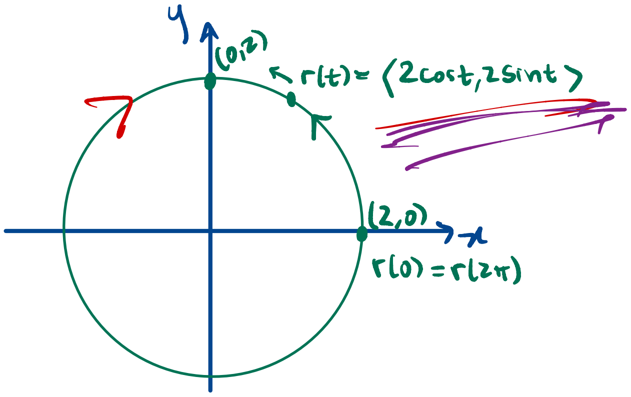
§16.5, 16.6 Surfaces & Surface Integrals

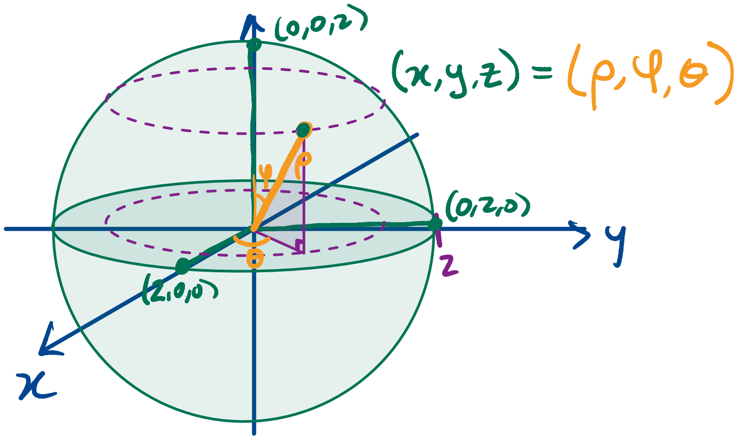
Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$	$z = f(x, y)$
Implicit:	$F(x, y) = 0$	$F(x, y, z) = 0$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$	

$r : \mathbb{R} \to \mathbb{R}^2$



$r : \mathbb{R}^2 \to \mathbb{R}^3$



We've already done a few surface parametrizations.

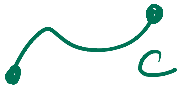
e.g.

⊛ Plane through the origin

$r(s,t) = s\vec{v}_1 + t\vec{v}_2$

⊛ spheres w/ fixed radius ρ using spherical coords.

§16.5, 16.6 Surfaces & Surface Integrals



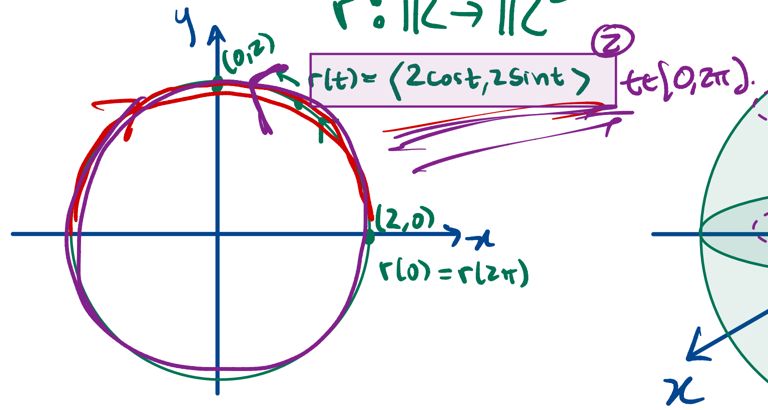
$$\left.\begin{array}{l}\int_C F\cdot T\,ds\\ \int_C f\,ds\end{array}\right\}\ \text{§16.1 - 16.4.}$$

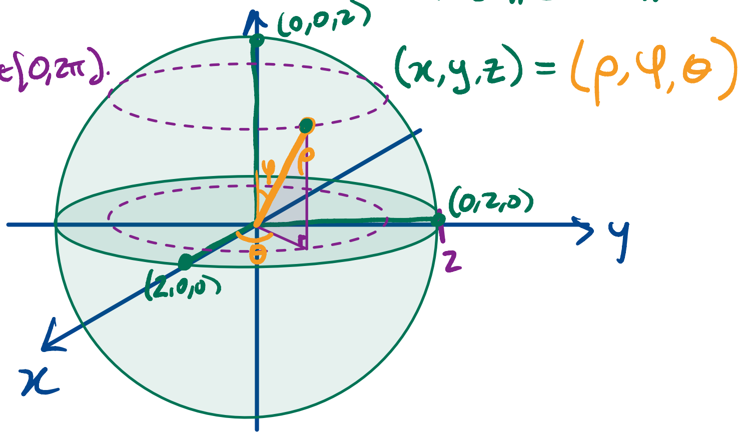
Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$ $y = \sqrt{4-x^2}$	$z = f(x, y)$ $z = \sqrt{4-x^2-y^2}$
Implicit:	$F(x, y) = 0$ $x^2+y^2=4$	$F(x, y, z) = 0$ $x^2+y^2+z^2=4$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t)\rangle$ ① $r(t) = \langle t, \sqrt{4-t^2}\rangle$ $t\in[-2,2]$	① $r(t,s) = \langle t, s, \sqrt{4-t^2-s^2}\rangle$ $(t,s)\in[-2,2]\times[-2,2]$

$r: \mathbb{R}\to\mathbb{R}^2$

$r: \mathbb{R}^2\to\mathbb{R}^3$





We've already done a few surface parametrizations.

e.g.

⊛ plane through the origin

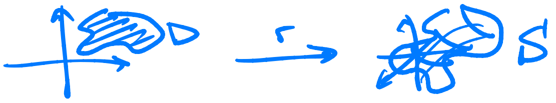
$r(s,t) = s\vec{v}_1 + t\vec{v}_2$

⊛ spheres w/ fixed radius ρ using spherical coords.

② $r(\varphi,\theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle$

$\varphi\in[0,\pi]$

$\theta\in[0,2\pi]$



GOAL: $r: \mathbb{R}^2 \to \mathbb{R}^3$ such that range of r is surface.

Example 144. Give parameteric representations for the surfaces below.

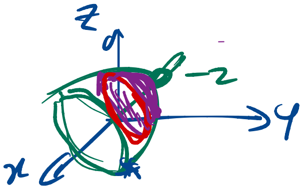
x is given explicitly as a function $f(y,z)$ of y & z.

a) $x = y^2 + \frac{1}{2}z^2 - 2$

$y=s, z=t$

let y, z be your parameters for your parameterization

① $r(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle$, $\underline{(s,t)} \in \mathbb{R}^2$. (ie $s \in \mathbb{R}, t \in \mathbb{R}$)

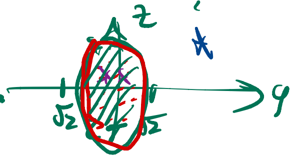


b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

we need only values of $x \leq 0$ (behind y-z plane).

@ $x=0$ $0 = y^2 + \frac{1}{2}z^2 - 2 \Rightarrow 2 = y^2 + \frac{1}{2}z^2$ ellipse.

