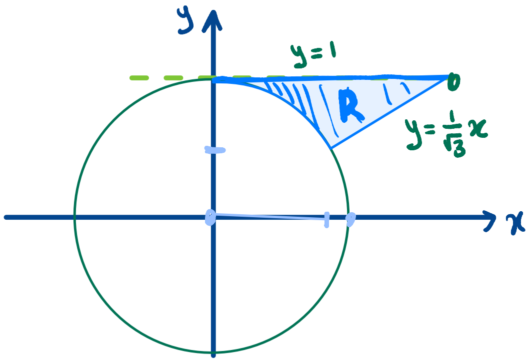
$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta\left[(1+\cos\theta)^3 - 1\right] d\theta$$

oh lol. oops.

hmmm...

$$= \ldots = \boxed{5\pi/8}$$

**Example 100.** Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. Then, evaluate the original integral to find the value of $\iint_R f(x,y)\ dA$.




$$\int_{\pi/6}^{\pi/2} \int_{1}^{\csc\theta} r^2 \cos\theta\ dr\ d\theta$$

## Tips and tricks

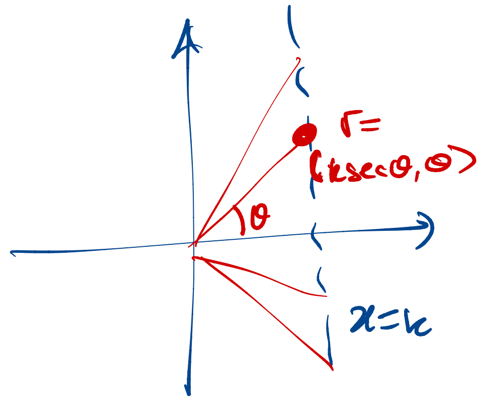
① $x = r\cos\theta$
② $y = r\sin\theta$

For horizontal lines such as $x = 2$:

use ① get $2 = r\cos\theta \Rightarrow r = 2\sec\theta$

in general

If $x = k \Rightarrow r = k\sec\theta$



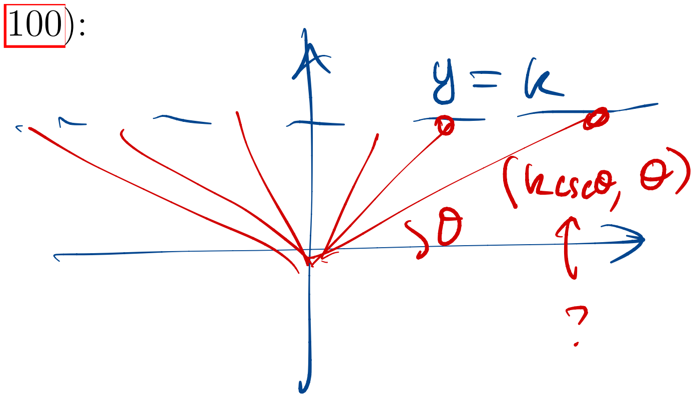

For vertical lines such as $y = 1$ (e.g., Example 100):

use ② get

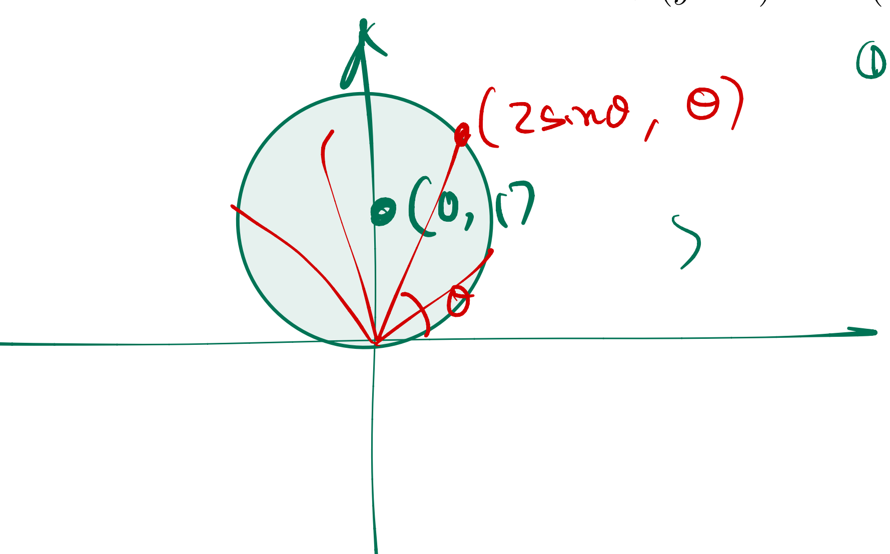
$1 = r\sin\theta$

So $r = \csc\theta$

So in general $y = k$
$\Rightarrow r = k\csc\theta$




For off-set circles such as $x^2 + (y-1)^2 = 1$ (e.g., Example 98):



