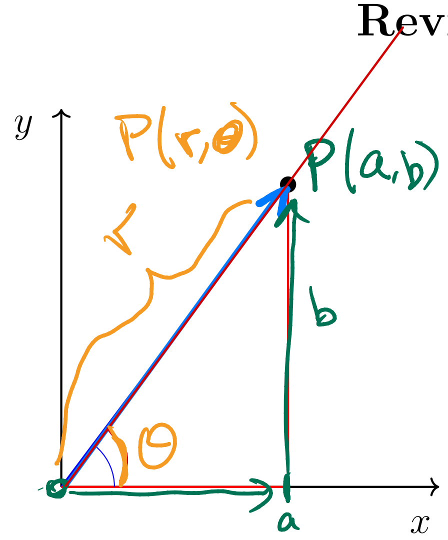
§15.4 Double Integrals in Polar Coordinates

Review of Polar Coordinates

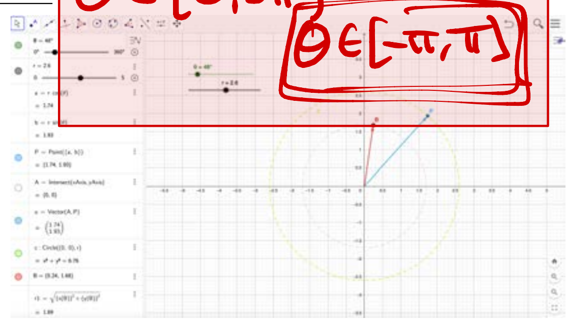


aka rectangular coordinates.

Cartesian coordinates: Give the distances in x and y directions from $(0,0)$

Polar coordinates:

- r = distance from $(0,0)$ to $P(a,b)$ $\boxed{r \geq 0}$
- θ = angle between the ray $\overline{OP}$ and the positive x-axis $\boxed{\theta \in [0, 2\pi) \text{ or } \boxed{\theta \in [-\pi, \pi]}}$



We can use trigonometry to go back and forth.

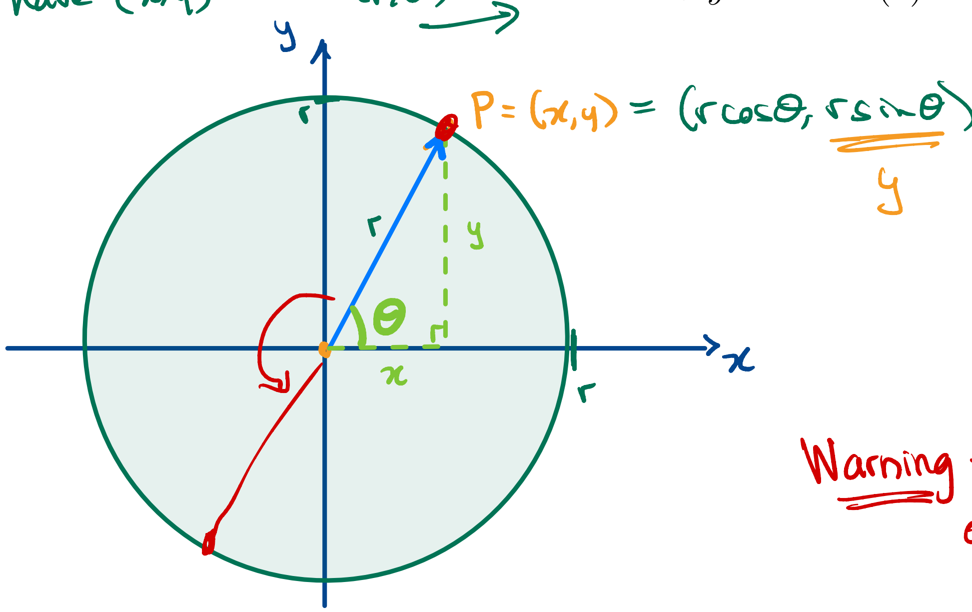
https://www.geogebra.org/classic/thaxxzzp

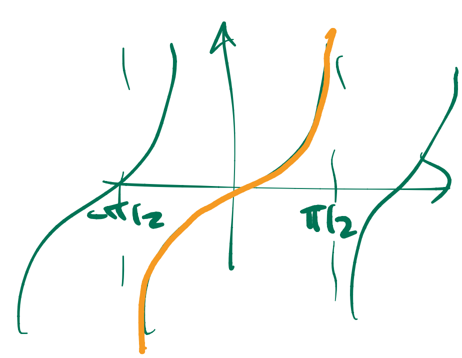
Polar to Cartesian:

have (r,θ) want (x,y) Then use $\rightarrow$ $x = r\cos(\theta)$ $y = r\sin(\theta)$

Cartesian to Polar:

have (x,y) want (r,θ) $\rightarrow$ $r^2 = x^2 + y^2$ $\tan(\theta) = \dfrac{y}{x}$ So $\theta = \underline{\arctan(y/x)}$.