Choose (y,z) s.t. ~~$x=$~~ $y^2 + \frac{1}{2}z^2 - 2 \leq 0 \Rightarrow \boxed{y^2 + \frac{1}{2}z^2 \leq 2}$



$y \in [-\sqrt{2}, \sqrt{2}]$ $z \in (-\sqrt{4-2y^2}, \sqrt{4-2y^2})$

$4 = 2y^2 + z^2$

$\Rightarrow z = \pm\sqrt{4-2y^2}$

① $r(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle$

where $s \in [-\sqrt{2}, \sqrt{2}]$, $t \in [-\sqrt{4-2s^2}, \sqrt{4-2s^2}]$.

c) $x^2 + y^2 + z^2 = 9$

① use rectangular coord

(top half only) $r(s,t) = \langle s, t, \sqrt{9 - s^2 - t^2} \rangle$

$s \in [-3, 3]$ $t \in (-\sqrt{9-s^2}, \sqrt{9-s^2}]$

② use spherical coords.

$r(s,t) \overset{✓}{=} \langle 3\sin(s)\cos(t), 3\sin(s)\sin(t), 3\cos(s) \rangle$ $s \in [0, \pi]$ ✓
$t \in [0, 2\pi]$

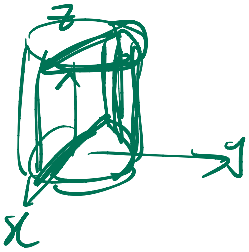
$(\rho = 3)$
$(s = \varphi)$
$(t = \theta)$

d) $x^2 + y^2 = 25$ (hint surface in $\mathbb{R}^3$ — cylinder opening in z-direction) horizontal cross sections are circles of radius $r=5$ for every $z = $ const.

① rectangular coords

$r(s,t) = \langle s, \sqrt{25 - s^2}, t \rangle$, $s \in [-5, 5]$. $t \in (-\infty, \infty)$

$y \geq 0$



② cylindrical coords

$(s = \theta)$ $(t = z)$.

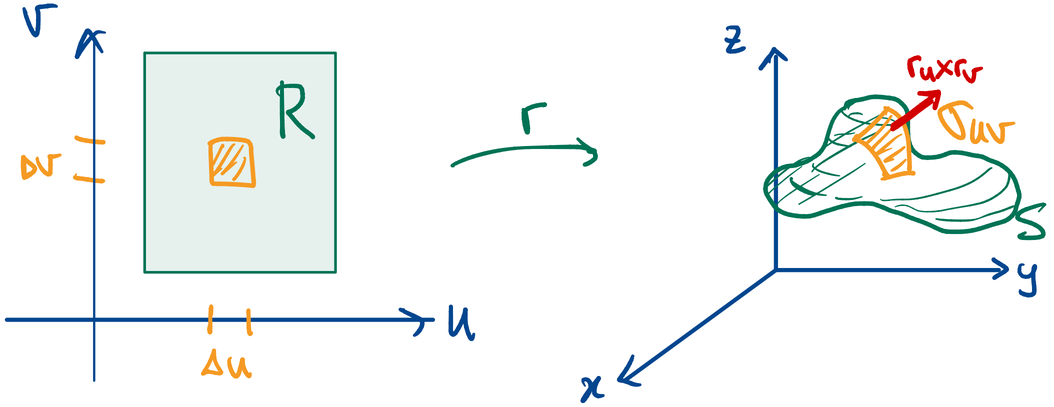
$r(s,t) = \langle 5\cos(s), 5\sin(s), t \rangle$, $s \in [0, 2\pi]$, $t \in \mathbb{R}$

Mass of a wire $= \int_C \delta(x,y)\,ds$ Mass of a surface??

What can we do with this? Surface area / Mass of a surface

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface.
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a rectangle of size $\|r_u \times r_v\| \Delta u \Delta v$ on the surface in $\mathbb{R}^3$.



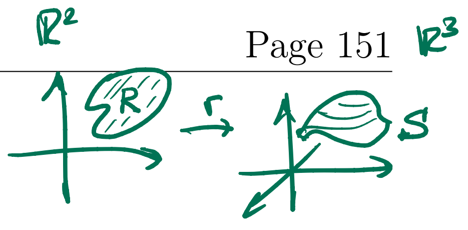
$$\text{Area } A = \|r_u \Delta u \times r_v \Delta v\|$$
$$= \|r_u \times r_v\| \Delta u \Delta v$$
$$\approx \text{Area } \sigma_{uv}$$

- Thus, Area$(S) = \iint_S 1\, d\sigma = \iint_R \|r_u \times r_v\|\, dA$

surface measure

Example 145. *You try it!* Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

$$\boxed{\begin{gathered} r(\theta,t) = \langle 5\cos\theta, 5\sin\theta, t\rangle \\ \theta \in [0, 2\pi] \quad t \in [?, ?] \end{gathered}}$$

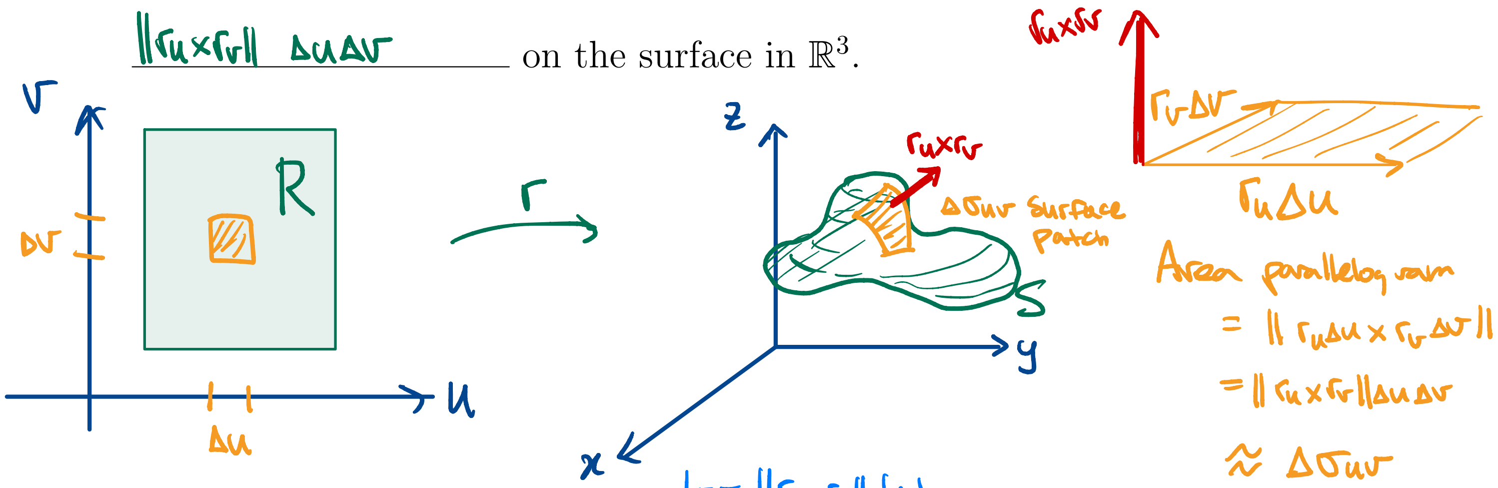


What can we do with this? Surface area

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

$S: r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
$u,v \in R$

- $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a rectangle of size $\|r_u \times r_v\| \Delta u \Delta v$ on the surface in $\mathbb{R}^3$.



