§16.3 Conservative Vector Fields & Fundamental Theorem

Definition 132. A vector field $\mathbf{F}$ is **path independent** on an open region D if ______________________________ for all paths C in the region that have the same endpoints.

When $\mathbf{F}$ is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F} \cdot d\mathbf{r}.$$

Example 133. If C is any closed path and $\mathbf{F}$ is path independent on a region containing C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} =$$

Question: Given $\mathbf{F}$, how do we tell if it is path independent on a particular region?

For example, is $\mathbf{F}(x, y) = \langle x, y\rangle$ a path independent vector field on its domain?

Example 134. *You try it!* Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x\rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

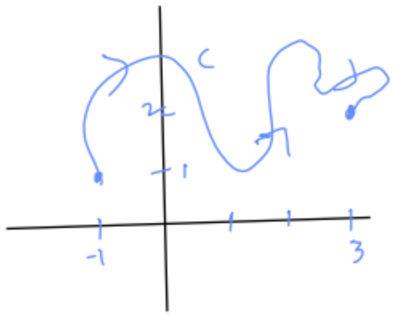
A different idea: Suppose $\mathbf{F}$ is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a ________________________ for $\mathbf{F}$. In this case we also say that $\mathbf{F}$ is **conservative.**

Is $\mathbf{F}(x, y) = \langle x, y \rangle$ conservative?

Theorem 135 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

$$\int_C \nabla f \cdot \mathbf{T}\, ds = f(B) - f(A)$$

Example 136. Compute $\int_C \langle x, y\rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.



It follows that **every conservative field is path independent.**

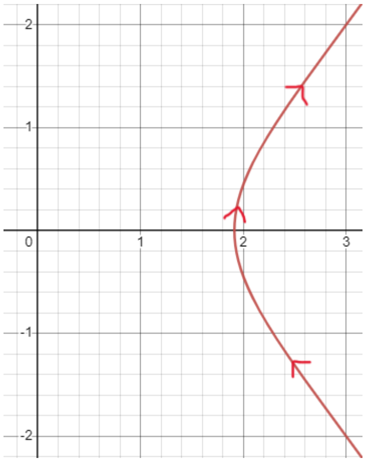
In fact, by carefully constructing a potential function, we can show the converse is also true: ________________________________

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle 0, 0, 0 \rangle$, then $\mathbf{F}$ is conservative.

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} =$
- This is also called the **mixed-partials test**, because

Example 137. Evaluate $\int_C (10x^4 - 2xy^3)\ dx - 3x^2y^2\ dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$.



§16.4 Divergence, Curl, Green's Theorem

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} =$

- $\nabla \times \mathbf{F} =$

How do we measure the change of a vector field?

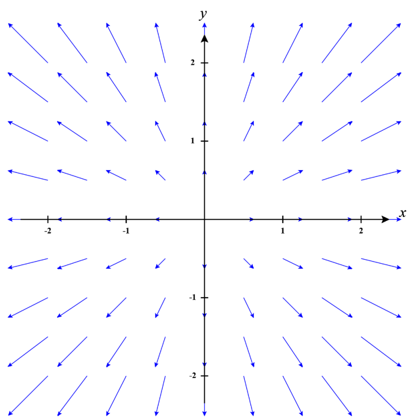
1. **Curl** (in $\mathbb{R}^3$)

   - Tells us ______________________
   - Measures ____
   - Is a _____________
   - Direction gives ______________________
   - Magnitude gives ______________________
   - $\operatorname{curl}\mathbf{F} =$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle$

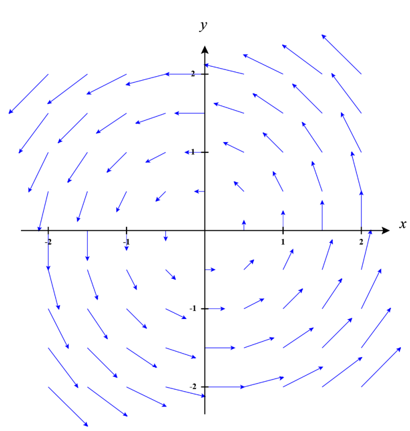
2. **Divergence** (in any $\mathbb{R}^n$)

   - Tells us ______________________
   - Measures ____
   - Is a _____________
   - $\operatorname{div}\mathbf{F} =$

Example 138. Let $\mathbf{F}(x, y) = \langle x, y \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Example 139. *You try it!* Let $\mathbf{F}(x, y) = \langle -y, x \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Question: How is this useful?

Answer: We can relate ________________________________ inside a region to the behavior of the vector field on the boundary of the region.

Theorem 140 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$. *If $\mathbf{F} = \langle P, Q \rangle$ has continuous partial derivatives around R, then*

a) Circulation form:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C P\, dx + Q\, dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R Q_x - P_y\, dA$$

b) Flux form:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\, dy - Q\, dx = \iint_R (\nabla \cdot \mathbf{F})\, dA = \iint_R P_x + Q_y\, dA$$

Example 141. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0,$ and $y = 1$ oriented counterclockwise.

Example 142. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first octant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.

Example 143. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \leq t \leq \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y \rangle$.

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.