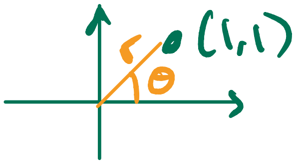




Warning: many possible (r,θ) for one particular (x,y).

Formula: $r^2 = x^2+y^2$ $\quad \theta = \arctan(y/x)$ for quadrants I/IV.

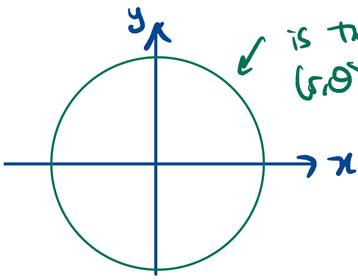
Example 94. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

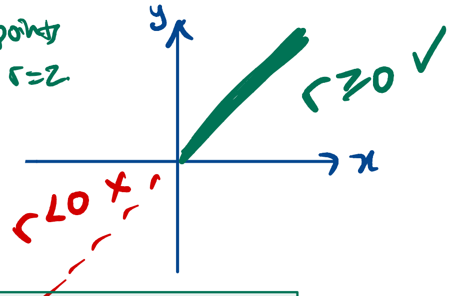


$r^2 = (1)^2+(1)^2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$.

$\theta = \arctan(1/1) = \pi/4$.

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the xy-plane.**





c) Write the function $\boxed{f(x, y) = \sqrt{x^2 + y^2}}$ in polar coordinates. $\boxed{f(r,\theta) = r}$

Idea: substitute
$x = r\cos\theta$
$y = r\sin\theta$

$f(r,\theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = |r| = r$ ($r \geq 0$)

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$. rewrite $\boxed{r = 2\sin\theta}$ using x's & y's.

$r\sin\theta = y$

$r^2 = x^2+y^2$

$\Rightarrow r^2 = 2r\sin\theta$

$\Rightarrow \boxed{x^2+y^2 = 2y}$

$\Rightarrow x^2 + y^2 - 2y + 1 = 1$

$\Rightarrow x^2 + (y-1)^2 = 1$

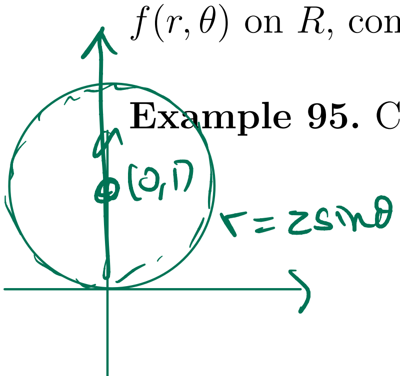
① $r = \sqrt{x^2+y^2}$

② $y = r\sin\theta \Rightarrow \sin\theta = y/r$

$r\sqrt{x^2+y^2} = 2\frac{y}{\cancel{r}} \Rightarrow \boxed{x^2+y^2 = 2y}$

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

$A = \pi R^2 = 25\pi$ $\quad R=5$

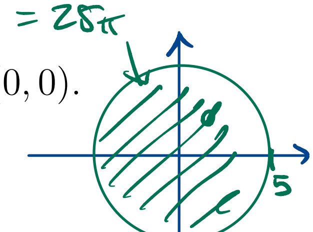


Example 95. Compute the area of the disk of radius 5 centered at $(0, 0)$.

Naive approach

$A \overset{??}{=} \iint_R 1\, dA = \int_0^{2\pi}\int_0^5 1\, dr\, d\theta = \int_0^{2\pi} r\Big|_0^5 d\theta = \int_0^{2\pi} 5\, d\theta$

r values in $[0,5]$
θ values in $[0,2\pi]$



Remember: In polar coordinates, the area form $dA =$? ________

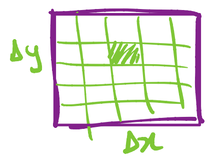
$= 5\theta\Big|_0^{2\pi} = \boxed{10\pi}$

$dA \neq dr\, d\theta$

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r,\theta)$ on R, compute $\iint_R f(r,\theta)\, dA$.