Area parallelogram
$= \|r_u\Delta u \times r_v \Delta v\|$
$= \|r_u \times r_v\| \Delta u \Delta v$
$\approx \Delta\sigma_{uv}$

- Thus, Area(S) = $\iint_S 1 \, d\sigma = \iint_R \|r_u \times r_v\| \, dA$

$d\sigma = \|r_u \times r_v\| dx dy$

$dA = dx dy = r dr d\theta$ etc

Surface integral w/ surface measure $d\sigma$

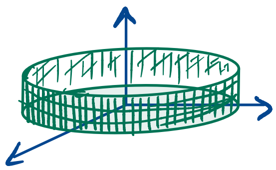
double integral w/ area measure

Example 145. *You try it!* Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

$\vec{r}(\theta, t) = \langle 5\cos\theta, 5\sin\theta, t \rangle$, $\theta \in [0, 2\pi]$, $z \in [0,1]$

$\vec{r}_\theta = \langle -5\sin\theta, 5\cos\theta, 0 \rangle$

$\vec{r}_t = \langle 0, 0, 1 \rangle$



So $n = r_\theta \times r_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5\sin\theta & 5\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 5\cos\theta, -(-5\sin\theta), 0 \rangle$

So $\|\vec{n}\|^2 = 25\cos^2\theta + 25\sin^2\theta + 0 = 25$, $\|n\| = 5$.

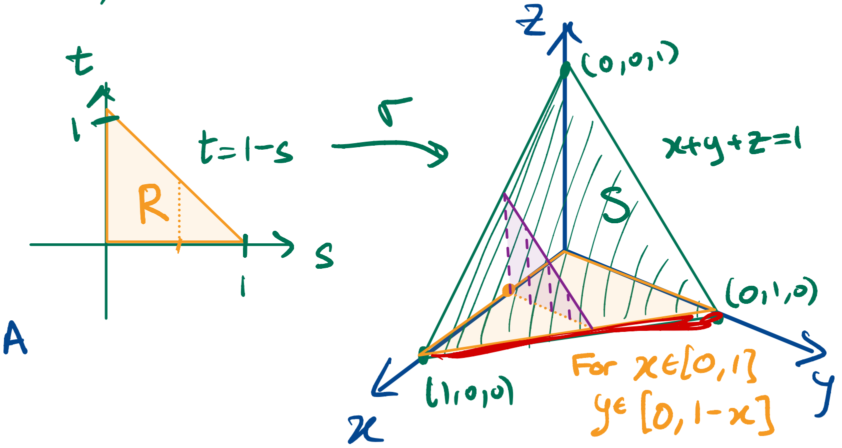
Area $S = \iint_R 5 \, dA = \int_0^{2\pi}\int_0^1 5 \, dt \, d\theta = \int_0^{2\pi} 5t\Big|_0^1 d\theta = \int_0^{2\pi} 5 \, d\theta = 5 * 2\pi = \boxed{10\pi}$

$$\text{Mass} = \iint_S \delta(x,y,z)\, d\sigma$$

Example 146. Suppose the density of a thin plate S in the shape of the portion of the plane $\boxed{x + y + z = 1 \text{ in}}$ the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

solve for $z = f(x,y)$

$$M = \iint_S \delta(x,y,z)\, d\sigma$$



Step 1: parametrize S

Step 2: Compute $d\sigma = \|r_u \times r_v\|\, dA$

Step 3: Substitute

① S is the graph of $z = f(x,y)$ where $f(x,y) = 1 - x - y$ over some triangle R in $\mathbb{R}^2$ (xy-plane)

If @ $y=0, z=0$ $x+y+z=1 \Rightarrow x=1$, Sim @ $x=0, z=0 \Rightarrow y=1$
@ $x=0, y=0 \Rightarrow z=1$

vertices are $(0,0,1), (0,1,0), (1,0,0)$.

$z = 1 - x - y$ where $x \in [0,1]$, $y \in [0, 1-x]$.

$\boxed{r(s,t) = \langle s, t, 1-s-t \rangle}$, $s \in [0,1]$, $t \in [0, 1-s]$.

② $r_s = \langle 1, 0, -1 \rangle$
$r_t = \langle 0, 1, -1 \rangle$

$$r_s \times r_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0+1, -(-1), 1 \rangle = \langle 1, 1, 1 \rangle$$

and $\|r_s \times r_t\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

③ $$\text{Mass} = \iint_S \delta\, d\sigma = \iint_R 6xy \sqrt{3}\, dA = \int_0^1 \int_0^{1-s} 6st * \sqrt{3}\, dt\, ds$$

$n = \langle 1,1,1\rangle - \langle 2,2,2\rangle$

$2x+2y+2z=2$

Example 146. Suppose the density of a thin plate S in the shape of the portion of the plane $\boxed{x + y + z = 1}$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

③ $\text{Mass} = \iint_S \delta \, d\sigma = \iint_R 6xy \sqrt{3} \, dA = \int_0^1 \int_0^{1-s} 6st * \sqrt{3} \, dt \, ds$

$= \int_0^1 3\sqrt{3} s t^2 \Big|_0^{1-s} ds = \int_0^1 3\sqrt{3} \, s(1-s)^2 \, ds$

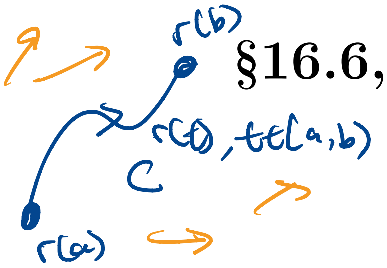
$= \int_0^1 3\sqrt{3} s \left(s^2 - 2s + 1\right) ds = \int_0^1 3\sqrt{3}\left(s^3 - 2s^2 + s\right) ds$

$= 3\sqrt{3}\left(\frac{1}{4}s^4 - \frac{2}{3}s^3 + \frac{1}{2}s^2\right)\Big|_0^1 = 3\sqrt{3}\left[\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) - 0\right]$

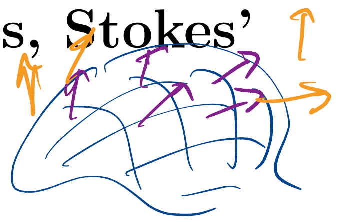
$= 3\sqrt{3}\left(\frac{3}{4} - \frac{2}{3}\right) = 3\sqrt{3}\left(\frac{9-8}{12}\right) = \frac{3\sqrt{3}}{12} = \boxed{\sqrt{3}/4}$

F Flow of F along C / Flux of F across C.

$\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ / $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$.



§16.6, 16.7 Flux Surface Integrals, Stokes' Theorem



Goal: If **F** is a vector field in $\mathbb{R}^3$, find the total flux of **F** through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of Outward pointing normal vector to your surface.

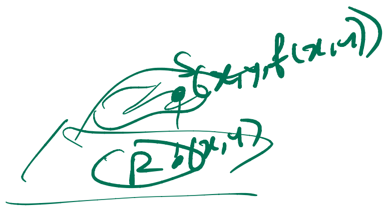
If $\mathbf{r}(u,v)$ is a smooth parameterization of S with domain R, we have

unit normal

plug in parametrization into F

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F}\cdot\mathbf{n})\,d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot\underbrace{(\mathbf{r}_u\times\mathbf{r}_v)}\,dA.$$

$\vec{n} = \frac{r_u \times r_v}{\|r_u\times r_v\|}$ so ↑ is $n * \|r_u\times r_v\|$.



