§15.5-15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$M \approx \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

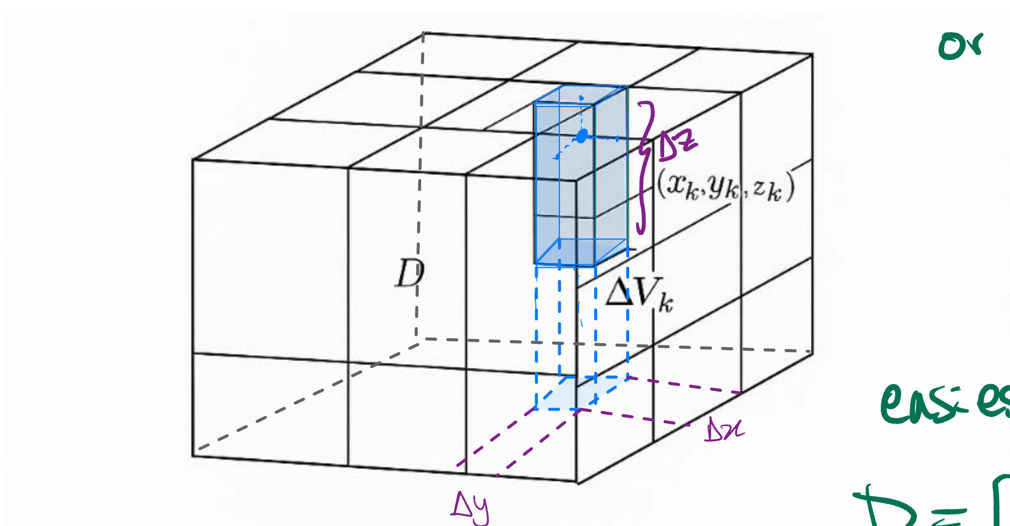
by breaking D into small rectangular prisms ΔV_k.

Idea $f: \mathbb{R}^3 \to \mathbb{R}$

or $f: D \to \mathbb{R}$

$D \subseteq \mathbb{R}^3$

domain of f.



easiest case

$D = [a,b] \times [c,d] \times [e,g]$

$\Delta V_k = \Delta x \Delta y \Delta z$

* plural-rectangles
* "boxes"
* rectangular prism.

triple integral

$$M = \iiint_D f(x,y,z)\, dV$$

"density function"

volume measure

Taking the limit gives a

triple integral : $\iiint_D f(x, y, z)\, dV$ ← $D \subseteq \mathbb{R}^3$

Important special case:

$$\iiint_D 1 \, dV = \text{volume.}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

$$\int_a^b \int_c^d \int_e^g f(x,y,z)\, dz\, dy\, dx = \int_c^d \int_e^g \int_a^b f(x,y,z)\, dx\, dz\, dy$$

Other important spatial applications:

§15.6

doesn't seem to appear in any sample exams?

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\quad$ $\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\, dV, \qquad M_{xz} = \iiint_D y\,\delta\, dV, \qquad M_{xy} = \iiint_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta \, dA$ $\quad$ $\delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\,\delta\, dA, \qquad M_x = \iint_R y\,\delta\, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

Shouldn't depend on z b/c current integration variable

Example 102. 1. **How to do the computation:**

Compute $M=\int_0^1 \int_0^{2-x} \left(\int_0^{2-x-y} 1 \, dz \right) dy\, dx.$

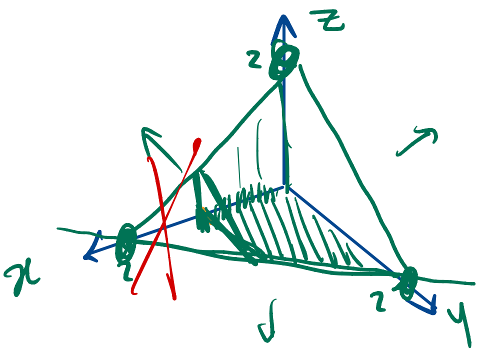
$$M=\int_0^1\int_0^{2-x}\left(z\Big|_0^{2-x-y}\right) dy\, dx = \int_0^1\int_0^{2-x} (2-x-y)-0\, dy\, dx$$

$$= \int_0^1 2y-xy-\tfrac12 y^2 \Big|_0^{2-x} dx = \int_0^1 \Big(2(2-x)-x(2-x)-\tfrac12\underbrace{(2-x)^2}_{4-4x+x^2}\Big)-0\, dx$$

$$= \int_0^1 4-2x-\cancel{2x}+x^2-2+\cancel{2x}-\tfrac12 x^2\, dx = \int_0^1 2-2x+\tfrac12 x^2\, dx$$

$$= 2x-x^2+\tfrac16 x^3\Big|_0^1 = \left(2-1+\tfrac16\right)-0 = \boxed{7/6}$$

2. **What does it mean:** What shape is this the volume of?



z-range $z\in[0,\, 2-x-y]$ top $z=2-x-y$

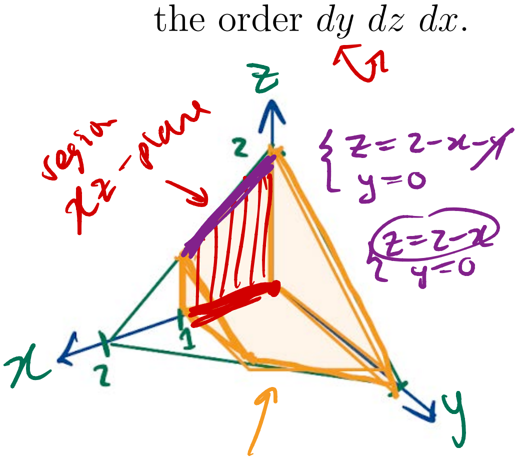
$\Rightarrow$ $\boxed{x+y+z=2}$

$\{y=2x,\ z=0\}$

Shouldn't depend on inner most integration variable.