Example 96. Compute the area of the disk of radius 5 centered at $(0,0)$.
Cont.

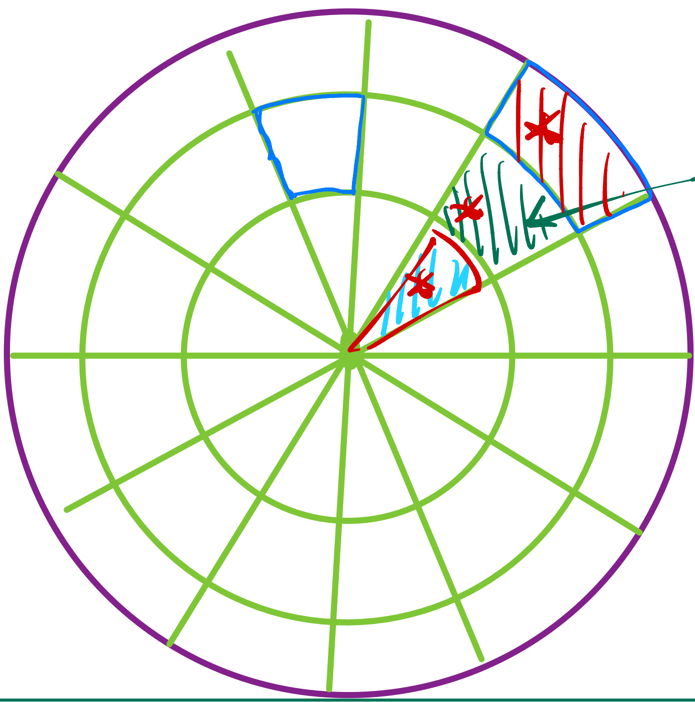
recall.



$dA_k = \Delta x \Delta y$

So $\iint_R 1\, dA = \int_a^b \int_c^d 1\, dx\, dy$

but now.



current radius

$A_k = r_1 \nabla\theta \nabla r$

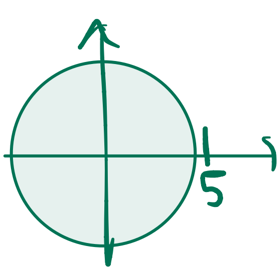
$$\boxed{dA = r\, d\theta\, dr}$$

So $\iint_R 1\, dA = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} 1 \ldots ?$

Let's try it out!

integration factor

Disk area $\iint_R 1\, dA = \int_0^{2\pi} \int_0^5 1 * \underbrace{\boxed{r\, dr\, d\theta}}_{dA}$



$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^5 \, d\theta = \int_0^{2\pi} \frac{25}{2}\, d\theta$

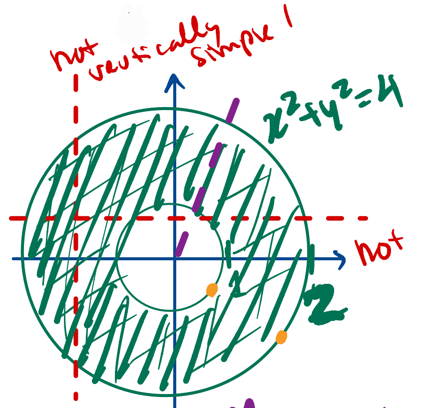
$= \frac{25}{2}\theta \Big|_0^{2\pi} = \frac{25}{2} * 2\pi = \boxed{25\pi}$

Remember: In polar coordinates, the area form $\boxed{dA = r\, dr\, d\theta}$

EXAM 2! on Thursday! @2pm in lecture hall

$f(x,y) = f(r,\theta)$ $\qquad r^2 = x^2+y^2$

Example 96. Compute $\iint_D e^{-(x^2+y^2)}\, dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.



R is radially simple.