Example 147. Find $\mathbf{r}_u\times\mathbf{r}_v$ and $\|\mathbf{r}_u\times\mathbf{r}_v\|$ when $z = f(x,y)$ so that S is the graph of a scalar function with domain in $\mathbb{R}^2$.

arbitrary graph

① $r(s,t) = \langle s, t, f(s,t)\rangle$, $(s,t)\in R$ in $\mathbb{R}^2$.

$r_s = \langle 1, 0, f_s\rangle$

$r_t = \langle 0, 1, f_t\rangle$

$r_s\times r_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_s \\ 0 & 1 & f_t \end{vmatrix} = \langle 0-f_s, -(f_t-0), 1\rangle = \langle -f_s, -f_t, 1\rangle.$

$\|r_s\times r_t\| = \sqrt{(-f_s)^2+(-f_t)^2+1^2} = \sqrt{1+f_s^2+f_t^2}$

Example 148. Find $\mathbf{r}_u \times \mathbf{r}_v$ and $\|\mathbf{r}_u \times \mathbf{r}_v\|$ when S is a portion of a sphere of radius $\rho = a$, for some fixed constant a, using the standard spherical coordinates for your parametrization.

spherical coord.

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad @\ \rho = a \ \text{(constant)}$$

$$r(\phi,\theta) = \langle a \sin\phi \cos\theta,\ a\sin\phi \sin\theta,\ a\cos\phi \rangle, \quad \theta \in [0, 2\pi],\ \phi \in [0,\pi]$$

$$r_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi \rangle$$

$$r_\theta = \langle -a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0 \rangle$$

$$r_\phi \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \langle 0 - (-a^2\sin^2\phi\cos\theta),\ -(0 - a^2\sin^2\phi\sin\theta),\ a^2\cos\phi\sin\phi\cos^2\theta - (-a^2\cos\phi\sin\phi\sin^2\theta) \rangle$$

$$= \langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\underbrace{\cos\phi\sin\phi}\underline{\cos^2\theta} + a^2\underbrace{\cos\phi\sin\phi}\underline{\sin^2\theta} \rangle$$

$$= \langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\cos\phi\sin\phi \rangle$$

$$\|r_\phi \times r_\theta\|^2 = a^4\underbrace{\sin^4\phi}\underline{\cos^2\theta} + a^4\underbrace{\sin^4\phi}\underline{\sin^2\theta} + a^4\cos^2\phi\sin^2\phi$$

$$= a^4\sin^4\phi + a^4\cos^2\phi\sin^2\phi = a^4\sin^2\phi\,\cancelto{1}{(\sin^2\phi + \cos^2\phi)}$$

$$= a^4\sin^2\phi$$

$$a \geq 0 \quad \phi \in [0,\pi] \ (\sin\phi \geq 0)$$

$$\|r_\phi \times r_\theta\| = \sqrt{a^4\sin^2\phi} \overset{a\geq 0}{=} a^2\sin\phi.$$

$$\boxed{dV = \underline{\rho^2\sin\phi}\, d\rho\, d\phi\, d\theta.}$$

Example 149. Find the flux of $\mathbf{F} = \langle x, -y, z\rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.

Want to compute $\text{Flux} = \iint_S F \cdot n \, d\sigma$

① parametrize S

② Compute partials & cross product $r_u, r_v, r_u \times r_v$

③ substitute & integrate.

$$= \iint_R F(r(u,v)) \cdot (r_u \times r_v) \, dA.$$

S is a sphere of radius 2.

① $r(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$ $(a=2)$ (whole sphere)

② $r_\phi \times r_\theta = \langle 4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$

$\phi \in [0, \pi/2]$
$\theta \in [0, 2\pi]$

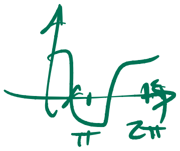
③ Flux

$$= \int_0^{2\pi}\int_0^{\pi/2} \langle 2\sin\phi\cos\theta, -2\sin\phi\sin\theta, 2\cos\phi\rangle \cdot \langle 4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, 4\sin\phi\cos\theta\rangle \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin^3\phi\cos^2\theta + 8\sin^3\phi\sin^2\theta + 8\cos\phi\sin\phi\cos\phi \; d\phi \; d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin^3\phi + 8\cos^2\phi\sin\phi \; d\phi \; d\theta = \int_0^{2\pi}\int_0^{\pi/2} 8\sin\phi\,\overset{1}{(\sin^2\phi + \cos^2\phi)} \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin\phi \; d\phi \, d\theta = \int_0^{2\pi} -8\cos\phi \Big|_0^{\pi/2} d\theta$$



$$= \int_0^{2\pi} -8(\cos\pi/2) - (-8\cos 0)\, d\theta = \int_0^{2\pi} 8\, d\theta = 8\theta\Big|_0^{2\pi} = \boxed{16\pi}$$

$\cos(\pi/2) = 0$
$\cos(0) = 1$

$M = \iint_S G\, d\sigma$ Integrate G over S

Example 150. *You try it!* ~~Compute $\iint_S G \cdot \mathbf{n}\, d\sigma$ the flux of G across the surface~~ S.

$$G(x, y, z) = x^2, \quad S:\ x^2 + y^2 + z^2 = 1$$

$$r(\phi,\theta) = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \rangle \qquad \phi \in [0,\pi),\ \theta \in [0,2\pi).$$

$$\iint_S G\, d\sigma = \iint_R G(r(u,v))\, \underline{\|r_u \times r_v\|}\, \underline{dA}$$

use hint: $\|r_\phi \times r_\theta\| = a^2 \sin\phi$

① parametrize S

~~② Compute partials & cross product $r_u,\ r_v,\ r_u \times r_v$~~

③ substitute & integrate.

$$= \int_0^{2\pi}\int_0^{\pi} (\overset{x}{\sin\phi\cos\theta})^2 \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\, d\phi\, d\theta$$

$$\sin^3\phi = \sin^2\phi\sin\phi = (1-\cos^2\phi)\sin\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} (1-\cos^2\phi)\,\underline{\sin\phi}\,\cos^2\theta\, d\phi\, d\theta$$

$u = \cos\phi$

$du = -\underline{\sin\phi\, d\phi}$

$\phi = 0 \quad u = 1$

$\phi = \pi \quad u = -1$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_0^{2\pi}\int_1^{-1} -(1-u^2)\cos^2\theta\, du\, d\theta$$

$$= \int_0^{2\pi} -u + \frac{1}{3}u^3 \Big|_1^{-1} \cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} \left[\left(-(-1) + \tfrac{1}{3}(-1)^3\right) - \left(-1 + \tfrac{1}{3}(1)^3\right)\right]\cos^2\theta\, d\theta = \frac{4}{3}\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= \frac{4}{3}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{4}{3}\left(\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{4}{3}\left(\left(\pi + \frac{1}{4}\sin 4\pi\right) - (0+0)\right)$$

$$= \boxed{\frac{4}{3}\pi}$$

$M = \iint_R G\, d\sigma$ Integrate G over S

Example 150. *You try it!* ~~Compute $\iint_S G \cdot \mathbf{n}\, d\sigma$ the flux of G across the~~ surface S.