3. **How to reorder the differentials:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

$$\int_0^1 \int_0^{2-x} \int_0^{\boxed{2-x-z}} 1\ dy\, dz\, dx$$



$\begin{cases} x\in[0,1) \\ z\in[0,2-x] \end{cases}$ $(y=0)$

region in the xz-plane

$z=2-x-y$

$\Rightarrow y=2-x-z$ ("top")

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\ dy\ dz$

(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\ dx\ dy\ dz$

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\; dy\; dz$

$$= \int_1^e \int_1^{e^2} \frac{1}{yz} \ln(x)\Big|_1^{e^3} dy\, dz = \int_1^e \int_1^{e^2} \frac{1}{yz} * 3 \; dy\, dz$$

$$= \int_1^e \frac{3}{z} \ln y \Big|_1^{e^2} dz = \int_1^e \frac{3}{z}(2-0)\, dz = 6 \ln z \Big|_1^e = 6 - 0$$

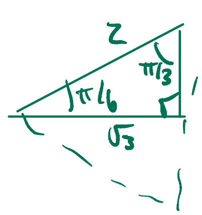
$$= \boxed{6}$$

(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\; dx\; dy\; dz$

$$= \int_0^{\pi/3} \int_0^1 y \sin z \cdot x \Big|_{-2}^3 dy\, dz = \int_0^{\pi/3} \int_0^1 5y \sin z\, dy\, dz$$

$$= \int_0^{\pi/3} \frac{5}{2} \sin z \cdot y^2 \Big|_0^1 dz = \int_0^{\pi/3} \frac{5}{2} \sin z\, dz$$

$$= -\frac{5}{2} \cos z \Big|_0^{\pi/3} = -\frac{5}{2}\left[\cos(\pi/3) - \cos(0)\right]$$



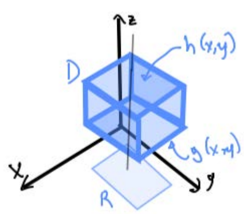
$$= -\frac{5}{2}\left(\frac{1}{2} - 1\right)$$

$$= \boxed{5/4}$$

We will think about converting triple integrals to iterated integrals in terms of the projection of D on one of the coordinate planes.

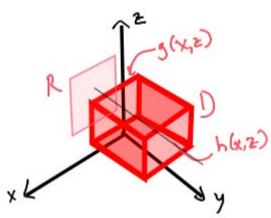
Case 1: z**-simple**) region. If R is the projection of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz \right)\ dy\ dx$$



Case 2: y**-simple**) region. If R is the projection of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} \underline{f(x,y,z)\ dy} \right)\ dz\ dx$$

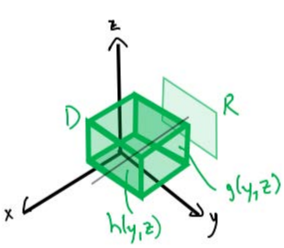


inner bounds shouldn't have y-variable

also shouldn't have y-variable.
(all bounds depend on x, z only)

Case 3: x**-simple**) region. If R is the projection of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\ dx \right)\ dz\ dy$$



$x \geq 0, y \geq 0, z \geq 0$

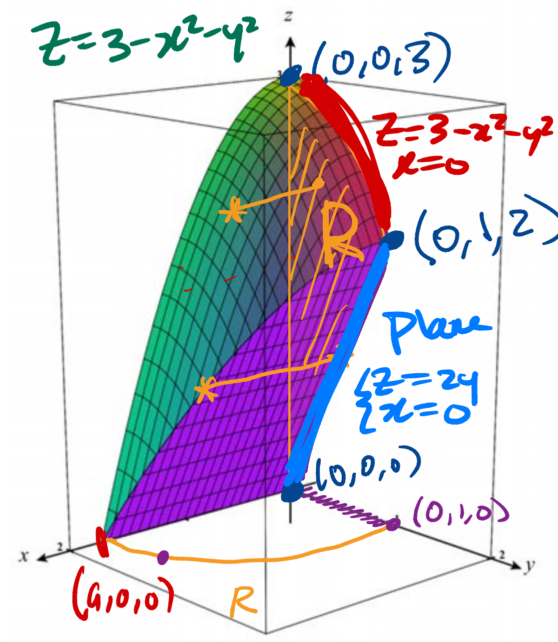
bot top

Example 104. Write an integral for the mass of the solid D in the first octant with $\boxed{2y} \leq z \leq \boxed{3 - x^2 - y^2}$ with density $\boxed{\delta(x, y, z) = x^2y + 0.1}$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

Case 1:

x-simple ✓

$$\iint_R \int_{g(y,z)}^{h(y,z)} \delta(x,y,z)\, dx\, dz\, dy = M$$



middle intersection @ $x = 0$ and

$$2y = 3 - x^2 - y^2$$

$$\Rightarrow 2y = 3 - y^2$$

$$\Rightarrow y^2 + 2y - 3 = 0$$

$$\Rightarrow (y+3)(y-1) = 0 \quad y = -3, \; y = 1$$

Express R in yz-plane.

$y \in [0,1]$ then $z \in [\underbrace{2y}_{\text{bot}}, \underbrace{3-y^2}_{\text{top}}]$ then top is

$$z = 3 - x^2 - y^2$$

$$\Rightarrow x^2 = 3 - z - y^2$$

$$\Rightarrow x = \pm\sqrt{3 - z - y^2}$$

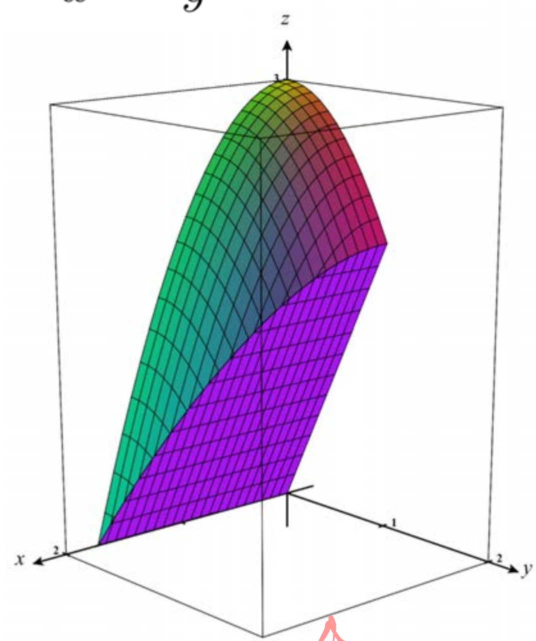
$$\boxed{M = \int_0^1 \int_{2y}^{3-y^2} \int_0^{\sqrt{3-z-y^2}} x^2y + 0.1 \; dx\, dz\, dy}$$

