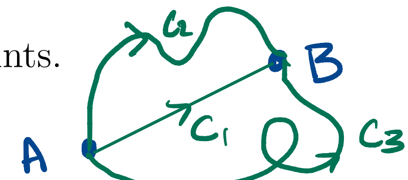
§16.3 Conservative Vector Fields & Fundamental Theorem

Definition 132. A vector field **F** is **path independent** on an open region D if $\int_C F \cdot T\, ds$ same for all paths C in the region that have the same endpoints.



$$\int_{C_1} F\cdot T\,ds = \int_{C_2} F\cdot T\,ds = \int_{C_3} F\cdot T\,ds = \dots$$

etc.

When **F** is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

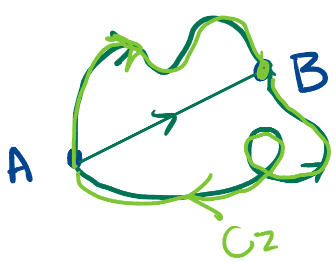
$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F}\cdot\mathbf{T}\,ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F}\cdot d\mathbf{r}.$$

Example 133. If C is any closed path and **F** is path independent on a region containing C, then

loop $C = C_1 \cup C_2$

C_1

since



$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$$

$$= \int_A^B F\cdot dr + \int_B^A F\cdot dr$$

$$= \int_A^B F\cdot dr - \int_A^B F\cdot dr$$

$$= 0.$$

Question: Given **F**, how do we tell if it is path independent on a particular region?

Several options!

① Check some paths C_1, C_2 if $\int_{C_1} F \cdot T\, ds \neq \int_{C_2} F \cdot T\, ds$, then F is NOT path independent.

② check some CLOSED LOOPS C to check if $\int_C F \cdot T\, ds = 0$.

doesn't always work?

For example, is $\mathbf{F}(x, y) = \langle x, y\rangle$ a path independent vector field on its domain?

Try ② w/ $C: r(t) = \langle \cos t, \sin t\rangle$, $t \in [0, 2\pi]$ unit circle.

$$\text{Then } \int_C F \cdot T\, ds = \int_0^{2\pi} \langle \cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$$

$$= \int_0^{2\pi} -\sin t \cos t + \sin t \cos t\, dt$$

$$= \int_0^{2\pi} 0\, dt = \boxed{0}$$

Still can NOT conclude path ind.

Example 134. *You try it!* Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x\rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

Question: Given $\mathbf{F}$, how do we tell if it is path independent on a particular region?

Several options:

① Check some paths C_1, C_2 if $\int_{C_1} F \cdot T\, ds \neq \int_{C_2} F \cdot T\, ds$, then F is <u>NOT</u> path independent.

② check some CLOSED LOOPS C to check if $\int_C F \cdot T\, ds = 0$.

For example, is $\mathbf{F}(x, y) = \langle x, y\rangle$ a path independent vector field on its domain?

Try ② w/ $C: r(t) = \langle \cos t, \sin t\rangle$, $t \in [0, 2\pi]$

Then $\int_C F \cdot T\, ds = \int_0^{2\pi} \langle \cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$

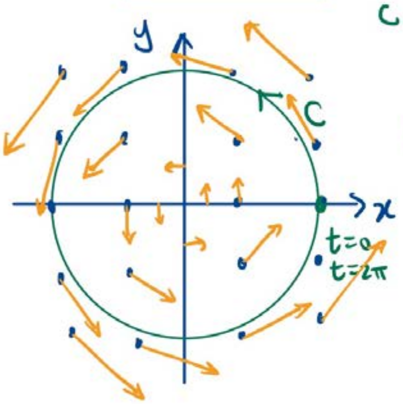
ANS: Not sure yet!

$= \int_0^{2\pi} -\sin t \cos t + \sin t \cos t\, dt = \int_0^{2\pi} 0\, dt$

$= 0$. Ok but other loops?

Example 134. *You try it!* Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x\rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

Example 129. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x\rangle$ cm/s around the unit circle, parameterized counterclockwise.