$$G(x,y,z) = x^2, \quad S:\ x^2+y^2+z^2 = 1$$

S: unit sphere $\rho = 1$ Then surface measure is the standard spherical element $d\sigma = \|\vec{r}_\varphi \times \vec{r}_\theta\| = \rho^2 \sin\varphi\, d\varphi\, d\theta$

$r(\varphi,\theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$

R: $\varphi \in [0,\pi]$, $\theta \in [0, 2\pi]$

u-sub

$u = \cos\varphi$
$du = -\sin\varphi$

$\varphi = 0 \Rightarrow u =$
$\varphi = \pi \Rightarrow u$

So $$M = \iint_S G\, d\sigma = \int_0^{2\pi}\int_0^{\pi} (\sin\varphi\cos\theta)^2 \sin\varphi\, d\varphi\, d\theta$$

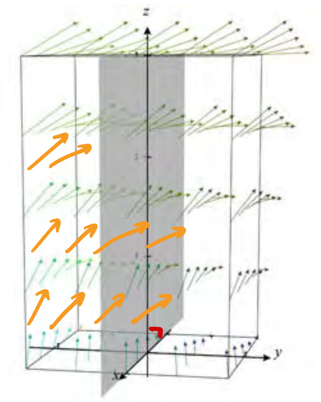
$$= \int_0^{2\pi}\int_0^{\pi} (1-\cos^2\varphi)\cos^2\theta \sin\varphi\, d\varphi = \int_0^{2\pi}\int_1^{-1} -(1-u^2)\cos^2\theta\, du\, d\theta = \int_0^{2\pi}\int_{-1}^{1} (1-u^2)\cos^2\theta\, du\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \left(u - \tfrac{1}{3}u^3 \Big|_{-1}^{1}\right) d\theta = \int_0^{2\pi} 2\cos^2\theta\left(1-\tfrac{1}{3}\right) d\theta = \frac{4}{3}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{2}{3}\left(\theta - \tfrac{1}{2}\cos 2\theta \Big|_0^{2\pi}\right)$$

$$= \frac{2}{3}\left[\left(2\pi - \tfrac{1}{2}\right) - \left(0 - \tfrac{1}{2}\right)\right] = \boxed{\frac{4\pi}{3}}$$

Example 151. *You try it!* Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\underbrace{\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma}_{\text{Flux.}} > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

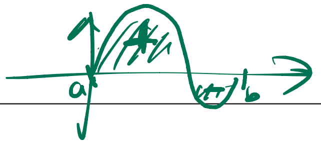
Example 152. *You try it!* Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ positive, negative, or zero?



Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

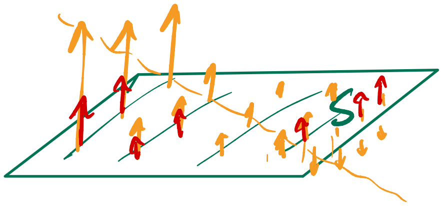
1. *divergence:* $\nabla\cdot\mathbf{F} = P_x + Q_y + R_z$

2. *curl:* $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$



$\int_a^b f(x)\,dx$
total area.
pos(above x-axis), neg(below x-axis)

Example 151. *You try it!* Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

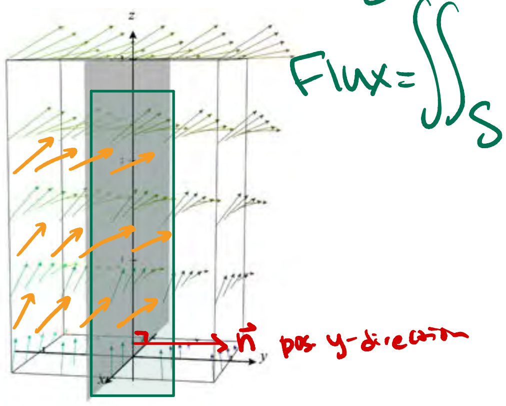


False just need "more work done" in the direction of $\vec{n}$ as opposed to the opposite direction.

Example 152. *You try it!* Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ positive, negative, or zero?

guess

Flux$=\iint_S \mathbf{F}\cdot\vec{n}\,ds$ is positive



Since vectors are going in same direction as $\vec{n}$.

So $\mathbf{F}\cdot\vec{n} \geq 0$

Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla\cdot\mathbf{F} = P_x + Q_y + R_z$
2. *curl:* $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

$\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

✓

On formula sheet

Practice w/ div & curl.

$F = \langle P, Q, R \rangle$. ① $\nabla \times F = G$ ② $\nabla \cdot G$.

Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

$$\nabla \times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = G = \text{curl}(F)$$

Fubini Thm
$P_{yz} = P_{zy}$
$R_{yx} = R_{xy}$
$Q_{zx} = Q_{xz}$

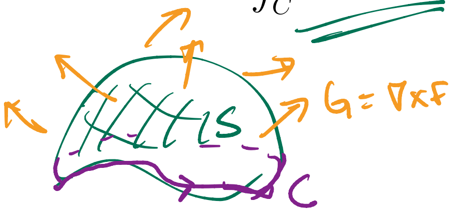
$$\nabla \cdot G = \text{div}(\text{curl} F) = (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz})$$

(by Fubini's) $= 0$

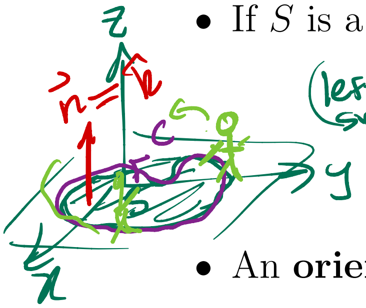
Theorem 154 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \overbrace{(\nabla \times \mathbf{F})}^{G} \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

Flux of $G = \text{curl}(F)$ across the surface S



Flow around a closed loop C, where C is the boundary of surface S.



- If S is a region R in the xy-plane, then we get:

(left side) $\iint_S \text{curl} F \cdot \hat{k}\, dA = \int_C F \cdot T\, ds$ (Green's Theorem).

- An **oriented surface** is one where normal vector stays consistently pointed as you move along the surface

✗ mobius strip not oriented.

- S and C are oriented compatibly if:

walking along curve C then surface is always on the left.

(i.e. walking counter-clockwise relative to S)

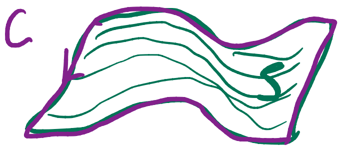
10 min

Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

$$\nabla \cdot (\nabla \times F) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = \boxed{0} \text{ by Fubini's Thm!}$$

Theorem 154 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

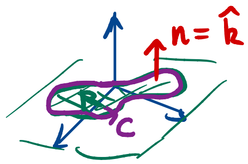


$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Flux of Curl(F) across surface S

Circulation (Flow) around closed loop C which is the boundary of S

- If S is a region R in the xy-plane, then we get:



$$\iint_R \operatorname{Curl} F \cdot \hat{k} \, dA = \oint_C F \cdot T \, ds \quad \text{Green's Theorem!}$$

- An **oriented surface** is one where normal vector stays consistent as you move along surface

* mobius strip is <u>NOT</u> oriented.

