§14.2 Limits & Continuity

Definition 37. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y) \to (x_0,y_0)} f(x, y) = L$ if and only if $f(x, y)$ ____________________________ regardless of how we approach (x_0, y_0).

Definition 38. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. ____________________________
2. ____________________________
3. ____________________________

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

Example 39. Evaluate $\lim_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Example 40. *You try it!* Evaluate $\lim_{(x,y)\to(\frac{\pi}{2},0)} \dfrac{\cos y + 1}{y - \sin x}$, if it exists.

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Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 41. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.

This idea is called the **two-path test:**

If we can find ____________________________ to (x_0, y_0) along which ______________________ takes on two different values, then ______________________________.

Example 42. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

Example 43. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test.

Example 43. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test.

Example 44. [Challenge:] Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4 + y^2}$$

does exist using the Squeeze Theorem.

Theorem 45 (Squeeze Theorem)**.** *If* $f(x, y) = g(x, y)h(x, y)$, *where* $\lim_{(x,y)\to(a,b)} g(x, y) = 0$ *and* $|h(x, y)| \leq C$ *for some constant* C *near* (a, b), *then* $\lim_{(x,y)\to(a,b)} f(x, y) = 0$.

Math 2551 Worksheet 8 - Review for Exam 1

1. Set up the integral to find the arc length of the curve $y = e^x$ from the point $(0, 1)$ to the point $(1, e)$. Focus on finding a parameterization, and on what values of t give these two points. Is this an integral you would want to compute? Why or why not?

2. Parameterize the line tangent to the curve

$$\mathbf{r}(t) = \langle \cos^2(t), \sin(t)\cos(t), \cos(t)\rangle$$

at the point where $t = \pi/2$.

3. Compute the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ to the circle

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle.$$

Before checking, should the normal vector be pointing into or out of the circle? Why?

4. We have seen that the curvature of a circle with radius a is $1/a$. Thinking about the geometry of a helix with radius a, do you think its curvature will be greater than or less than $1/a$? Why? Compute the curvature using the parameterization

$$\mathbf{r}(t) = \langle a\cos(t), t, a\sin(t)\rangle$$

to confirm or challenge your intuition.

5. The function $\ell(t)$ below describes a line. There is a particular plane that $\ell(t)$ is normal to at the point $t = 0$. Find an equation of this plane.

$$\ell(t) = \langle 3 - 3t, 2 + t, -2t\rangle.$$

Where does this line intersect the different plane $3x - y + 2z = -7$?

6. Find and sketch the domain of each of the following functions of two variables:

   (a) $\sqrt{9 - x^2} + \sqrt{y^2 - 4}$

   (b) $\arcsin(x^2 + y^2 - 2)$

   (c) $\sqrt{16 - x^2 - 4y^2}$

7. Solve the differential equation below, together with its given initial conditions. Remember that this means finding all functions $\mathbf{r}(t)$ which satisfy the given equations.

$$\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}, \quad \mathbf{r}'(1) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(1) = \mathbf{i} + \mathbf{j}$$

8. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?