$C: r(t) = \langle \cos t, \sin t\rangle$ $t \in [0, 2\pi]$

$r'(t) = \langle -\sin t, \cos t\rangle$ $|r'(t)| = 1$

$\text{Flow} = \int_a^b F(\vec{r}(t)) \cdot r'(t)\, dt$

$\text{Flow} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$

$= \int_0^{2\pi} \underbrace{\sin^2 t + \cos^2 t}_{1}\, dt$

$= \int_0^{2\pi} 1\, dt = t \Big|_0^{2\pi} = \boxed{2\pi}$

If we reverse orientation of C, then what would Flow value be?

(pick) $0, 2\pi, \boxed{-2\pi}$?

So $F(x, y) = \langle -y, x\rangle$ is <u>NOT</u> path independent.

A different idea: Suppose **F** is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a potential function for **F**. In this case we also say that **F** is **conservative.**

Is $\mathbf{F}(x, y) = \langle x, y\rangle$ conservative? Need to find $f(x,y)$ s.t. $\nabla f = F$.

$$f(x,y) = \underbrace{\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2}_{\text{Scalar function}} \checkmark \qquad \text{Check: } f_x = x, \ f_y = y \quad \nabla f = \langle x, y\rangle \checkmark$$

Step 1. we know $\frac{\partial}{\partial x} f = x$ So $f = \int x\, dx = \frac{1}{2}x^2 + C(y)$

Step 2 $\frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + C(y)\right) = y$ but $\frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + C(y)\right) = 0 + C'(y) = y.$

So $C'(y) = y \Rightarrow \int y\, dy = \frac{1}{2}y^2 + C = C(y)$

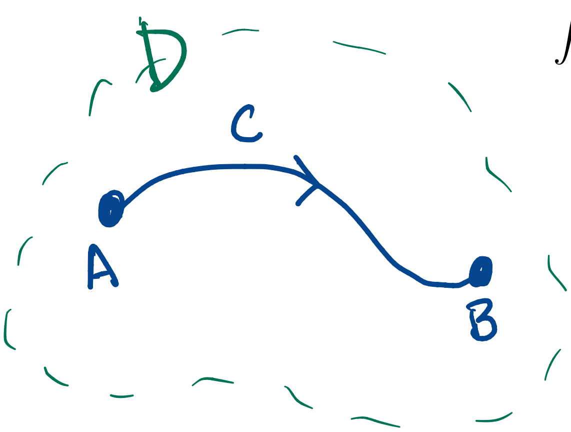
So $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

Theorem 135 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

$$\int_C \overset{F}{\nabla f} \cdot \mathbf{T}\, ds = \underbrace{f(B) - f(A)}$$

Compare to FTC

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

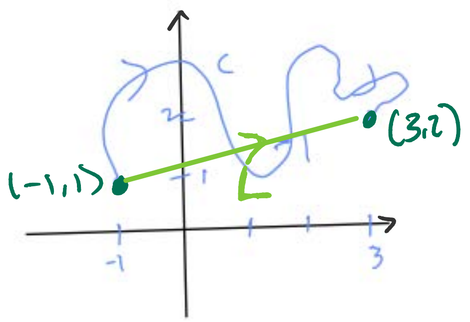


doesn't depend on the curve C at all, only start & end.

So conservative $F = \nabla f$ is automatically path ind.