- S and C are oriented compatibly if:

↑ surface ↑ boundary

Walking along C keeps S to your LEFT

(ie walking "counter clockwise" relative to S)

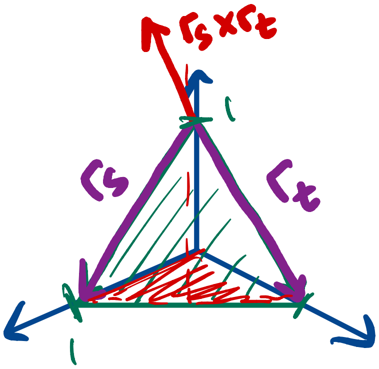
Example 155. Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ by calculating the flux across the interior of C.

right side of Stokes thm.

$$\mathbf{F} = \langle y, xz, x^2\rangle = \langle P, Q, R\rangle$$

C : boundary of $x+y+z=1$ in first octant,

oriented counter-clockwise from above.



S's Thm.

$$\int_C F\cdot dr \overset{\downarrow}{=} \iint_S \text{curl } F \cdot n \, d\sigma$$

C: boundary of $x+y+z=1$ boundary of S

S: $r(s,t) = \langle s, t, 1-s-t\rangle$, $s\in[0,1]$, $t\in[0,1-s]$

$r_s = \langle 1, 0, -1\rangle$

$r_t = \langle 0, 1, -1\rangle$

$$r_s \times r_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1\rangle$$

$R_y = 0$, $Q_z = x$

$P_z = 0$, $R_x = 2x$

$Q_x = z$, $P_y = 1$

$$\text{Curl } F = \nabla \times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0 - x, 0 - 2x, z - 1\rangle$$

$$= \langle -x, -2x, z-1\rangle.$$

Example 156. *You try it!* Use Stokes' Theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ the flux of $\mathbf{F}$ across S by calculating the circulation line integral around the boundary curve C of S.

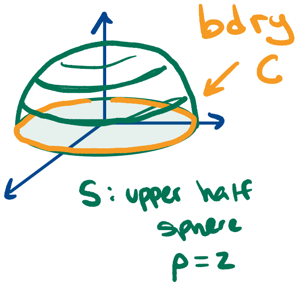
$$\begin{aligned}\mathbf{F} &= \langle 2z, 3x, 5y\rangle\\ S :&\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, (4-r^2)\rangle\\ R :&r\in[0,2],\ \theta\in[0,2\pi]\end{aligned}$$

Example 156. *You try it!* Use Stokes' Theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$ the flux of $\mathbf{F}$ across S by calculating the circulation line integral around the boundary curve C of S.

$$\mathbf{F} = \langle 2z, 3x, 5y\rangle$$
$$S : \mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, (4-r^2)\rangle$$
$$R : r \in [0,2],\ \theta \in [0,2\pi]$$

S: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4-r^2\rangle$ R: $r\in[0,2]$, $\theta\in[0,2\pi]$

C: $\hat{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0\rangle$, $\theta\in[0,2\pi]$



$\hat{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0\rangle$

$F = \langle 2z, 3x, 5y\rangle$

bdry circle of radius 2 in xy-plane

So Flux thru S $= \iint_S \nabla\times F\cdot n\, d\sigma = \oint_C F\cdot dr$

$$= \int_0^{2\pi} \langle 0, 6\cos\theta, 10\sin\theta\rangle\cdot\langle -2\sin\theta, 2\cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} 12\cos^2\theta\, d\theta = \int_0^{2\pi} 6(1+\cos 2\theta)\, d\theta$$

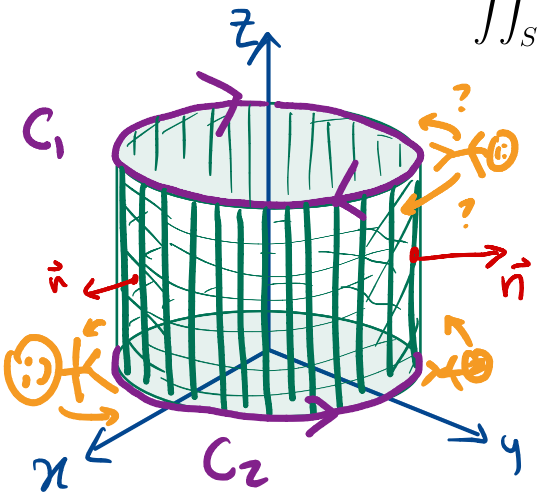
$$= 6\theta + 3\sin 2\theta \Big|_0^{2\pi} = \left(6(2\pi) + 3\sin 4\pi\right) - \left(6(0) + 3\sin 0\right)$$

$$= \boxed{12\pi}$$

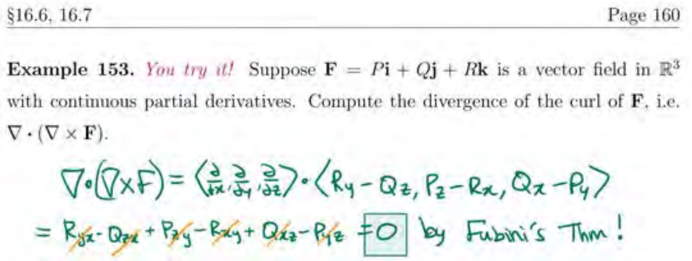
§16.7 Stokes' Theorem

Theorem 152 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let* **F** *be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$



Recall: the boundary is compatibly oriented if walking along the boundary with your head "up" in the direction of $\vec{n}$ The normal vector of S, then your LEFT HAND is pointing over S.



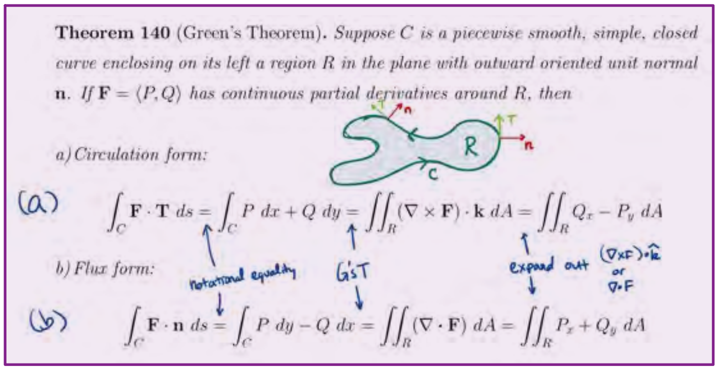
§16.6, 16.7 Page 160

Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of **F**, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

$$\nabla \cdot (\nabla \times F) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= R_{\cancel{yx}} - Q_{\cancel{zx}} + P_{\cancel{zy}} - R_{\cancel{xy}} + Q_{\cancel{xz}} - P_{\cancel{yz}} = \boxed{0} \text{ by Fubini's Thm!}$$

From last time:



Theorem 140 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* **n**. *If $\mathbf{F} = \langle P, Q \rangle$ has continuous partial derivatives around R, then*

a) Circulation form:

(a) $$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C P\, dx + Q\, dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R Q_x - P_y\, dA$$

b) Flux form:

notational equality

G's T

expand out

$(\nabla \times F) \cdot \hat{k}$ or $\nabla \cdot F$

(b) $$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\, dy - Q\, dx = \iint_R (\nabla \cdot \mathbf{F})\, dA = \iint_R P_x + Q_y\, dA$$

Summary:

If curl $F = \vec{0}$ then F is conservative
and $F = \nabla f$. (and FToLI holds etc.)