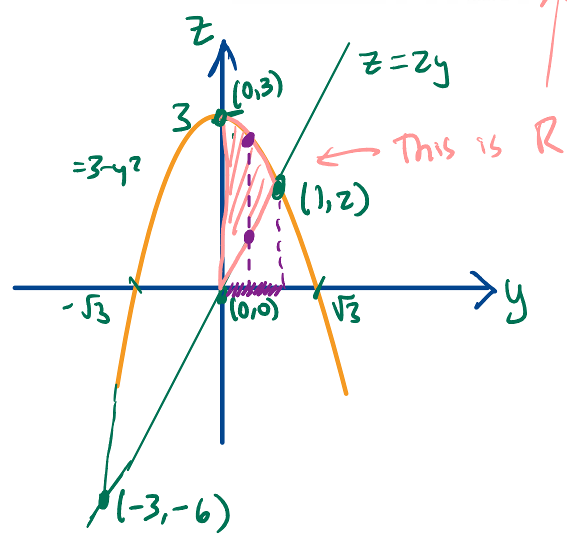
Example 104 (cont.) D: $2y \le z \le 3 - x^2 - y^2$

Case 3:

(b) x-simple

$$M = \iint_R \int_{g(x,y)}^{h(x,y)} \delta(x,y,z)\, dx\, dz\, dy$$





$-z$

a Few Short Words...

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

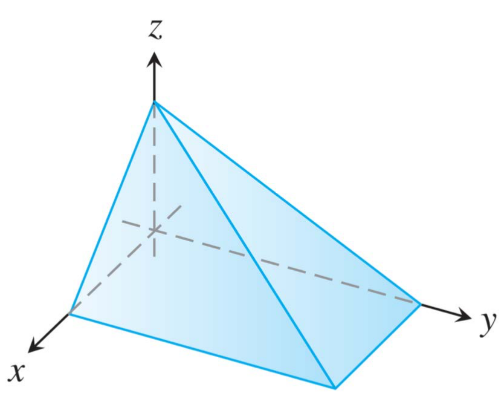
Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 105. *You try it!* ~~Find~~ Set up. the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$.



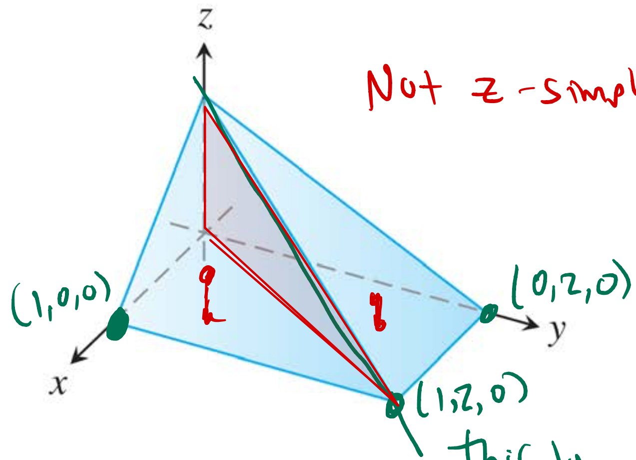
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$

quadrant

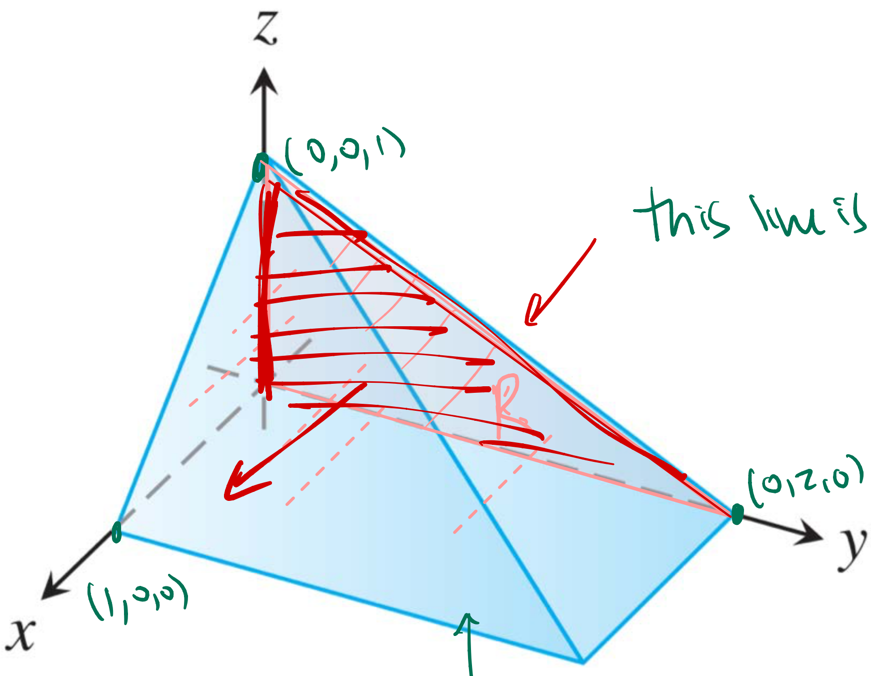
$$\int_*^* \int_*^* \int_*^* 1 \, dz \, dx \, dy$$

bad news.

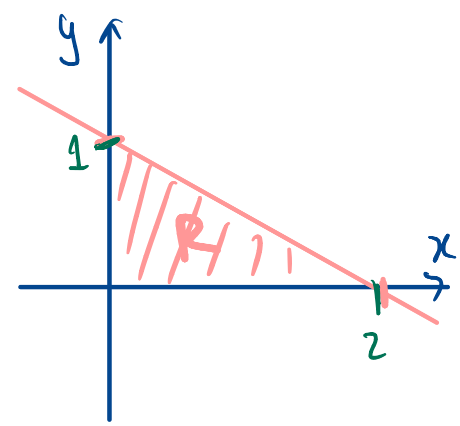
Not z-simple!