Flow $= \int_C \langle x,y\rangle \cdot dr = \boxed{\int_C F\cdot T\, ds}$ $\quad F = \langle x,y\rangle.$

Example 136. Compute $\int_C \langle x, y\rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.



use straight line instead since $F = \langle x,y\rangle$ is conservative w/ potential function $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

$$\int_C \langle x,y\rangle \cdot dr \overset{\text{path independence of } F}{=} \int_L \langle x,y\rangle \cdot dr$$

$$\overset{\text{notation}}{=} \int_{(-1,1)}^{(3,2)} \langle x,y\rangle \cdot dr$$

$$\overset{\text{FToLI}}{=} f(3,2) - f(-1,1)$$

$$= \left(\tfrac{1}{2}(3)^2 + \tfrac{1}{2}(2)^2 + C\right) - \left(\tfrac{1}{2}(-1)^2 + \tfrac{1}{2}(-1)^2 + C\right)$$

$$= \frac{9}{2} + \frac{4}{2} - \left(\frac{1}{2} + \frac{1}{2}\right) = \boxed{11/2}$$

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1$$

NOTE: trying the same trick to find f w/ $\nabla f = \langle -y, x\rangle$ MUST FAIL (why?)

Let's try anyways.

① $\frac{\partial}{\partial x} f = -y \Rightarrow f = -yx + C_1(y)$

② $\frac{\partial}{\partial y} f = x \Rightarrow f = yx + C_2(x)$

no way to choose C_1 & C_2 so that you get f here.

(This is a strategy for showing F is not conservative, so also not path independent)

$\left(F = \nabla f\right) \Rightarrow \left(\int_{C_1} F \cdot T\, ds = \int_{C_2} F \cdot T\, ds\right)$ for any C_1, C_2.

It follows, from THM 135, that **every conservative field is path independent.**

Does the implication go the other way, too? (Iff?)

In fact, by carefully constructing a potential function, we can show the converse is also true: every path independent F is Conservative.

Idea: $f(x,y) = \int_{(a,b)}^{(x,y)} F \cdot T\, ds$

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0, 0, 0\rangle$, then $\mathbf{F}$ is conservative.

$F(x,y) = \langle P(x,y), Q(x,y), 0\rangle \qquad R = 0$

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} = \langle 0-0, 0-0, Q_x - P_y\rangle = \langle 0,0,0\rangle$
- This is also called the **mixed-partials test**, because So $\boxed{Q_x = P_y}$

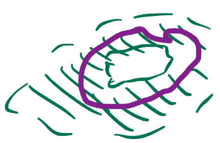
① $F(x,y) = \langle x, y\rangle$ ($\downarrow P \downarrow Q$)

$Q_x = 0$, $P_y = 0$ ✓ conservative F

② $F(x,y) = \langle -y, x\rangle$

$Q_x = 1$, $Q_y = -1$ not conservative F.

Simply connected means "no holes"



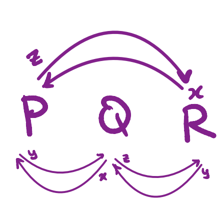
loops have to not be able to go around parts of the domain that aren't included.

Not simply connected ✗

Simply Connected ✓

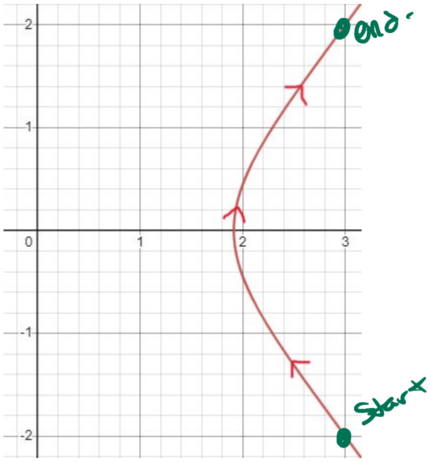
textbook calles the Curl test the Component test/mixed partials

$F = \langle P, Q, R\rangle$ conservative on simply connected D iff $R_y = Q_z$, $P_z = R_x$, $Q_x = R_y$



NEW NOTATION: $\int_C P\,dx + Q\,dy = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where $\mathbf{F} = \langle P, Q\rangle$

Example 137. Evaluate $\int_C (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$ where C is the part of the curve $\boxed{x^5 - 5x^2y^2 - 7x^2 = 0}$ from $(3,-2)$ to $(3,2)$.



Let's check the Curl Test for $F = \langle P, Q\rangle$.