$R = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 1 \le r \le 2\}$

"polar rectangle" $R = [0,2\pi] \times [1,2]_p$

$$V = \int_0^{2\pi}\int_1^2 e^{-r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\int_{-1}^{-4} -\tfrac{1}{2} e^u\, du\, d\theta$$

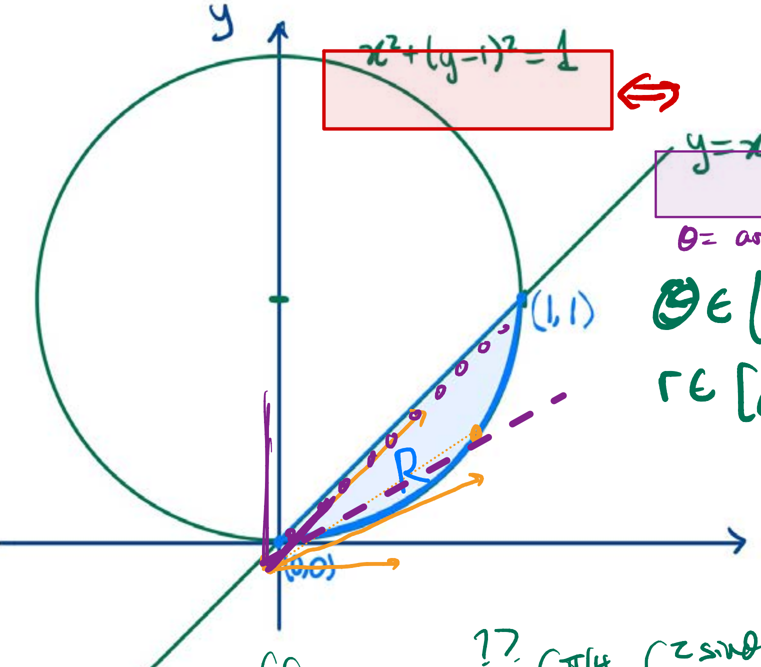
u-sub BOX: $u = -r^2$, $du = -2r\,dr$, $-\tfrac{1}{2}du = r\,dr$

$r=1 \Rightarrow u=-1$
$r=2 \Rightarrow u=-4$

$$= \int_0^{2\pi} -\tfrac{1}{2} e^u \Big|_{-1}^{-4} d\theta = \int_0^{2\pi} -\tfrac{1}{2}e^{-4} - \left(-\tfrac{1}{2}e^{-1}\right) d\theta$$

$$= -\tfrac{1}{2}e^{-4}\theta + \tfrac{1}{2}e^{-1}\theta \Big|_0^{2\pi} = \boxed{-\pi e^{-4} + \pi e^{-1}}$$

Example 97. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$. Step 1: use $x = r\cos\theta$, $y = r\sin\theta$



$(r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + \cancel{1} = \cancel{1}$

$\Rightarrow r^2 - 2r\sin\theta = 0$

$\Rightarrow r^2 = 2r\sin\theta$

$\Rightarrow \boxed{r = 2\sin\theta}$ $(r \ne 0)$

$\theta \in [0, \pi/4]$

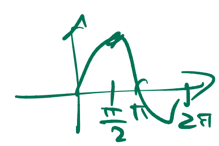
$r \in [0, 2\sin\theta]$

power reducing formula $\boxed{\sin^2\theta = \dfrac{1-\cos 2\theta}{2}}$

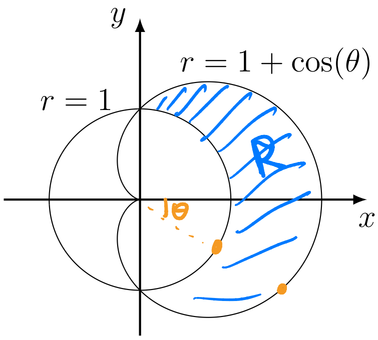
$$\text{Area} = \iint_R 1\, dA \overset{??}{=} \int_0^{\pi/4}\int_0^{2\sin\theta} 1\, r\, dr\, d\theta = \int_0^{\pi/4} \tfrac{1}{2}r^2 \Big|_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi/4} \tfrac{1}{2}(2\sin\theta)^2\, d\theta = \int_0^{\pi/4} 2\sin^2\theta\, d\theta = \int_0^{\pi/4} \cancel{2}\left(\frac{1-\cos 2\theta}{\cancel{2}}\right) d\theta$$

