§14.8 Constrained Optimization, Lagrange Multipliers

Goal: Maximize or minimize $f(x, y)$ or $f(x, y, z)$ subject to a *constraint*, $g(x, y) = c$.

Example 77. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

Objective function:

Constraint equation:

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(Cont.)

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

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(Cont.)

Example 79. *You try it!* Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

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§15.1 Double Integrals, Iterated Integrals, Change of Order

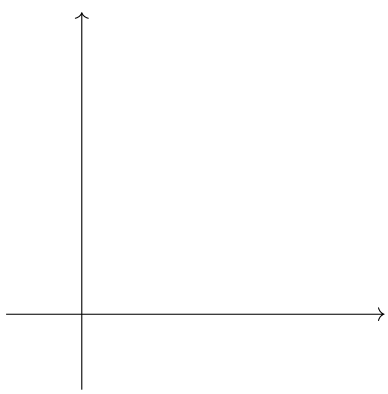
Recall: Riemann sum and the definite integral from single-variable calculus.

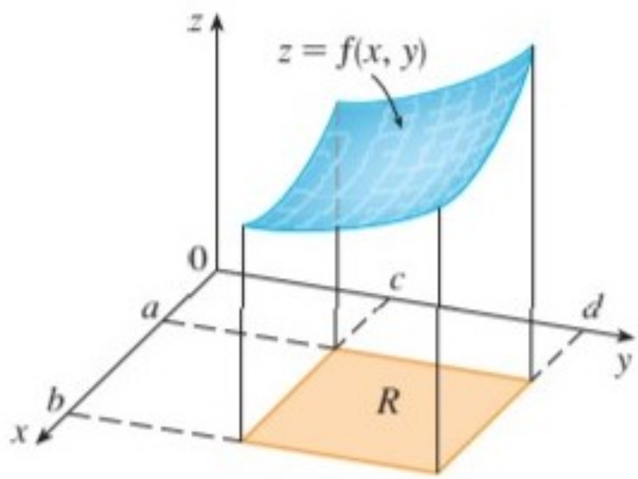
Double Integrals

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x, y)$ be a function defined on R such that $f(x, y) \geq 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

Definition 79. The ______________________ of $f(x, y)$ over a rectangle R is

$$\iint_R f(x, y)\ dA = \lim_{|P| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

if this limit exists.

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Question: How can we compute a double integral?

Answer:

Let $f(x, y) = 2xy$ and lets integrate over the rectangle $R = [1, 3] \times [0, 4]$.

We want to compute $\int_1^3 \int_0^4 f(x, y)\ dy\ dx$, but lets consider the slice at $x = 2$.

What does $\int_0^4 f(2, y)\ dy$ represent here?

In general, if $f(x,y)$ is integrable over $R=[a,b]\times[c,d]$, then $\int_c^d f(2,y)\,dy$ represents:

What about $\int_c^d f(x,y)dy$?

Let $A(x)=\int_c^d f(x,y)dy$. Then,

$$= \int_a^b A(x)dx =$$

This is called an ____________________.

Example 80. Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y\ dy\ dx.$

Theorem 81 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R=[a,b]\times[c,d]$, then*

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 82. *You try it!* Integrate:

a) $\displaystyle\int_0^2 \int_{-1}^1 x - y \; dy \; dx$ **easy**

b) $\displaystyle\int_0^1 \int_0^1 \frac{y}{1+xy} \; dx \; dy$ **medium**

c) $\displaystyle\int_1^4 \int_1^e \frac{\ln x}{xy} \; dx \; dy$ HARD!

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c) $\int_1^4 \int_1^e \frac{\ln x}{xy} \; dx \; dy$ HARD!

Example 83. Compute $\iint_R xe^{e^{e^y}}\ dA$, where R is the rectangle $[-1,1]\times[0,4]$.

Hint: Fubini's Theorem.