$Q(x,y) = -3x^2y^2$

$Q_x = -6xy^2$

$\boxed{Q_x = P_y}$ Curl test

$P(x,y) = 10x^4 - 2xy^3$

$P_y = 0 - 6xy^2$

$Q_x = P_y$ so $F = \langle 10x^4 - 2xy^3, -3x^2y^2\rangle$ is conservative.

to use FToLI we need $f(x,y)$ s.t. $\nabla f = F$.

① $f_x = P = 10x^4 - 2xy^3 \Rightarrow f = 2x^5 - x^2y^3 + C(y)$

② $f_y = \frac{\partial}{\partial y}\left(2x^5 - x^2y^3 + C(y)\right) = -3x^2y^2 + C'(y) \overset{?}{=} Q = -3x^2y^2$

So $C'(y) = 0 \Rightarrow C(y) = k$ constant

$f(x,y) = 2x^5 - x^2y^3 + \cancel{k}$ (can choose $k=0$)

$$\int_C F\cdot dr \overset{\text{path ind}}{=} \int_{(3,-2)}^{(3,2)} F\cdot dr \overset{\text{FToLI}}{=} f(3,2) - f(3,-2)$$

$$= \left[\left(\cancel{2(3)^5} - (3)^2(2)^3\right) - \left(\cancel{2(3)^5} - (3)^2(-2)^3\right)\right]$$

$$= -3^2 2^4 = \boxed{-144}$$

§16.4 Divergence, Curl, Green's Theorem

\nabla

not really a vector except formally.

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x,y,z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field: $\leftarrow F = \langle P, Q, R\rangle$

- $\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R = P_x + Q_y + R_z$

  $= \text{div}(F)$ by definition

  outputs a scalar

- $\nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \hat{i}(R_y - Q_z) - \hat{j}(R_x - P_z) + \hat{k}(Q_x - P_y)$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= \text{curl}(F) \text{ by defn.}$$

outputs a vector.

Ex. Find div(F) & curl(F), $F = \langle \overset{P}{xy}, \overset{Q}{2y^2}, \overset{R}{x+z} \rangle$

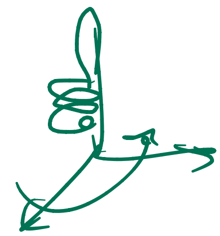
$$\text{div} F = P_x + Q_y + R_z = y + 4y + 1 = \boxed{5y + 1}$$

$$\text{curl} F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle 0 - 0, 0 - 1, 0 - x \rangle$$

$$= \boxed{\langle 0, -1, -x \rangle}$$

How do we measure the change of a vector field?

1. **Curl** (in $\mathbb{R}^3$)



   - Tells us Flow / Circulation density
   - Measures local circulation at a point
   - Is a vector
   - Direction gives axis of rotation (using right hand rule)
   - Magnitude gives rotation rate
   - $\operatorname{curl} \mathbf{F} = \nabla \times F$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle$

THM: $\operatorname{Curl}(F) = \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, 0 \rangle$ iff $\boxed{F = \langle P, Q \rangle \text{ has a potential function}}$ & F is path independent

F conservative $\longrightarrow$

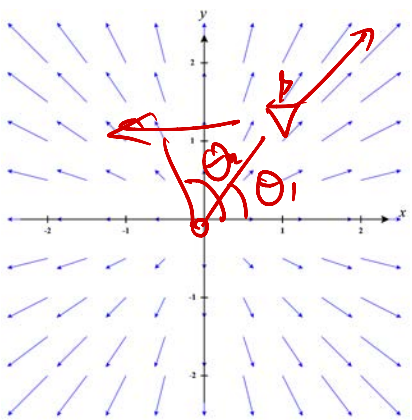
2. **Divergence** (in any $\mathbb{R}^n$)

   - Tells us Flux density
   - Measures Compression/expansion at a point
   - Is a Scalar
   - $\operatorname{div} \mathbf{F} = \nabla \cdot F$

Geometric interpretation of $\operatorname{div} F = 0$.

F is "incompressible"

Example 138. Let $\mathbf{F}(x, y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.

