

§16.5, 16.6 Surfaces & Surface Integrals

Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$	$z = f(x, y)$
Implicit:	$F(x, y) = 0$	$F(x, y, z) = 0$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$	

Example 144. Give parametric representations for the surfaces below.

a) $x = y^2 + \frac{1}{2}z^2 - 2$

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz -plane.

c) $x^2 + y^2 + z^2 = 9$

d) $x^2 + y^2 = 25$

What can we do with this?

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is _____
- A rectangle of size $\Delta u \times \Delta v$ in the uv -domain is mapped to a rectangle of size _____ on the surface in \mathbb{R}^3 .
- Thus, $\text{Area}(S) =$

Example 145. *You try it!* Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

Example 146. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

§16.6, 16.7 Flux Surface Integrals, Stokes' Theorem

Goal: If \mathbf{F} is a vector field in \mathbb{R}^3 , find the total flux of \mathbf{F} through a surface S .

Note: If the flux is positive, that means the net movement of the field through S is in the direction of _____

If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R , we have

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.$$

Example 147. Find $\mathbf{r}_u \times \mathbf{r}_v$ and $\|\mathbf{r}_u \times \mathbf{r}_v\|$ when $z = f(x, y)$ so that S is the graph of a scalar function with domain in \mathbb{R}^2 .

Example 148. Find $\mathbf{r}_u \times \mathbf{r}_v$ and $\|\mathbf{r}_u \times \mathbf{r}_v\|$ when S is a portion of a sphere of radius $\rho = a$, for some fixed constant a , using the standard spherical coordinates for your parametrization.

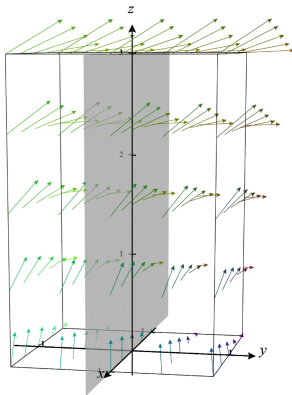
Example 149. Find the flux of $\mathbf{F} = \langle x, -y, z \rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.

Example 150. *You try it!* Compute $\iint_S G \cdot \mathbf{n} \, d\sigma$ the flux of G across the surface S .

$$G(x, y, z) = x^2, \quad S : x^2 + y^2 + z^2 = 1$$

Example 151. *You try it!* Suppose S is a smooth surface in \mathbb{R}^3 and \mathbf{F} is a vector field in \mathbb{R}^3 . **True or False:** If $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma > 0$, then the angle between \mathbf{F} and \mathbf{n} is acute at all points on S .

Example 152. *You try it!* Based on the plot of the vector field \mathbf{F} and the surface S below, oriented in the positive y -direction, is the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$ positive, negative, or zero?



Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$

2. *curl:* $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in \mathbb{R}^3 with continuous partial derivatives. Compute the divergence of the curl of \mathbf{F} , i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

Theorem 154 (Stokes' Theorem). *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let \mathbf{F} be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

- If S is a region R in the xy -plane, then we get:

- An **oriented surface** is one where _____

- S and C are oriented compatibly if:

Example 155. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by calculating the flux across the interior of C .

$$\mathbf{F} = \langle y, xz, x^2 \rangle$$

C : boundary of $x + y + z + 1$ in first octant,
oriented counter-clockwise from above.

Example 156. *You try it!* Use Stokes' Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$ the flux of \mathbf{F} across S by calculating the circulation line integral around the boundary curve C of S .

$$\mathbf{F} = \langle 2z, 3x, 5y \rangle$$

$$S : \mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, (4 - r^2) \rangle$$

$$R : r \in [0, 2], \theta \in [0, 2\pi]$$

§16.7 Stokes' Theorem

Theorem 152 (Stokes' Theorem). *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let \mathbf{F} be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Example 153 (DD). Let $\mathbf{F} = \langle -y, x + (z-1)x^{x \sin(x)}, x^2 + y^2 \rangle$. Find $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$ over the surface S which is the part of the sphere $x^2 + y^2 + z^2 = 2$ above $z = 1$, oriented away from the origin.

Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C ?

Example 154. Suppose $\text{curl } \mathbf{F} = \langle y^{y^y} \sin(z^2), (y-1)e^{x^{x^x}} + 2, -ze^{x^{x^x}} \rangle$. Compute the net flux of the curl of \mathbf{F} over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y -axis in the plane $y = 1$.

§16.8 Divergence Theorem

Theorem 155 (Divergence Theorem). *Let S be a closed surface oriented outward, D be the volume inside S , and \mathbf{F} be a vector field with continuous partial derivatives. Then*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

Example 156. Let $\mathbf{F} = \langle y^{1234} e^{\sin(yz)}, y - x^{z^x}, z^2 - z \rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z -axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of \mathbf{F} through S .