This line $\begin{cases} x+z=1 \\ y+2z=2 \end{cases}$ @ $z=0$: $x=1$, $y=2$ @ $x=y=0$: $z=1$



this line is $\begin{cases} x=0 \\ y+2z=2 \end{cases} \Rightarrow \begin{cases} x=0 \\ z=1-\frac{1}{2}y \end{cases}$



this plane is $x+z=1$

$\Leftrightarrow x=1-z$

So

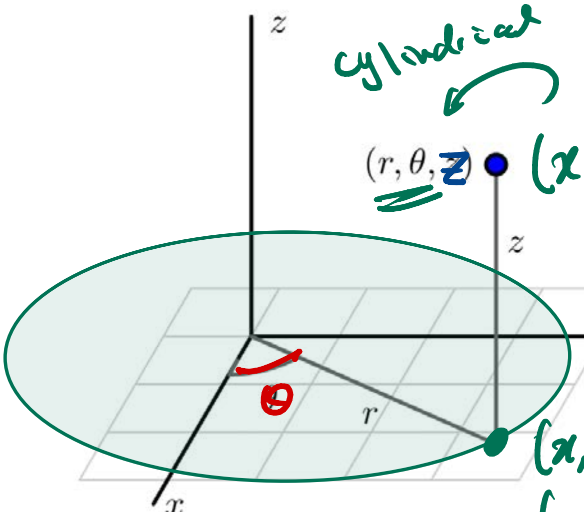
dy dz

$$\boxed{\text{Vol} = \int_0^2 \int_0^{1-\frac{1}{2}y} \int_0^{1-z} 1 \, dx \, dz \, dy}$$

§15.7 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinate System

cylindrical



(x, y, z)

$(x,y,0)$ in floor

$(r,\theta,0)$ using polar coord

Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

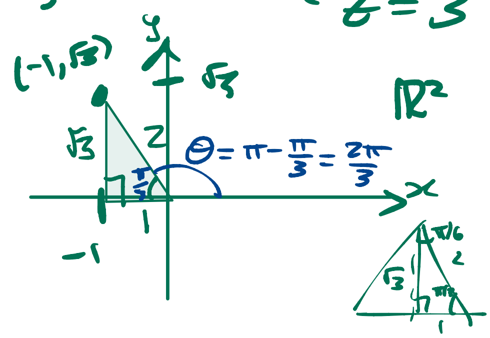
Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

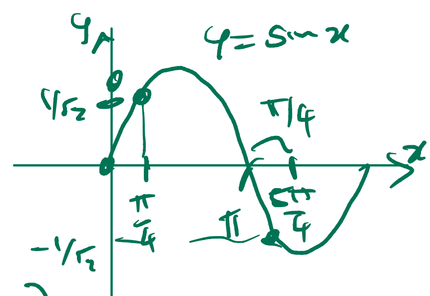
Conventions: As before for double-integrals: $r \geq 0$ and $\theta \in [0, 2\pi]$ or $\theta \in [-\pi, \pi]$

Example 108. a) Find cylindrical coordinates for the point with Cartesian coordinates $\boxed{(-1, \sqrt{3}, 3)_e}$. $= (2, 2\pi/3, 3)_{cy}$

$$\begin{cases} x = -1 \\ y = \sqrt{3} \\ z = 3 \end{cases} \xrightarrow{\text{cylindrical}} \begin{cases} r = 2 \\ \theta = 2\pi/3 \\ z = 3 \end{cases}$$



b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.
$\uparrow_r \quad \uparrow_\theta \quad \uparrow_z$

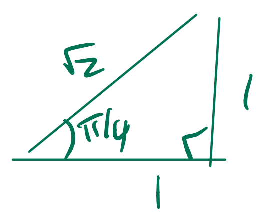


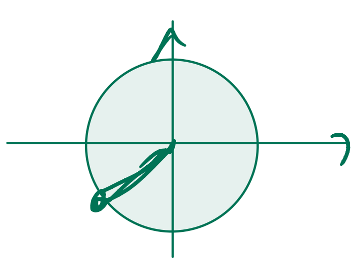
formulas:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

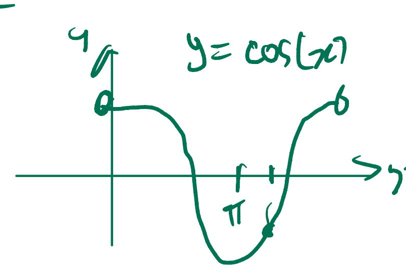
@ $(r,\theta,z) = (2, 5\pi/4, 1)$

get $x = 2\cos 5\pi/4 = 2 \times (-1/\sqrt{2})$
$y = 2\sin 5\pi/4 = 2 \times (-1/\sqrt{2})$
$z = 1$