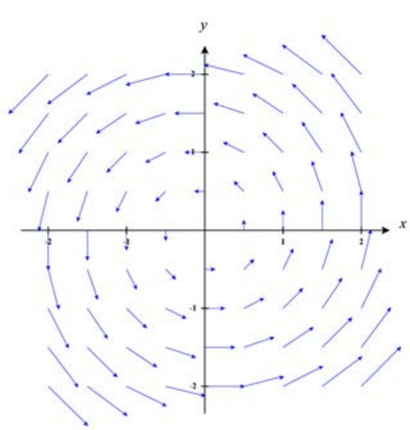


lots of expansion, so div(F) should be positive at every point. (and Flux $= \oint_C \mathbf{F}\cdot\mathbf{n}\, ds > 0$)

no rotation, so curl(F) should be the zero vector.
(and $F = \nabla f$ & Flow $= \oint_C \mathbf{F}\cdot\mathbf{T}\, ds = f(A) - f(A) = 0$)

Check: $\operatorname{div} F = P_x + Q_y = 1 + 1 = 2$ ✓
$\operatorname{curl} F = \langle 0, 0, Q_x - P_y\rangle = \langle 0, 0, 0 - 0\rangle$
$= \langle 0, 0, 0\rangle$ ✓

Example 139. *You try it!* Let $\mathbf{F}(x, y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



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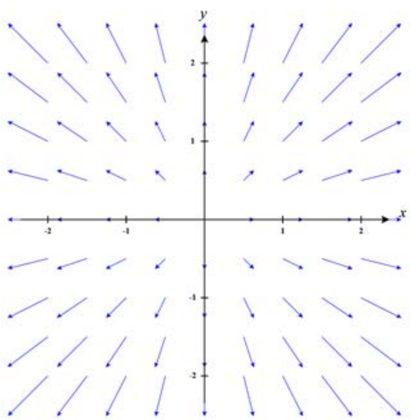
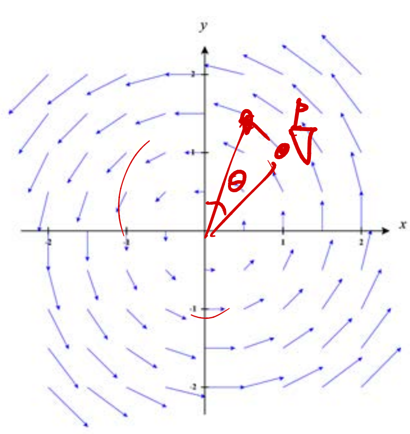


lots of expansion, so div(F) should be positive at every point. (and Flux = $\oint_C F\cdot n\,ds > 0$)

no rotation, so curl(F) should be the zero vector.
(and $F = \nabla f$ & Flow = $\oint_C F\cdot T\,ds = f(A) - f(A) = 0$)

Check: $\operatorname{div} F = P_x + Q_y = 1 + 1 = 2$ ✓
$\operatorname{curl} F = \langle 0, 0, Q_x - P_y\rangle = \langle 0, 0, 0 - 0\rangle$
$= \langle 0, 0, 0\rangle$ ✓

Example 139. *You try it!* Let $\mathbf{F}(x, y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



lots of rotation so curl(F) $\neq \vec{0}$
and should be pointing "up"
out of the page by RHR

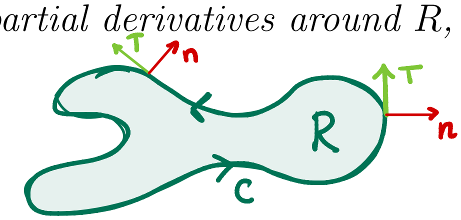
no compression, so div(F) = 0

Check $\operatorname{curl} F = \langle 0, 0, Q_x - P_y\rangle$
$= \langle 0, 0, 1 - (-1)\rangle = \langle 0, 0, 2\rangle$ "up" ✓
$\operatorname{div} F = P_x + Q_y = 0 + 0 = 0$ ✓

Question: How is this useful?