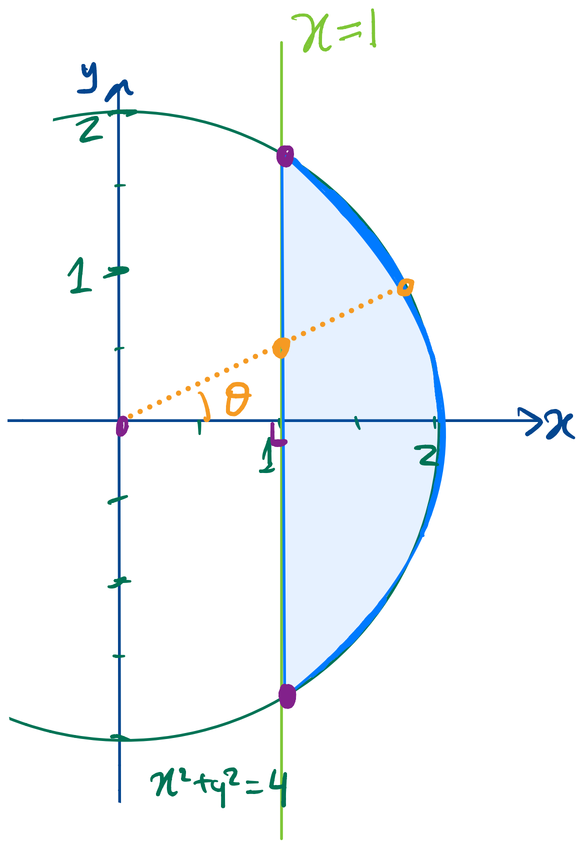

① & ② CTS.

$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$

$\Rightarrow r^2 - 2r\sin\theta = 0$

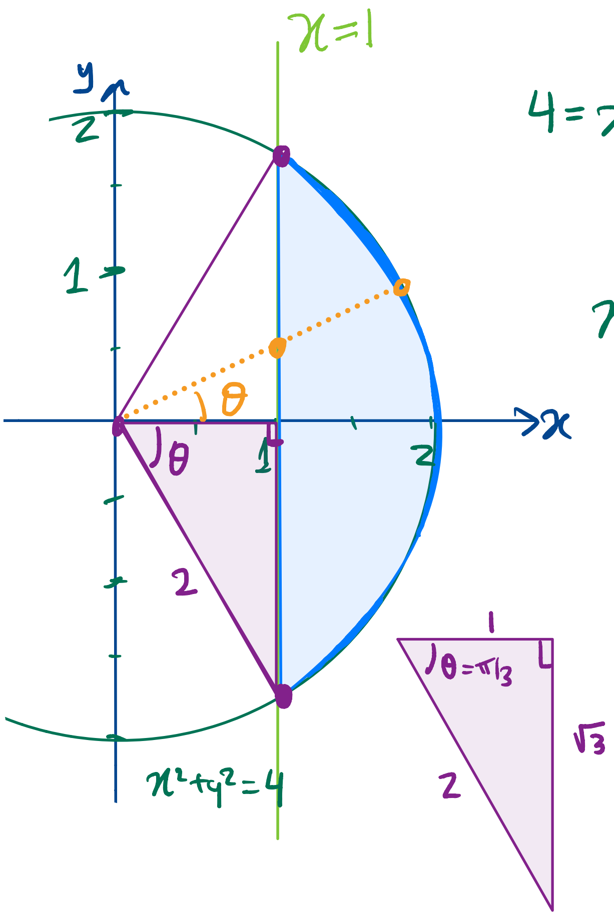
$\Rightarrow \boxed{r = 2\sin\theta}$

**Example 101.** *You try it!* Find the area of the region $R$ which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.



**Example 101.** *You try it!* Find the area of the region $R$ which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

The $\theta$-value is between $(1,-\sqrt{3})$ & $(1,\sqrt{3})$ so $\theta \in [-\pi/3, \pi/3]$.




$4 = x^2 + y^2 = r^2 \Rightarrow r = \pm 2,\ r \geq 0$

so $r = 2$ on circle.

$x = r\cos\theta \Rightarrow r = \dfrac{x}{\cos\theta} = x\sec\theta$

@ $x=1$ $\quad r = \sec\theta$ on line $x=2$.

So $\theta \in [-\pi/3, \pi/3]$
and $r \in [\sec\theta, 2]$

So $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$

$$\text{Area} = \int_{-\pi/3}^{\pi/3} \int_{\sec\theta}^{2} 1 * r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2} r^2 \Big|_{\sec\theta}^{2} d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(\sec^2\theta - 4\right) d\theta$$

$$= \frac{1}{2}\tan\theta - 2\theta \Big|_{-\pi/3}^{\pi/3} = \tan(\pi/3) - 4 * \pi/3 = \boxed{\sqrt{3} - 4\pi/3}$$

Function is odd

# Math 2551 Worksheet: Exam 2 Review

1. Which of the following statements are true if $f(x, y)$ is differentiable at $(x_0, y_0)$? Give reasons for your answers.
   (a) If $\mathbf{u}$ is a unit vector, the derivative of $f$ at $(x_0, y_0)$ in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.
   (b) The derivative of $f$ at $(x_0, y_0)$ in the direction of $\mathbf{u}$ is a vector.
   (c) The directional derivative of $f$ at $(x_0, y_0)$ has its greatest value in the direction of $\nabla f$.
   (d) At $(x_0, y_0)$, the vector $\nabla f$ is normal to the curve $f(x, y) = f(x_0, y_0)$.
2. Find $dw/dt$ at $t = 0$ if $w = \sin(xy + \pi), x = e^t$, and $y = \ln(t + 1)$.
3. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.
4. Test the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points and find the function's value at these points.
5. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the $xy$-plane.
6. Evaluate the integral $\displaystyle\int_0^1 \int_{2y}^2 4\cos(x^2)\, dx\, dy$. Describe why you made any choices you did in the course of evaluating this integral.
7. If $f(x, y) \geq 2$ for all $(x, y)$, is it possible that the average value of $f(x, y)$ on a unit disk centered at the origin is $\dfrac{2}{\pi}$?
8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.