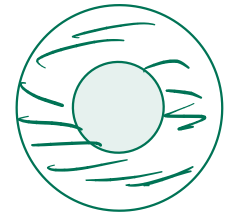
$$\boxed{(-\sqrt{2}, -\sqrt{2}, 1)_e}$$

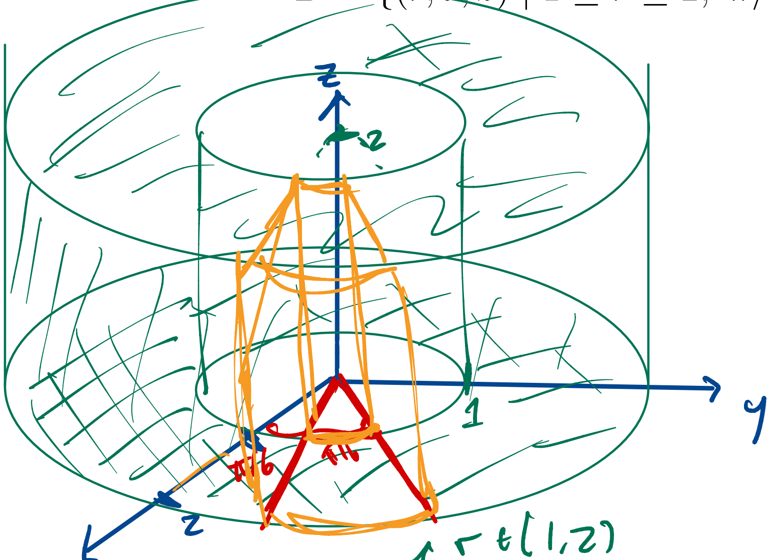


$\cos(\pi/4) = 1/\sqrt{2}$
$\sin(\pi/4) = 1/\sqrt{2}$

Example 109. In xyz-space sketch the *cylindrical box*

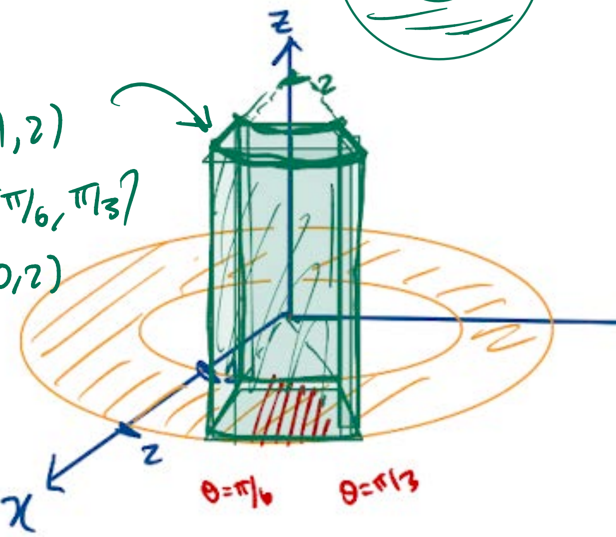
$$B = \{(r, \theta, z) \mid 1 \le r \le 2,\ \pi/6 \le \theta \le \pi/3,\ 0 \le z \le 2\}.$$





$r \in [1,2]$
$\theta \in [0, 2\pi]$
$z = 0$

$r \in [1,2]$
$\theta \in [\pi/6, \pi/3]$
$z \in [0,2]$



Triple Integrals in Cylindrical Coordinates

integration factor
same as polar

We have $dV =$ $r\,dz\,d\theta\,dr$

$$\iiint_D f(x,y,z)\,dV \overset{??}{=} \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\int_{z_1}^{z_2} f(r,\theta,z)\, r\, dz\, d\theta\, dr$$

volume measure

Example 110. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $\boxed{z = x + 2}$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

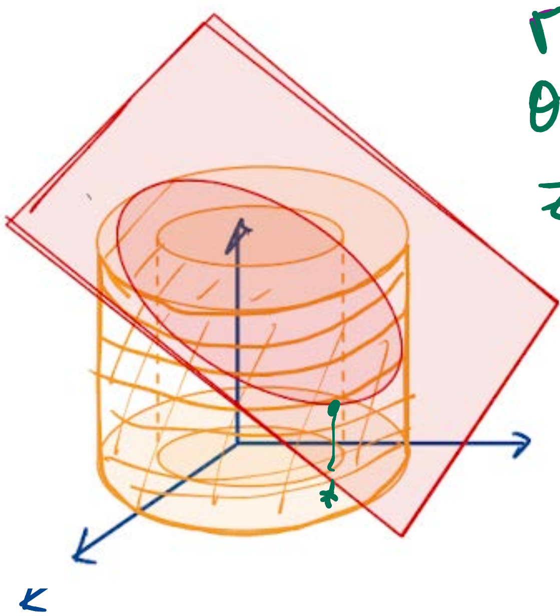
not 4. b/c $x^2+y^2=r^2$

$r \in [1,2]$ ✓

$\theta \in [0, 2\pi]$ or $[-\pi, \pi]$ ✓

$z \in [0, x+2]$ or $z \in [0, r\cos\theta + 2]$

put $x = r\cos\theta$ into cylind. coords.

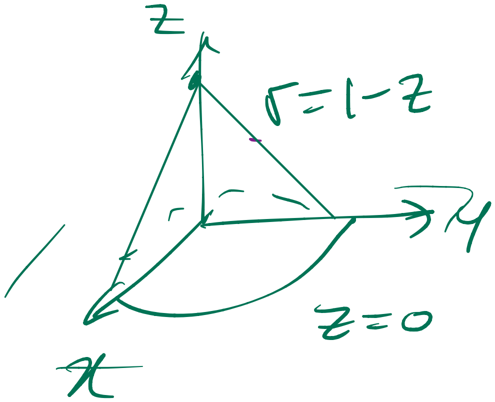


$$\mathrm{Vol}(D) = \iiint_D 1\, dV$$

integration factor

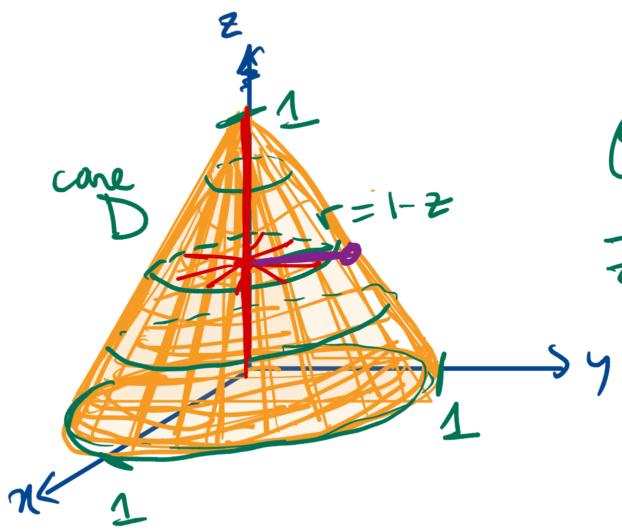
$$= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} 1 * r\, dz\, dr\, d\theta$$

Example 111. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.