Answer: We can relate rates of change of F inside a region to the behavior of the vector field on the boundary of the region.

Theorem 140 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$*. If* $\mathbf{F} = \langle P, Q \rangle$ *has continuous partial derivatives around R, then*



a) Circulation form:

(a)
$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C P\, dx + Q\, dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R Q_x - P_y\, dA$$

notational equality G's T expand out

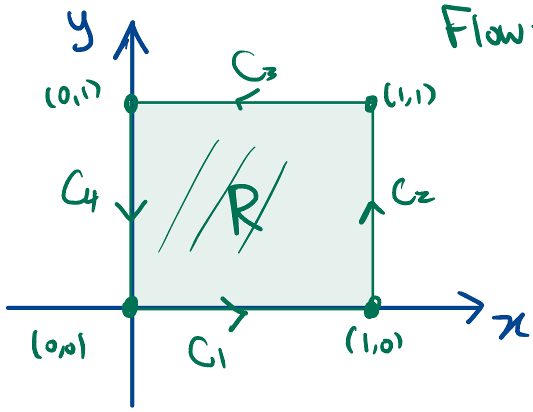
b) Flux form:

(b)
$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\, dy - Q\, dx = \iint_R (\nabla \cdot \mathbf{F})\, dA = \iint_R P_x + Q_y\, dA$$

(a) Says: The $\boxed{\text{Flow}}$ across a closed simple loop C is the double-integral over the interior R of C of $\boxed{\text{the } \hat{k}\text{-component of Curl } F.}$

(b) Says: The $\boxed{\text{Flux}}$ across a Closed Simple loop C is the double-integral of the interior R of C of $\boxed{\text{div } F}$

Example 141. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0$, and $y = 1$ oriented counterclockwise. $C = C_1 \cup C_2 \cup C_3 \cup C_4$ yuck!



$$\text{Flow} = \oint_C F \cdot T\, ds = \int_{C_1} F\cdot T\, ds + \int_{C_2} F\cdot T\, ds + \int_{C_3} F\cdot T\, ds + \int_{C_4} F\cdot T\, ds$$

4 times as much work as a single flow problem (not great).

use G-T.

$$\int_C F\cdot T\, ds = \iint_R \operatorname{curl} F \cdot \hat{k}\, dA \quad \text{(better idea)}$$

$$F = \langle P, Q \rangle = \langle -y^2, xy \rangle$$

$$P_y = -2y$$
$$Q_x = y$$

$$= \iint_R Q_x - P_y\, dA = \iint_R y - (-2y)\, dA$$

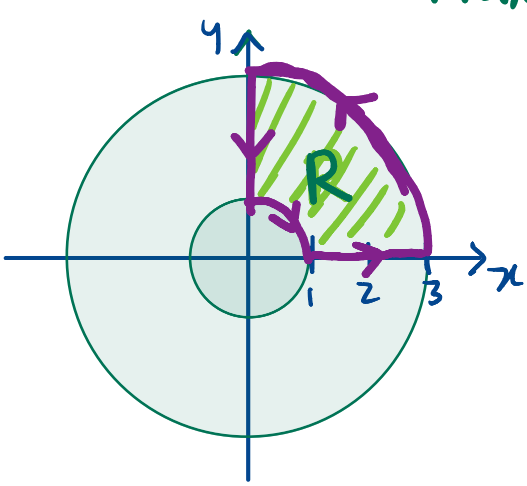
$$= \iint_R 3y\, dA = \int_0^1 \int_0^1 3y\, dy\, dx \qquad \boxed{3/2}$$

$$= \int_0^1 \frac{3}{2} y^2 \Big|_0^1 dx = \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}\Big|_0^1 = \frac{3}{2} - 0$$

Example 142. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first ~~octant~~ quadrant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.