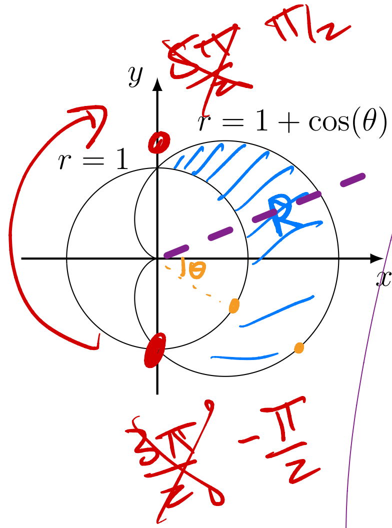
$$= \theta - \tfrac{1}{2}\sin 2\theta \Big|_0^{\pi/4} = \left(\frac{\pi}{4} - \frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right)\right) - \left(0 - \tfrac{1}{2}\cancel{\sin 0}\right) = \boxed{\frac{\pi}{4} - \frac{1}{2}}$$



Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



$\theta \in [-\pi/2, \pi/2]$

$r \in [1, 1+\cos\theta]$

~~$\theta \in \left[\frac{3\pi}{2}, \frac{\pi}{2}\right]$~~ ???

So $f(x,y) = x$

$f(r,\theta) = r\cos\theta$

don't forget r!!

$$\text{So } \iint_R f(x,y)\,dA = \boxed{\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} r\cos\theta * r\,dr\,d\theta}$$

Stop here ☺

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \cos\theta \Big|_1^{1+\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\left[(1+\cos\theta)^3\cos\theta - \cos\theta\right] d\theta$$

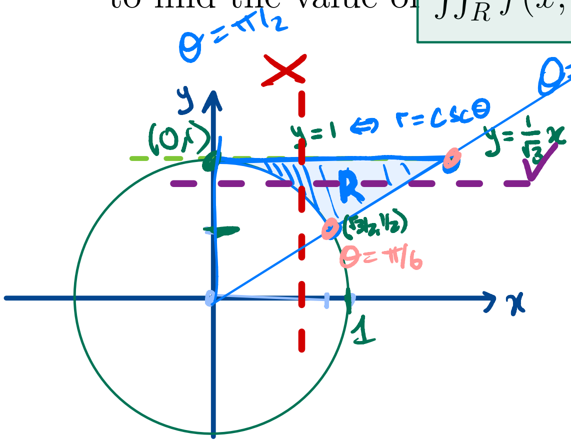
$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta\left[(1+\cos\theta)^3 - 1\right] d\theta$$

hmmm...

Oh lol. oops.

$$= \dots = \boxed{5\pi/8}$$

Example 100. (a) Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. (b) Then, evaluate the original integral to find the value of $\boxed{\iint_R f(x,y)\, dA.}$



$$\int_{\pi/6}^{\pi/2}\int_{1}^{\csc\theta} \underline{r^2\cos\theta}\; dr\, d\theta$$

f??

$\underbrace{r\cos\theta}\, * r$

$f(r,\theta) = r\cos\theta = x$

(a) META

① Figure out bounds for x, y to describe the region R

② Change f into cartesian coords

(a) R in polar coords $\theta \in [\pi/6, \pi/2]$ $r \in [1, \csc\theta]$

R rectangular $y \in [1/2, 1]$ $x \in [\sqrt{1-y^2}, \sqrt{3}y]$

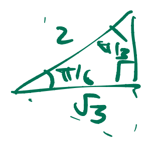
if $\theta = \pi/6$ and $r = 1$

Then $x = 1\cos(\pi/6)$ $y = 1 \sin \pi/6$

$x = \sqrt{3}/2$ $y = 1/2$

so line passing through $(0,0)$ & $(\sqrt{3}/2, 1/2)$

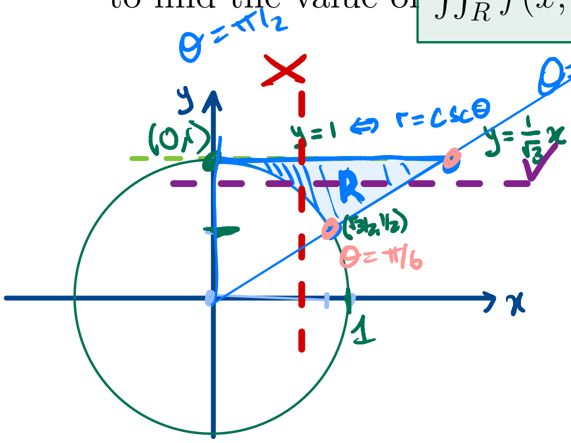
$y = mx \Rightarrow m = \frac{1/2}{\sqrt{3}/2} = 1/\sqrt{3}$



(a)

$$V = \int_{1/2}^{1}\int_{\sqrt{1-y^2}}^{\sqrt{3}\,y} \underbrace{x}_{f(x,y)}\; dx\, dy$$