Example 111. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

$\delta = z \stackrel{?}{=} 1 - r$



$\theta \in [0, 2\pi]$

$z \in [0, 1]$

$r \in [0, 1-z]$

non constant density function

So $M = \iiint_D \delta(r, \theta, z)\, r \, dr\, dz\, d\theta$

$$M = \boxed{\int_0^{2\pi} \int_0^1 \int_0^{1-z} z\, r \, dr\, dz\, d\theta}$$

$(1-r)$?

other options also work

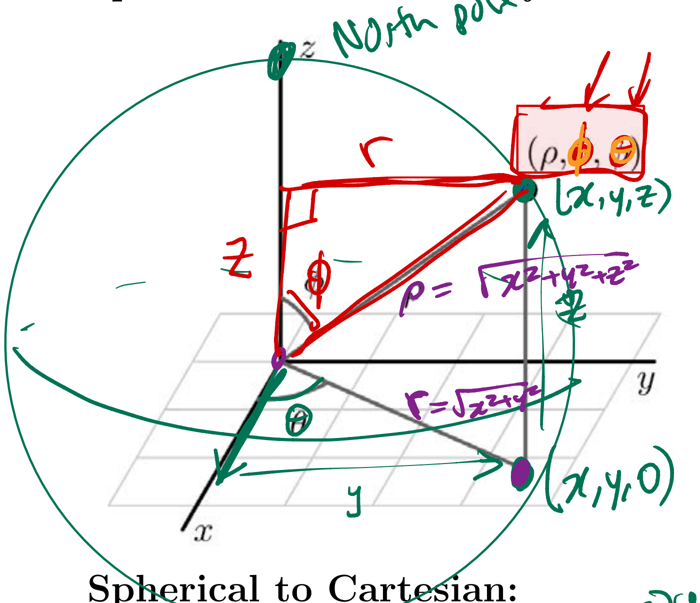
$$M = \int_0^{2\pi} \int_0^1 \int_0^{1-r} z\, r \, dz\, dr\, d\theta$$

or even

$$M = \int_0^1 \int_0^{1-r} \int_0^{2\pi} z\, r \, d\theta\, dz\, dr$$ also fine!

Polar & cylindrical $\theta \in [0, 2\pi)$ ✓ or $\theta \in [-\pi, \pi)$ ✓

Spherical Coordinate System



Conventions: $\rho \geq 0$ (same as r)

$\theta \in [0, 2\pi)$ or $[-\pi, \pi)$ (same)

$\phi \in [0, \pi)$ always.

WARNING: $\phi = 0$ is UP not along xy-plane

Spherical to Cartesian:

$$\boxed{\begin{aligned} x &= \rho\sin(\varphi)\cos(\theta) \\ y &= \rho\sin(\varphi)\sin(\theta) \\ z &= \rho\cos(\varphi) \end{aligned}}$$

Cartesian to Spherical:

$\boxed{\varphi \text{ vs. } \phi \text{ same}}$

$$\rho^2 = x^2 + y^2 + z^2$$

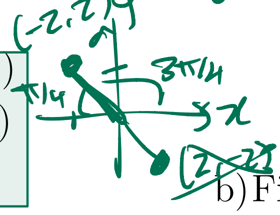
$$\tan(\theta) = \frac{y}{x}$$

$$\tan(\varphi) = \frac{\sqrt{x^2+y^2}}{z}$$

Example 112. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4+4+8} = \sqrt{16} = 4.$

$\tan\theta = y/x = 2/-2 = -1 \quad \theta = \cancel{-\pi/4} \text{ or } \frac{3\pi}{4}.$



$\tan\varphi = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{4+4}}{\sqrt{8}} = \frac{\sqrt{8}}{\sqrt{8}} = 1 \quad \varphi = \pi/4.$

$\boxed{(4, \pi/4, 3\pi/4)_S}$

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

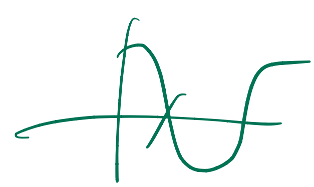
ϕ θ

$x = \rho\sin(\phi)\cos(\theta) = 2\sin(\pi/2)\cos(\pi/3) = 2(1)\cdot\frac{1}{2} = 1$

$y = \rho\sin(\phi)\sin\theta = 2\sin(\pi/2)\sin(\pi/3) = 2(1)\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

$z = \rho\cos(\phi) = 2\cos(\pi/2) = 0$

$(x,y,z) = \boxed{(1, \sqrt{3}, 0)_C}$



🌀 The two common lowercase phi symbols:

Fee ϕ

Fi φ

ϕ (curly phi) — sometimes called “script phi” or “open phi”

Looks like a curly or loopy “C” with a vertical line

Unicode: U+03D5

Often used in physics and engineering (e.g., magnetic flux)

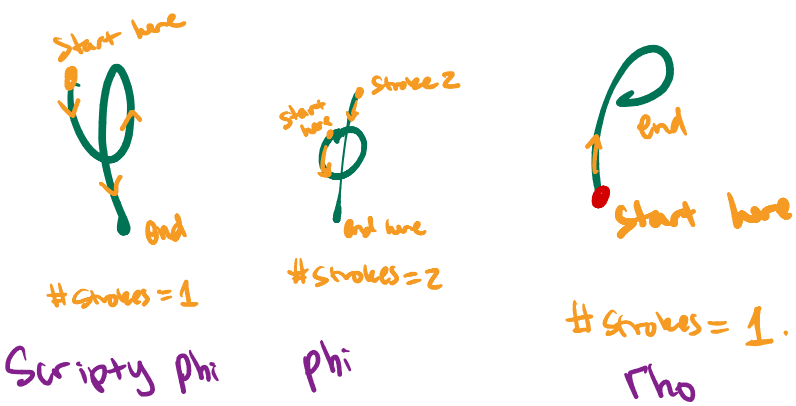
φ (straight phi) — often just “phi”

Looks like a circle with a vertical line through it

Unicode: U+03C6

Often used in math, philosophy, and logic

How to draw them:



P “pee”

ρ “rho”

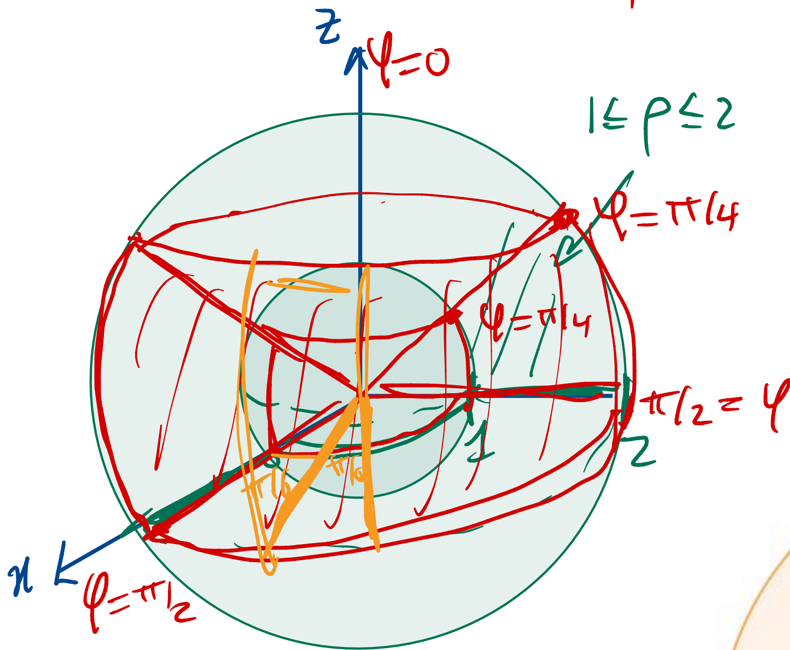
$\rho \in [1,2]$
$\varphi \in [\pi/4, \pi/2]$
$\theta \in [\pi/6, \pi/3]$