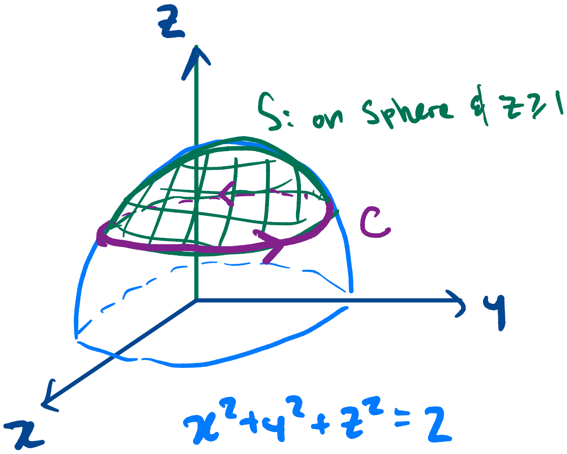
If div $F = 0$ and domain of F is simply connected, then F is the

curl of some other vector field G.

@ $z=1$ get 0

Example 153 (DD)**.** Let $\mathbf{F} = \langle -y, x+(z-1)x^{x\sin(x)}, x^2+y^2\rangle$. Find $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$ over the surface S which is the part of the sphere $x^2+y^2+z^2=2$ above $z=1$, oriented away from the origin.

(Option 1)



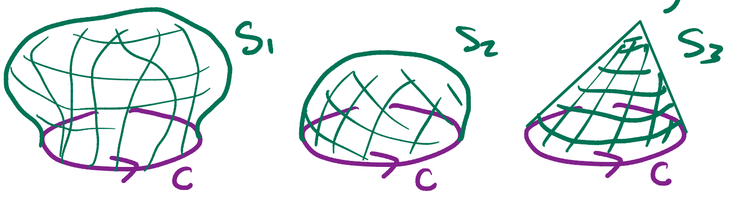
Flux of curl (across S) $= \iint_S (\nabla\times F)\cdot n\,d\sigma$

Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C?

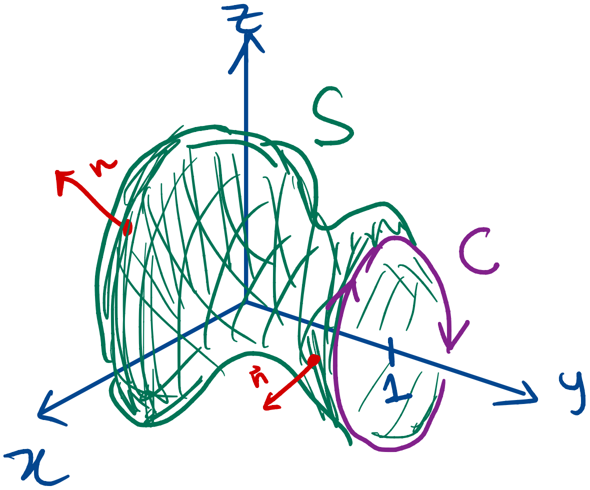
For any F, $\oint_C F \cdot T \, ds = \iint_{S_1} (\nabla \times F) \cdot n \, d\sigma$

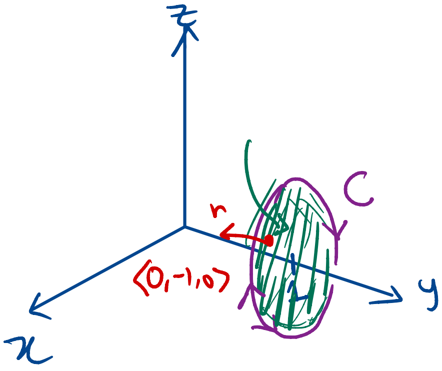
$= \iint_{S_2} (\nabla \times F) \cdot n \, d\sigma$

etc.



Example 154. Suppose $\operatorname{curl} \mathbf{F} = \langle y^{y^y} \sin(z^2), (y-1)e^{x^{x^x}} + 2, -ze^{x^{x^x}} \rangle$. Compute the net flux of the curl of $\mathbf{F}$ over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y-axis in the plane $y = 1$.





§16.8 Divergence Theorem

Theorem 155 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV.$$

Example 156. Let $\mathbf{F} = \langle y^{1234}e^{\sin(yz)}, y - x^{z^x}, z^2 - z\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.

Example 156. Let $\mathbf{F} = \langle y^{1234}e^{\sin(yz)}, y - x^{z^x}, z^2 - z\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.

Closed surface means: encloses a closed and bounded 3D region

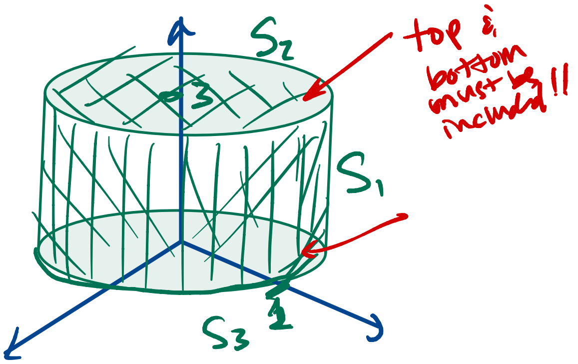
§16.8 Divergence Theorem

Theorem 155 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV. = \iiint_D \text{div}\,\mathbf{F}\,dV$$

Example 156. Let $\mathbf{F} = \langle y^{1234}e^{\sin(yz)}, y - x^{z^x}, z^2 - z\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.

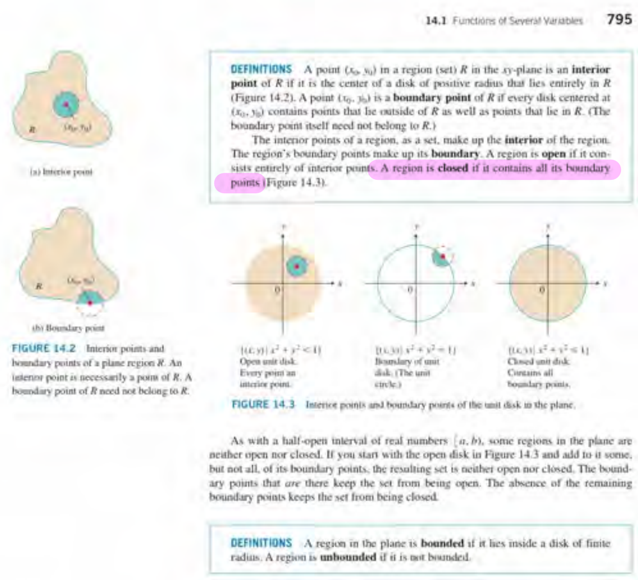
$S = S_1 \cup S_2 \cup S_3$



Cylindrical coords

$D: \theta \in [0, 2\pi]$
$r \in [0, 1]$
$z \in [0, 3]$

Closed, closed, or closed?



14.1 Functions of Several Variables **795**

(a) Interior point

(b) Boundary point

FIGURE 14.2 Interior points and boundary points of a plane region R. An interior point is necessarily a point of R. A boundary point of R need not belong to R.

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

$\{(x, y) \mid x^2 + y^2 < 1\}$ Open unit disk. Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$ Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \le 1\}$ Closed unit disk. Contains all boundary points.

