§15.2 Double Integrals on General Regions

Question: What if the region R we wish to integrate over is not a rectangle?

Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Example 84. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

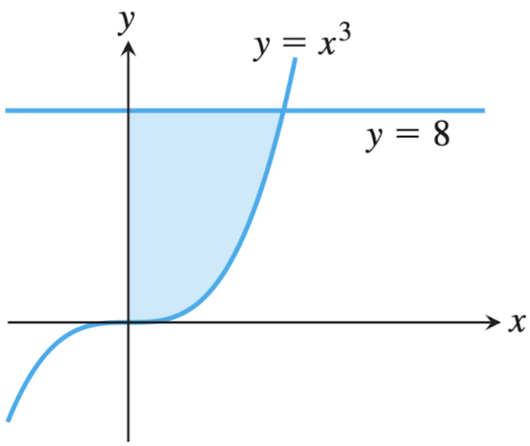
Vertically simple:

Horizontally simple:

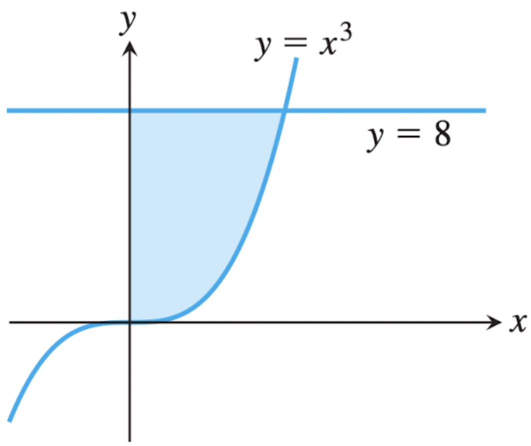
Example 85. Write the two iterated integrals for $\iint_R 1 \, dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Example 86. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2 y \, dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.

Example 87. *You try it!* Write the two iterated integrals for $\iint_R 1 \; dA$ for the region R which is bounded by $x = 0, y = 8$, and $y = x^3$.



Example 87. *You try it!* Write the two iterated integrals for $\iint_R 1 \; dA$ for the region R which is bounded by $x = 0, y = 8$, and $y = x^3$.



Example 88. Sketch the region of integration for the integral

$$\int_0^1 \int_{4x}^4 f(x, y)\ dy\ dx.$$

Then write an equivalent iterated integral in the order $dx\ dy$.

§15.3 Area & Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \rule{4cm}{0.4pt}$$

Example 89. Find the area of the region R bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

$$f_{avg} = \rule{5cm}{0.4pt}$$

Example 90. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.

Example 91. *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

Example 91. *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

Example 92. Find the average value of the function $f(x, y) = \sin(x + y)$ on (a) the region $R_1 = [0, \pi] \times [0, \pi]$, and (b) the region $R_2 = [0, \pi] \times [0, \pi/2]$.

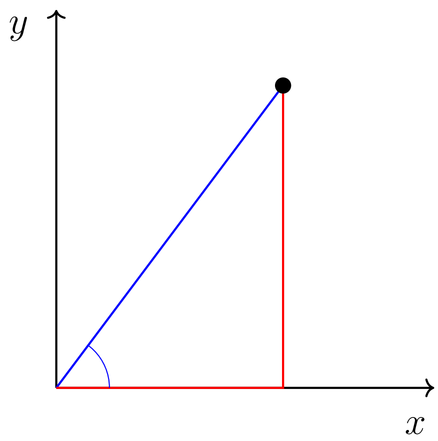
Hint: choose your order of integration carefully!

Example 93. *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of f over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.

Example 93. *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of f over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.

§15.4 Double Integrals in Polar Coordinates

Review of Polar Coordinates



Cartesian coordinates: Give the distances in ________ and ________ directions from ________

Polar coordinates:

- r = distance from ________ to ________
- θ = angle between the ray ________ and the positive ____________

We can use trigonometry to go back and forth.

Polar to Cartesian:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

Example 94. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the** xy**-plane.**

c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

Example 95. Compute the area of the disk of radius 5 centered at $(0, 0)$.

Remember: In polar coordinates, the area form $dA =$________________

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\ dA$.

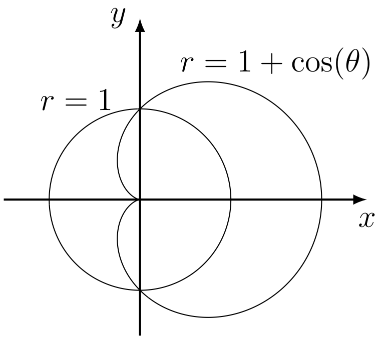
Example 96. Compute the area of the disk of radius 5 centered at $(0, 0)$.
Cont.

Remember: In polar coordinates, the area form $dA =$ ____________

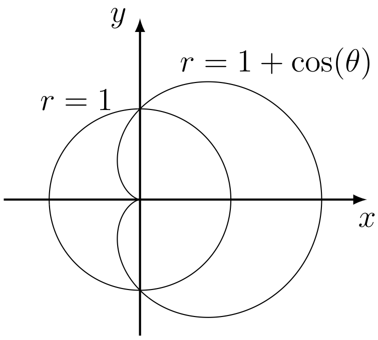
Example 96. Compute $\iint_D e^{-(x^2+y^2)}\, dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.

Example 97. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



Example 100. Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. Then, evaluate the original integral to find the value of $\iint_R f(x,y)\ dA$.

$$\int_{\pi/6}^{\pi/2} \int_{1}^{\csc\theta} r^2 \cos\theta\ dr\ d\theta$$

Tips and tricks

For horizontal lines such as $x = 2$:

For vertical lines such as $y = 1$ (e.g., Example 100):

For off-set circles such as $x^2 + (y - 1)^2 = 1$ (e.g., Example 98):

Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

Math 2551 Worksheet: Exam 2 Review

1. Which of the following statements are true if $f(x,y)$ is differentiable at (x_0,y_0)? Give reasons for your answers.
   (a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0,y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0,y_0)\mathbf{i}+f_y(x_0,y_0)\mathbf{j})\cdot\mathbf{u}$.
   (b) The derivative of f at (x_0,y_0) in the direction of $\mathbf{u}$ is a vector.
   (c) The directional derivative of f at (x_0,y_0) has its greatest value in the direction of ∇f.
   (d) At (x_0,y_0), the vector ∇f is normal to the curve $f(x,y)=f(x_0,y_0)$.
2. Find dw/dt at $t=0$ if $w=\sin(xy+\pi), x=e^t$, and $y=\ln(t+1)$.
3. Find the extreme values of $f(x,y)=x^3+y^2$ on the circle $x^2+y^2=1$.
4. Test the function $f(x,y)=x^3+y^3+3x^2-3y^2$ for local maxima and minima and saddle points and find the function's value at these points.
5. Find the points on the surface $xy+yz+zx-x-z^2=0$ where the tangent plane is parallel to the xy-plane.
6. Evaluate the integral $\displaystyle\int_0^1\int_{2y}^2 4\cos(x^2)\,dx\,dy$. Describe why you made any choices you did in the course of evaluating this integral.
7. If $f(x,y)\geq 2$ for all (x,y), is it possible that the average value of $f(x,y)$ on a unit disk centered at the origin is $\dfrac{2}{\pi}$?
8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.