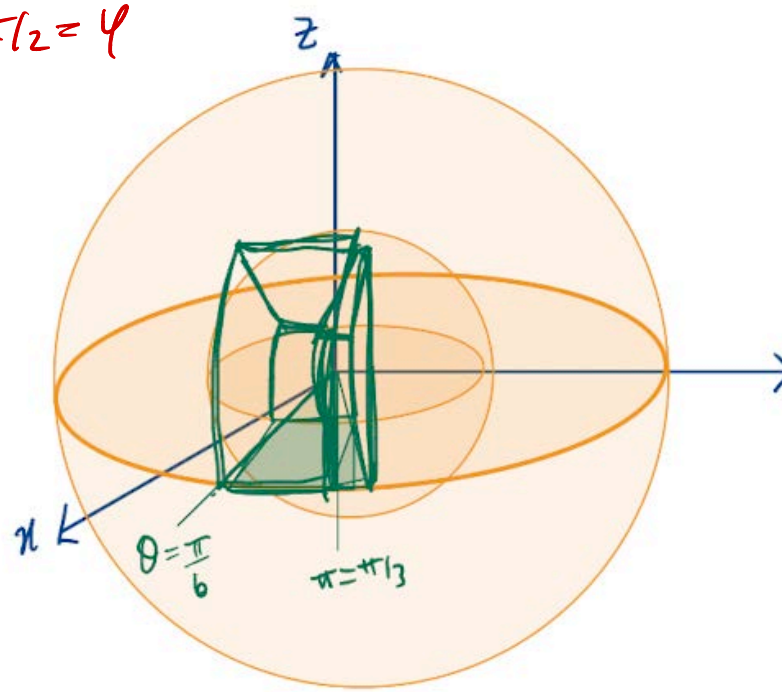
Example 113. In xyz-space sketch the *spherical box*

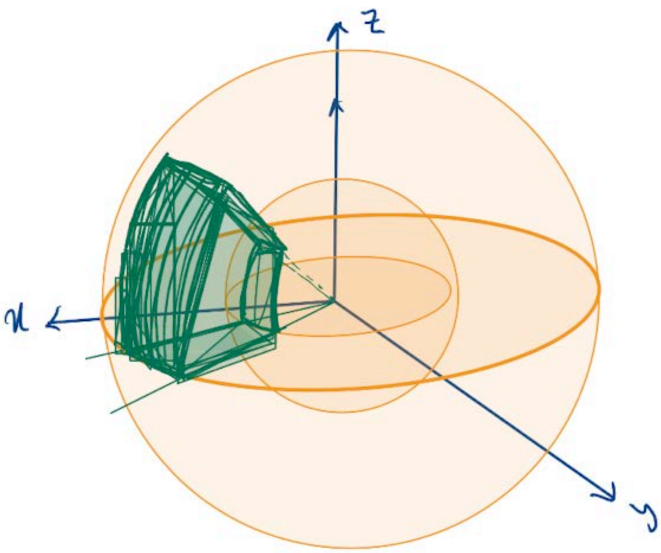
$$B = \{(\rho, \varphi, \theta) \mid 1 \le \rho \le 2, \cancel{0}\frac{\pi}{4} \le \varphi \le \pi/\cancel{4}2, \ \pi/6 \le \theta \le \pi/3\}.$$

* According to Dr. H.

"Spheres, curves, and planes through the z-axis are fair game."







integration factor?

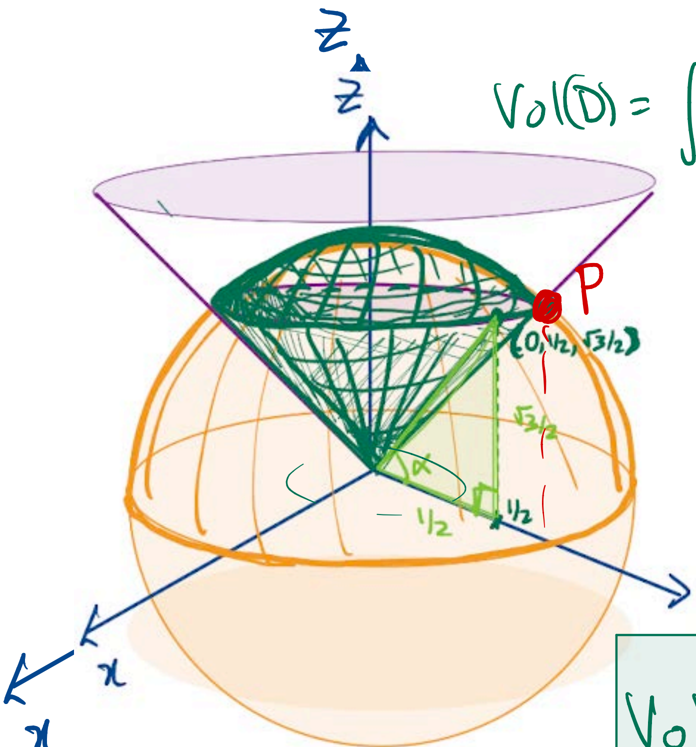
Triple Integrals in Spherical Coordinates

We have $dV = \rho^2 \sin\varphi \; d\varphi \, d\theta \, d\rho$, $\delta(x,y,z) = 1$.

Example 114. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.

$\rho^2 = 1 \Rightarrow \rho = 1$

$z = \sqrt{3}\, r$ (polar)



$\mathrm{Vol}(D) = \iiint_D 1 \; dV$

$\rho \in [0, 1]$?