Example 100. (a) Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. (b) Then, evaluate the original integral to find the value of $\boxed{\iint_R f(x,y)\,dA.}$



$$\int_{\pi/6}^{\pi/2}\int_{1}^{\csc\theta} r^2\cos\theta\;dr\;d\theta$$

(a) META

① Figure out bounds for x, y to describe the region R

② Change f into cartesian coords

(b) META

① integrate

② no step 2.

$$V = \int_{\pi/6}^{\pi/2}\int_{1}^{\csc\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} \frac{1}{3} r^3\cos\theta\Big|_{1}^{\csc\theta} d\theta$$

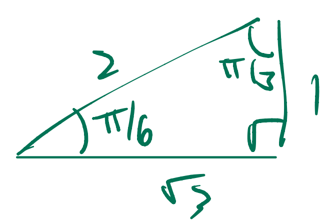
$$= \int_{\pi/6}^{\pi/2} \frac{1}{3}(\csc\theta)^3\cos\theta - \frac{1}{3}(1)^3\cos\theta\;d\theta$$

$$\csc^3\theta * \cos\theta = \frac{\cos\theta}{\sin^3\theta} = \csc^2\theta\cot\theta$$

$$\cancel{= \int_{\pi/6}^{\pi/2} \cot\theta\csc^2\theta - \frac{1}{3}\cos\theta\,d\theta}$$

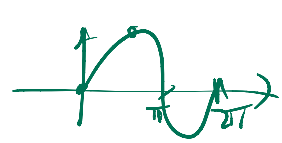
$$\boxed{\begin{aligned} u &= \sin\theta \\ du &= \cos\theta\,d\theta \end{aligned}}$$

$$= \int_{\pi/6}^{\pi/2} \frac{1}{3}\left(\frac{1}{\sin\theta}\right)^3\cos\theta\,d\theta - \int_{\pi/6}^{\pi/2}\frac{1}{3}\cos\theta\,d\theta$$



$$= \int_{*}^{*}\frac{1}{3}u^{-3}\,du - \int_{\pi/6}^{\pi/2}\frac{1}{3}\cos\theta\,d\theta = \frac{1}{3}*\frac{-1}{2}u^{-2}\Big|_{*}^{*} - \frac{1}{3}\sin\theta\Big|_{\pi/6}^{\pi/2}$$

$$= \frac{-1}{6}\sin^{-2}\theta\Big|_{\pi/6}^{\pi/2} - \left(\frac{1}{3}\sin\pi/2 * \frac{1}{3}\sin(\pi/6)\right)$$



$$= \left[-\frac{1}{6}(1) + \frac{1}{6}\left(\sin(\pi/6)\right)^{-2}\right] - \left(\frac{1}{3}(1) + \frac{1}{3}\,\frac{1}{2}\right)$$

$$= -\frac{1}{6} + \frac{1}{6}*\left(\frac{2}{1}\right)^2 - \frac{1}{3} + \frac{1}{6} = -\cancel{\frac{1}{6}} + \frac{4}{6} - \frac{1}{3} + \cancel{\frac{1}{6}} = \frac{2}{3} - \frac{1}{3} = \boxed{1/3}$$

① $x = r\cos\theta$
② $y = r\sin\theta$

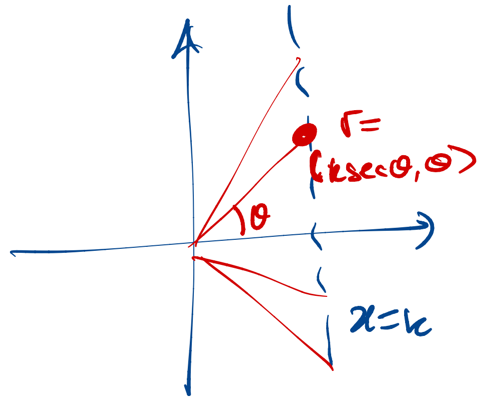
Tips and tricks

For ~~horizontal~~ vertical lines such as $x = 2$:

use ① get $2 = r\cos\theta \Rightarrow \boxed{r = 2\sec\theta}$

in general

If $\boxed{x = k \Rightarrow r = k\sec\theta}$



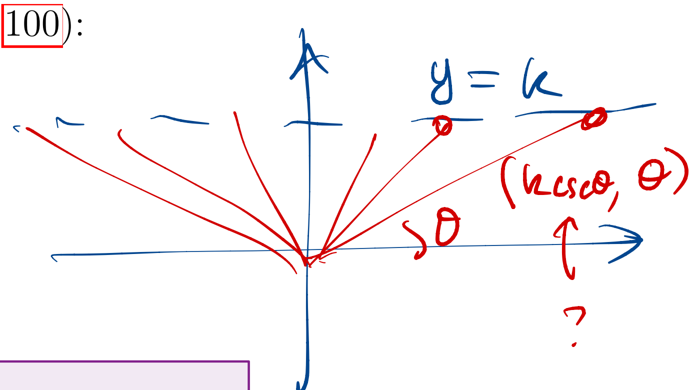
For ~~vertical~~ horizontal lines such as $y = 1$ (e.g., Example 100):

use ② get

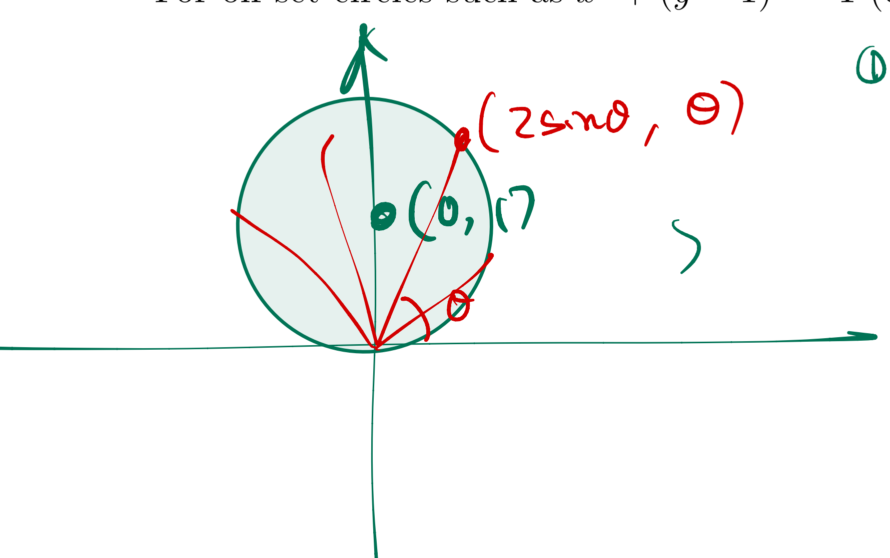
$1 = r\sin\theta$

So $r = \csc\theta$

So in general $\boxed{y = k \Rightarrow r = k\csc\theta}$



For off-set circles such as $x^2 + (y-1)^2 = 1$ (e.g., Example 98):



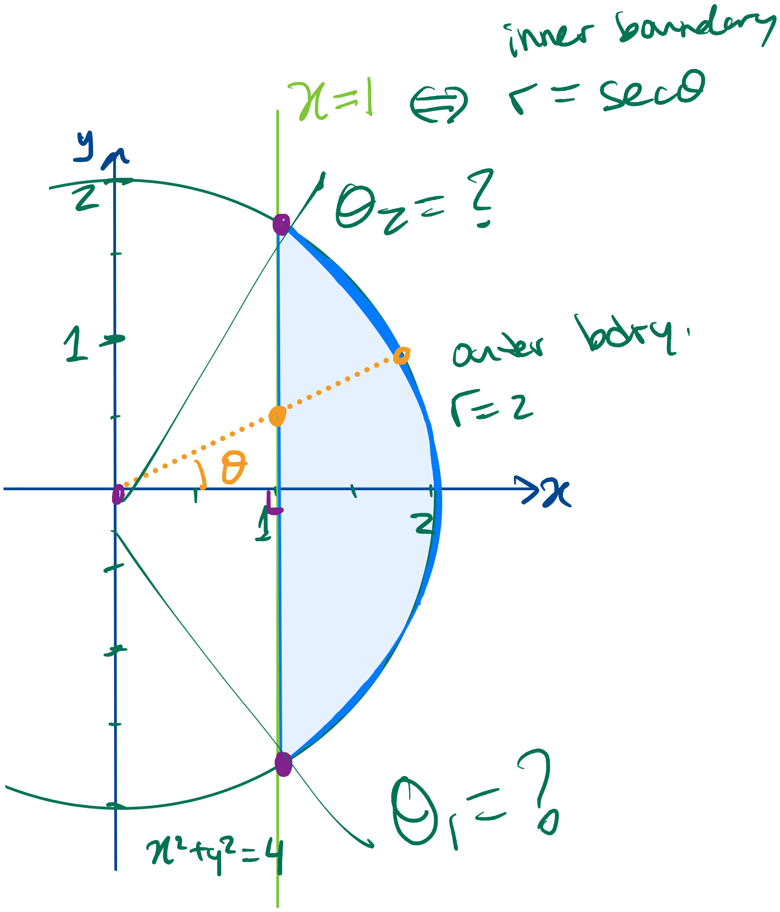
① & ② CTS.