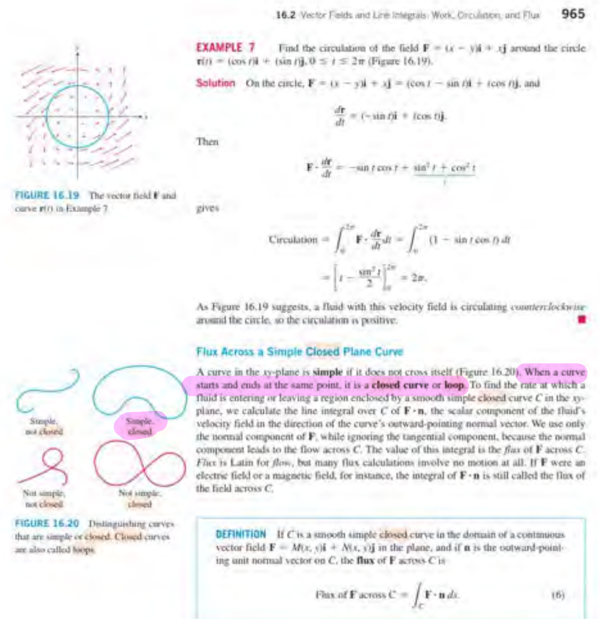
FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

As with a half-open interval of real numbers $[a, b)$, some regions in the plane are neither open nor closed. If you start with the open disk in Figure 14.3 and add to it some, but not all, of its boundary points, the resulting set is neither open nor closed. The boundary points that *are* there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

① a region in $\mathbb{R}^2$ is closed if it contains all its boundary points

e.g. absolute MAX/MIN on closed & bounded regions in $\mathbb{R}^2$



16.2 Vector Fields and Line Integrals: Work, Circulation, and Flux **965**

FIGURE 16.19 The vector field $\mathbf{F}$ and curve $\mathbf{r}(t)$ in Example 7.

EXAMPLE 7 Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$ (Figure 16.19).

Solution On the circle, $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$$

gives

$$\text{Circulation} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} (1 - \sin t \cos t)\,dt$$
$$= \left[t - \frac{\sin^2 t}{2}\right]_0^{2\pi} = 2\pi.$$

As Figure 16.19 suggests, a fluid with this velocity field is circulating *counterclockwise* around the circle, so the circulation is positive. ■

Flux Across a Simple Closed Plane Curve

Simple, not closed

Simple, closed

Not simple, not closed

Not simple, closed

FIGURE 16.20 Distinguishing curves that are simple or closed. Closed curves are also called loops.

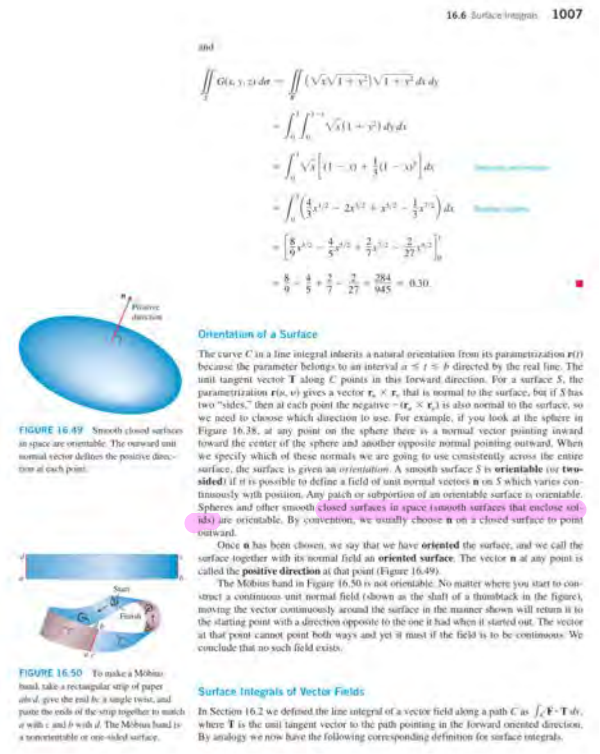
A curve in the xy-plane is **simple** if it does not cross itself (Figure 16.20). When a curve starts and ends at the same point, it is a **closed curve** or **loop**. To find the rate at which a fluid is entering or leaving a region enclosed by a smooth simple closed curve C in the xy-plane, we calculate the line integral over C of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. We use only the normal component of $\mathbf{F}$, while ignoring the tangential component, because the normal component leads to the flow across C. The value of this integral is the *flux* of $\mathbf{F}$ across C. *Flux* is Latin for *flow*, but many flux calculations involve no motion at all. If $\mathbf{F}$ were an electric field or a magnetic field, for instance, the integral of $\mathbf{F} \cdot \mathbf{n}$ is still called the flux of the field across C.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\,ds. \qquad (6)$$

② a loop is closed if it has same starting & ending point.

e.g. line integrals for Flux/Flow around closed loops like circles.



16.6 Surface Integrals **1007**

and

$$\iint_S G(x, y, z)\,d\sigma = \iint_R \left(\sqrt{x}\sqrt{1 + y^2}\right)\sqrt{1 + y^2}\,dx\,dy$$
$$= \int_0^1 \int_0^{1-x} \sqrt{x}(1 + y^2)\,dy\,dx$$
$$= \int_0^1 \sqrt{x}\left[(1 - x) + \frac{1}{3}(1 - x)^3\right]dx$$
$$= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2}\right)dx$$
$$= \left[\frac{8}{9}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{27}x^{9/2}\right]_0^1$$
$$= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945} = 0.30.$$
■

Positive direction

FIGURE 16.49 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

Orientation of a Surface

The curve C in a line integral inherits a natural orientation from its parametrization $\mathbf{r}(t)$ because the parameter belongs to an interval $a \le t \le b$ directed by the real line. The unit tangent vector $\mathbf{T}$ along C points in this forward direction. For a surface S, the parametrization $\mathbf{r}(u, v)$ gives a vector $\mathbf{r}_u \times \mathbf{r}_v$ that is normal to the surface, but if S has two "sides," then at each point the negative $-(\mathbf{r}_u \times \mathbf{r}_v)$ is also normal to the surface, so we need to choose which direction to use. For example, if you look at the sphere in Figure 16.38, at any point on the sphere there is a normal vector pointing inward toward the center of the sphere and another opposite normal pointing outward. When we specify which of these normals we are going to use consistently across the entire surface, the surface is given an *orientation*. A smooth surface S is **orientable** (or **two-sided**) if it is possible to define a field of unit normal vectors $\mathbf{n}$ on S which varies continuously with position. Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we usually choose $\mathbf{n}$ on a closed surface to point outward.

Once $\mathbf{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector $\mathbf{n}$ at any point is called the **positive direction** at that point (Figure 16.49).

The Möbius band in Figure 16.50 is not orientable. No matter where you start to construct a continuous unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out. The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exists.

Start

Finish

FIGURE 16.50 To make a Möbius band, take a rectangular strip of paper $abcd$, give the end bc a single twist, and paste the ends of the strip together to match a with c and b with d. The Möbius band is a nonorientable or one-sided surface.

Surface Integrals of Vector Fields

In Section 16.2 we defined the line integral of a vector field along a path C as $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$, where $\mathbf{T}$ is the unit tangent vector to the path pointing in the forward oriented direction. By analogy we now have the following corresponding definition for surface integrals.

③ closed surface in space $\mathbb{R}^3$ is a smooth surface that encloses a closed and bounded 3D region.

lol

④ closed region in $\mathbb{R}^3$ is same defn as ① (but in $\mathbb{R}^3$)