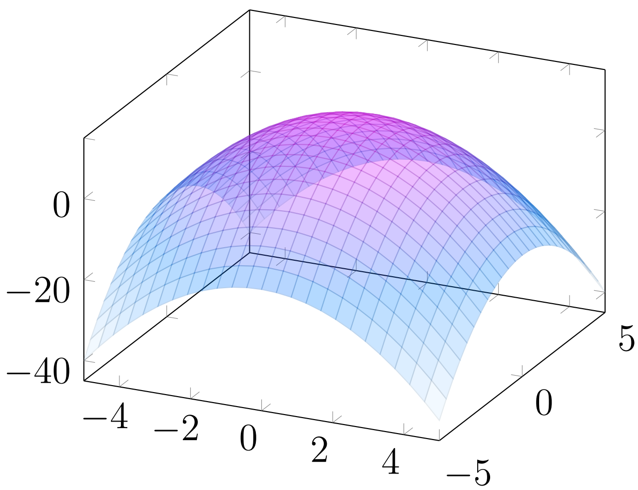
§14.6 Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$\boxed{x^2 + y^2 + z = 10, P = (-1, 3, 0)}$

Example 67. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

This should look familiar: it's ____________________

Example 68. *You try it!* Consider the surface in $\mathbb{R}^3$ containing the point P and defined by

$$x^2 + 2xy - y^2 + z^2 = 7, \ P(1, -1, 3).$$

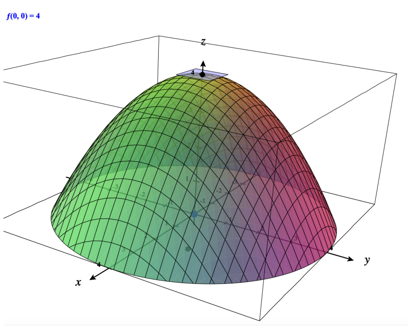
Identity the function $F(x, y, z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

§14.7 Optimization: Local & Global

Gradient: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

$\boxed{\text{Back to the hill}}$ $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$.

What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?



Definition 68. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a ____________________ value of f if $f(a, b)$ ____ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a ____________________ value of f if $f(a, b)$ ____ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $\boxed{z = x^2 - y^2}$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a ____________________

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

Definition 69. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ ______________ or where $Df(a, b)$ ____________________ is called a ____________________ of f.

Example 70. Find the critical points of the function

$$f(x, y) = x^3 + y^3 - 3xy.$$

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0, 0)$.

a) $f(x, y) = 3x + y^3 + 2y^2$

b) $g(x, y) = \cos(x) + \sin(x)$

c) $h(x, y) = \dfrac{4}{x^2 + y^2}$

d) $k(x, y) = x^2 + y^2$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x, y)$ at (a, b) is

$$Hf(a, b) =$$

Theorem 72 (2nd Derivative Test)**.** *Suppose (a, b) is a critical point of $f(x, y)$ and f has continuous second partial derivatives. Then we have:*

- *If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum*
- *If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum*
- *If $\det(Hf(a, b)) < 0$, f has a saddle point at (a, b)*
- *If $\det(Hf(a, b)) = 0$, the test is inconclusive.*

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 73. Classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$ from Example 70.

Two Local Maxima, No Local Minimum: The function $g(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1,0)$ and $(1,2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x,y)$ is like a local maximum, except we must have $f(a,b) \geq f(x,y)$ for **all** (x,y) in the domain of f. A global minimum is defined similarly.

Theorem 74. *On a closed & bounded domain, any continuous function $f(x,y)$ attains a global minimum & maximum.*

Closed:

Bounded:

Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 75. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Example 76. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

(Cont.)