$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$

$\Rightarrow r^2 - 2r\sin\theta = 0$

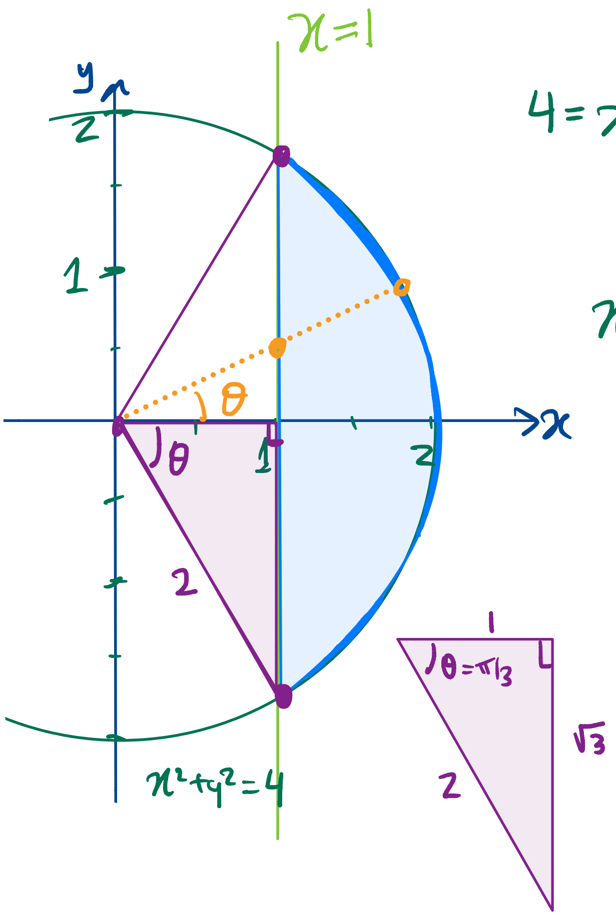
$\Rightarrow \boxed{r = 2\sin\theta}$

Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.



Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

The θ-value is between $(1,-\sqrt{3})$ & $(1,\sqrt{3})$ so $\theta \in [-\pi/3, \pi/3]$.



$4 = x^2 + y^2 = r^2 \Rightarrow r = \pm 2,\ r \geq 0$

so $r = 2$ on circle.

$x = r\cos\theta \Rightarrow r = \dfrac{x}{\cos\theta} = x\sec\theta$

@ $x=1$ $\quad r = \sec\theta$ on line $x = 2$.

So $\theta \in [-\pi/3, \pi/3]$

and $r \in [\sec\theta, 2]$

So $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$

$$\text{Area} = \int_{-\pi/3}^{\pi/3} \int_{\sec\theta}^{2} 1 * r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2} r^2 \Big|_{\sec\theta}^{2} d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(\sec^2\theta - 4\right) d\theta$$

$$= \frac{1}{2}\tan\theta - 2\theta \Big|_{-\pi/3}^{\pi/3} = \tan(\pi/3) - 4 * \pi/3 = \boxed{\sqrt{3} - 4\pi/3}$$

Function is odd

Math 2551 Worksheet: Exam 2 Review

1. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0)? Give reasons for your answers.

   (a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.

   (b) The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.

   (c) The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f.

   (d) At (x_0, y_0), the vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.

2. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi), x = e^t$, and $y = \ln(t + 1)$.

3. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.

4. Test the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points and find the function's value at these points.

5. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.

6. Evaluate the integral $\displaystyle\int_0^1 \int_{2y}^2 4\cos(x^2)\, dx\, dy$. Describe why you made any choices you did in the course of evaluating this integral.

7. If $f(x, y) \geq 2$ for all (x, y), is it possible that the average value of $f(x, y)$ on a unit disk centered at the origin is $\dfrac{2}{\pi}$?

8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.