§15.8 Change of Variables in Multiple Integrals

Thinking about single variable calculus: Compute $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

Theorem 116 (Substitution Theorem)**.** *Suppose* $\mathbf{T}(u, v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

$$\iint_R f(x, y)\ dx\ dy = \iint_G f(\mathbf{T}(u, v))|\det(D\mathbf{T}(u, v))|\ du\ dv.$$

Example 117. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x - y}{2}\ dx\ dy$ via the transformation $x = u + v$, $y = 2v$.

1. **Find T:**

2. **Find G and sketch:**

3. **Find Jacobian:**

4. **Convert and use theorem:**

Example 118. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v,\ \ y = v.$$

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2}\ dx\ dy?$$

i) $u = x, v = y$

ii) $u = \sqrt{x^2 + y^2}, v = \arctan(y/x)$

iii) $u = 2x - y,\ v = y^3$

iv) $u = y,\ v = 2x - y$

v) $u = 2x - y,\ v = y$

vi) $u = e^{(2x-y)^2},\ v = y^3$

Example 118. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v, \ \ y = v.$$

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2} \, dx \, dy?$$

i) $u = x, v = y$

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

iii) $u = 2x - y, \ v = y^3$

iv) $u = y, \ v = 2x - y$

v) $u = 2x - y, \ v = y$

vi) $u = e^{(2x-y)^2}, \ v = y^3$

Theorem 119 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane and* $T(u_0, v_0) = (x_0, y_0)$*, then we have*

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

Example 120. Let's evaluate $\displaystyle\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

1. **Find the rectangle G in the uv plane that is mapped to R**

2. **Evaluate $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ in terms of u and v without directly solving for $\mathbf{T}$ using the theorem above**

3. **Use the Substitution Theorem to compute the integral.**

§16.1 Line Integrals of Scalar Functions

Chapter 16: Vector Calculus

Goals:

- Extend __________ integrals to __________ objects living in higher-dimensional space
- Extend the __________________________ in new ways

We will use tools from everything we have covered so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Example 121. Suppose we build a wall whose base is the straight line from $(0, 0)$ to $(1, 1)$ in the xy-plane and whose height at each point is given by $h(x, y) = 2x + y^2$ meters. What is the area of this wall?

Definition 122. The **line integral** of a scalar function $f(x, y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x, y)\ ds =$$

What things can we compute with this?

- If $f = 1$:
- If $f = \delta$ is a density function:
- If f is a height:

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \leq t \leq b$
2. Compute $ds = \|\mathbf{r}'(t)\|\ dt$
3. Substitute: $\int_C f(x, y, z)\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt$
4. Integrate

Example 123. *You try it!* Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \leq t \leq \frac{1}{10}$.

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \leq t \leq b$

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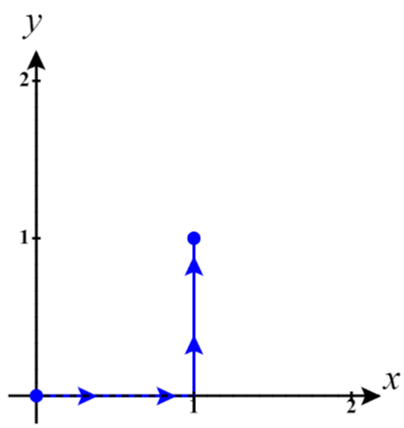
4. Integrate

Example 123. *You try it!* Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \leq t \leq \frac{1}{10}$.

Example 124. Compute $\int_C 2x + y^2 \; ds$ along the curve C pictured below.



Example 125. *You try it!* Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \leq t \leq b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \leq t \leq b$ is also a parameterization of C with the same orientation

b) $\mathbf{r}(2t)$ for $a/2 \leq t \leq b/2$ is also a parameterization of C with the same orientation

c) $\mathbf{r}(-t)$ for $a \leq t \leq b$ is also a parameterization of C with the opposite orientation

d) $\mathbf{r}(-t)$ for $-b \leq t \leq -a$ is also a parameterization of C with the opposite orientation

e) $\mathbf{r}(b-t)$ for $0 \leq t \leq b-a$ is also a parameterization of C with the opposite orientation

Example 125. *You try it!* Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \leq t \leq b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \leq t \leq b$ is also a parameterization of C with the same orientation

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e) $\mathbf{r}(b-t)$ for $0 \leq t \leq b-a$ is also a parameterization of C with the opposite orientation

Example 126. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2 + 1$ from $(1, 2)$ to $(0, 1)$ and use it to write the integral $\int_C x^2 + y^2 \, ds$ as an integral with respect to your parameter.

Example 126. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2+1$ from $(1, 2)$ to $(0, 1)$ and use it to write the integral $\int_C x^2+y^2\, ds$ as an integral with respect to your parameter.

§16.2 Vector Fields & Vector Line Integrals

Vector Fields:

Definition 127. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain.

Examples:

-
-
-
-
-

Graphically: For each point (a, b, c) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b, c)$ with its base at (a, b, c).

Tools: CalcPlot3d
Field Play

Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The ______________________________ by a field **F** on an object moving along a curve C is given by

Example 128. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x, y) = \langle x, y\rangle$ N. Find the work done by **F** on a moving object from $(0, 1)$ to $(1, 0)$ in a straight line, where x, y are measured in meters.

1. The ______________________ along a curve C of a velocity field $\mathbf{F}$ for a fluid in motion is given by

When C is ____________, this is called ____________________. C is called ________________ if it does not intersect itself.

Example 129. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x \rangle$ cm/s around the unit circle, parameterized counterclockwise.

Example 130. *You try it!* What is the circulation of $\mathbf{F}(x, y) = \langle x, y \rangle$ around the unit circle, parameterized counterclockwise?

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt$
4. Integrate

Example 130. *You try it!* What is the circulation of $\mathbf{F}(x, y) = \langle x, y \rangle$ around the unit circle, parameterized counterclockwise?

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt$
4. Integrate

Idea: ______________ across a plane curve of a 2D-vector field measures the flow of the field across that curve (instead of along it).

We compute this with the integral

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds.$$

The sign of the flux integral tells us whether the net flow of the field across the curve is in the direction of __________ or in the opposite direction.

We can choose $\mathbf{n}$ to be either of

Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \pm \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle \, dt$ (sign based on choice of normal)
4. Integrate

Example 131. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0\rangle$ cm/s across the quarter of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ in the first quadrant, oriented away from the origin.