G'sT

$$\text{Flux} = \oint_C F \cdot n\, ds = \iint_R \underbrace{P_x + Q_y}_{\operatorname{div}(F)}\, dA$$



$$\begin{cases} P = \frac{1}{3}x^3 \\ Q = \frac{1}{3}y^3 \end{cases}$$

$$= \iint_R x^2 + y^2\, dA$$

$$= \int_0^{\pi/2} \int_1^3 r^2 \cdot r\, dr\, d\theta = \int_0^{\pi/2} \int_1^3 r^3\, dr\, d\theta$$

$$\theta \in [0, \pi/2)$$
$$r \in [1, 3]$$

$$= \int_0^{\pi/2} \frac{1}{4} r^4 \Big|_1^3 d\theta = \int_0^{\pi/2} \frac{1}{4}(81 - 1)\, d\theta = \int_0^{\pi/2} 20\, d\theta$$

$$= 20\theta \Big|_0^{\pi/2} = \frac{20\pi}{2} = \boxed{10\pi}$$

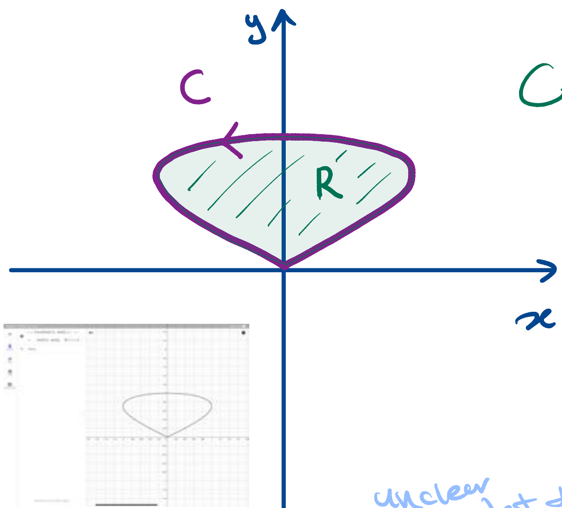
Example 143. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \le t \le \pi$. Find the area of R, using Green's Theorem applied to the vector field $\boxed{\mathbf{F} = \cancel{\frac{1}{2}\langle x, y \rangle}}$.

For Flux or $\boxed{F = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle}$ For Flow

Idea: $\text{Area } R = \iint_R 1\,dA = \iint \frac{1}{2} - \left(-\frac{1}{2}\right) dA$ (Q_x, P_y)

G'sT (Flow)

$$= \oint_C F \cdot T\,ds \qquad \boxed{\text{w/ } F = \langle -\tfrac{1}{2}y, \tfrac{1}{2}x \rangle}$$



C: $r(t) = \langle \sin 2t, \sin t \rangle$, $r'(t) = \langle 2\cos 2t, \cos t \rangle$
$t \in [0, \pi)$

Flow $\oint_C F \cdot T\,ds = \int_a^b F(r(t)) \cdot r'(t)\,dt$

$$= \int_0^\pi \langle -\tfrac{1}{2}\sin t, \tfrac{1}{2}\sin 2t \rangle \cdot \langle 2\cos 2t, \cos t \rangle dt$$

unclear what to do?

use identities
$\cos 2t = 2\cos^2 t - 1$
$\sin 2t = 2\sin t\cos t$

$$= \int_0^\pi -\sin t\cos 2t + \tfrac{1}{2}\sin 2t\cos t\;dt$$

$$= \int_0^\pi -\sin t\,(2\cos^2 t - 1) + \tfrac{1}{2}(2\sin t\cos t)\cos t\;dt$$

$u = \cos t$
$du = -\sin t\,dt$

$$= \int_0^\pi -2\cos^2 t\sin t + \sin t + \cos^2 t\sin t\;dt$$

$$= \int_0^\pi -\cos^2 t\sin t + \sin t\;dt = \int_0^\pi (-\cos^2 t + 1)\sin t\,dt$$