$\varphi \in [0, ?]$?

$\theta \in [0, 2\pi)$ ✓

$$\boxed{\mathrm{Vol}(D) = \int_0^{2\pi} \int_0^{1} \int_0^{\pi/6} 1 * \rho^2 \sin\varphi \; d\varphi \, d\rho \, d\theta}$$

point $P(x,y,z)$ above y-axis

so $x=0$ on sphere $x^2+y^2+z^2=1$ @ $x=0$

on cone $z = \sqrt{3}\sqrt{x^2+y^2}$

$\Rightarrow$ $y^2+z^2=1$

$z = \sqrt{3}\, y$ $(y \geq 0)$

$\Rightarrow$ $y^2 + 3y^2 = 1 \Rightarrow y^2 = 1/4$

$\Rightarrow y = 1/2$

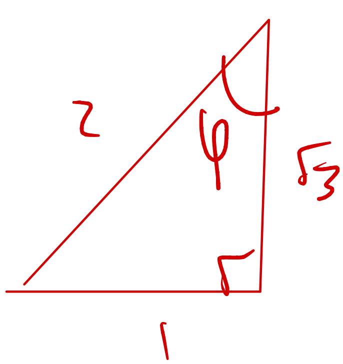
and $z = \sqrt{3}\, y$ so $z = \sqrt{3}/2$

$P(0, 1/2, \sqrt{3}/2)$

need φ

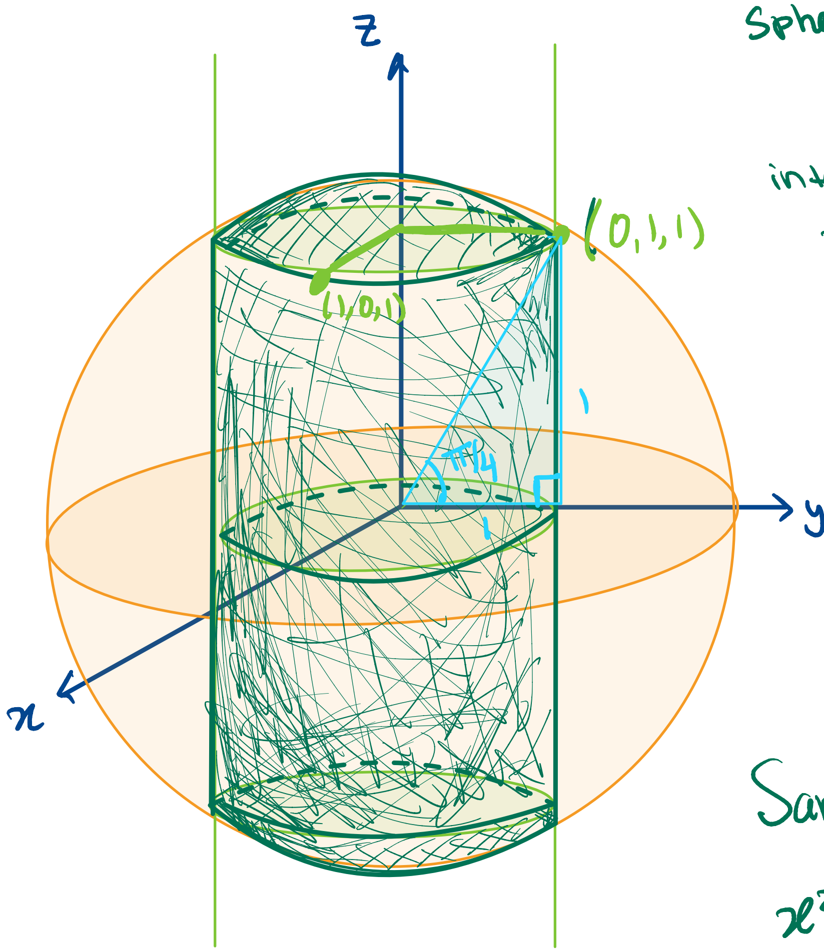
$\tan\varphi = \dfrac{1/2}{\sqrt{3}/2} = 1/\sqrt{3}$

$\varphi = \pi/6$



Example 115. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

Example 115. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.



Sphere:
$x^2+y^2+z^2=2 \iff \rho=\sqrt{2}$

intersection
$x^2+y^2+z^2=2$ & $x^2+y^2=1$

$\Rightarrow 1+z^2=2$

$\Rightarrow z^2=1 \Rightarrow z=\pm 1$

one point in intersection is $(0,1,1)_{\mathcal{C}} = (1, \frac{\pi}{2}, \pi/4)_{\mathcal{S}}$

Same idea as last example

$x^2+y^2=\rho^2\sin^2\varphi$ so

cylinder is $\rho^2\sin^2\varphi=1$

$\Rightarrow \rho\sin\varphi=1$

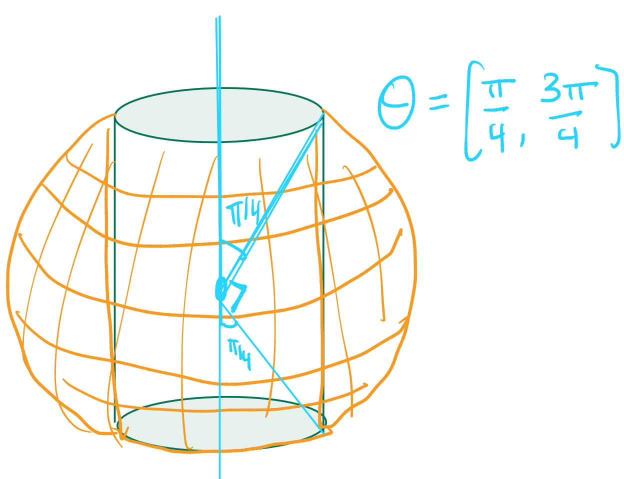
$\Rightarrow \rho=\csc\varphi$

we want outer part outside cylinder & inside sphere so

$\varphi\in\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$\rho\in[\csc\varphi, \sqrt{2}]$

and $\theta\in[0,2\pi)$



$$\boxed{\mathrm{Vol} = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_{\csc\varphi}^{\sqrt{2}} \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta}$$