$t = 0 \quad u = 1$
$t = \pi \quad u = -1$

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

$$= \int_1^{-1} (u^2 - 1)\,du = \tfrac{1}{3}u^3 - u \Big|_1^{-1}$$

$$= \left(\tfrac{1}{3}(-1)^3 - (-1)\right) - \left[\tfrac{1}{3} - 1\right) = \left(1 - \tfrac{1}{3}\right] + \left[1 - \tfrac{1}{3}\right] = \boxed{4/3}$$

THM (G'ST Area Formula) for Flux $\boxed{\text{Area } R = \frac{1}{2}\oint_C x\,dy - y\,dx}$

Example 143. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t)\rangle$ for $0 \leq t \leq \pi$. Find the area of R, using Green's Theorem applied to the vector field $\boxed{\mathbf{F} = \frac{1}{2}\langle x, y\rangle.}$

For Flux

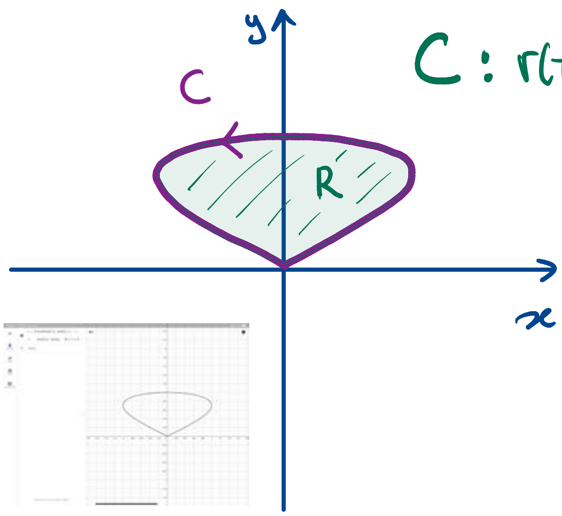
or $F = \langle -\frac{1}{2}y, \frac{1}{2}x\rangle$ For Flow

Idea: $\text{Area } R = \iint_R 1\,dA = \iint \frac{1}{2} + \frac{1}{2}\,dA$

$\uparrow P_x \quad \uparrow Q_y$

G'ST (Flux)

$= \oint_C F \cdot n\,ds \quad$ w/ $F = \langle \frac{1}{2}x, \frac{1}{2}y\rangle$



$C: r(t) = \langle \sin 2t, \sin t\rangle$

$t \in [0, \pi]$

$r'(t) = \langle 2\cos 2t, \cos t\rangle$

$n \sim \langle \cos t, -2\cos 2t\rangle$

$\text{Area } R = \oint_C \langle \frac{1}{2}x, \frac{1}{2}y\rangle \cdot n\,ds$

$= \int_0^\pi \langle \frac{1}{2}\sin 2t, \frac{1}{2}\sin t\rangle \cdot \langle \cos t, -2\cos 2t\rangle\,ds$

$= \int_0^\pi \frac{1}{2}\sin 2t\cos t - \sin t\cos 2t\,dt$

unclear how to integrate?

$\cos 2t = 2\cos^2 t - 1$

$\sin 2t = 2\sin t\cos t$

$= \int_0^\pi \frac{1}{2}(2\sin t\cos t)\cos t - \sin t(2\cos^2 t - 1)\,dt$

$= \int_0^\pi \cos^2 t\sin t - 2\cos^2 t\sin t + \sin t\,dt$

u-sub
$u = \cos t$
$du = -\sin t\,dt$

same

$= \int_0^\pi -\cos^2 t\sin t + \sin t\,dt = \frac{1}{3}\cos^3 t - \cos t\Big|_0^\pi = \left[\frac{1}{3}(\cos\pi)^3 - \cos\pi\right] - \left[\frac{1}{3}(\cos(0))^3 - \cos(0)\right]$

$-\frac{1}{3} \quad +1 \qquad \frac{1}{3} \quad -1$

$= \frac{2}{3} + \frac{2}{3} = \boxed{4/3}$

Note: This is the idea behind the operation of the measuring instrument known as a $\boxed{\textit{planimeter.}}$