MATH 2551 GT-E Midterm 1
VERSION A
Summer 2025
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-14.2

Full name: Key **GT ID:**

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	2
4	10
5	10
6	12
7	6
8	6
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{u} = \mathbf{0}$.

● **TRUE** ○ **FALSE**

2. (2 points) A particle moving at constant speed must have zero acceleration.

○ **TRUE** ● **FALSE**

e.g. $r(t) = \langle \cos t, \sin t \rangle$
$|r'(t)| \equiv 1$ for all t.

3. (2 points) Which of the following is true about the quadratic surface $x = y^2 + 2z^2$? [A]

○ **A)** It is a sphere, because all of it's cross sections in the $x = k$, $y = k$, and $z = k$ planes are circles.

○ **B)** It is a hyperbolic paraboloid, because its cross-sections are hyperbolas in the $x = k$ planes and parabolas in the $z = k$ and $y = k$ planes.

○ **C)** It is a cone, because its cross-sections are circles in the $x = k$ planes and straight lines in the $x = k$ and $y = k$ planes.

● **D)** It is an elliptical paraboloid, because its cross-sections are ellipses in the $x = k$ planes and parabolas in the $z = k$ and $y = k$ planes.

○ **E)** It is a plane, because all of its cross sections in the $x = k$, $y = k$, and $z = k$ planes are straight lines.

4. (10 points) Find the plane containing the lines parameterized by

$$\ell_1(t) = \langle 1,1,1\rangle + t\langle 2,1,0\rangle, \qquad -\infty < t < \infty$$
$$\ell_2(s) = \langle 0,-1,0\rangle + s\langle 1,2,1\rangle, \qquad -\infty < s < \infty$$

Give your answer in the form $Ax + By + Cz = D$ or $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.

[AJN]

Need ① $P(x_0, y_0, z_0)$ and ② $\vec{n} = \langle a,b,c\rangle = \vec{v}_1 \times \vec{v}_2$

To find intersection point P need t_0, s_0 s.t. $\ell_1(t_0) = \ell_2(s_0)$.

So $\ell_1(t) - \ell_2(s) = \langle 1+2t, 1-t, 1\rangle - \langle s, -1+2s, s\rangle = 0$

$\overset{\text{solve}}{\Rightarrow} \begin{cases} 1+2t-s=0 \\ 1-t+1-2s=0 \\ 1-s=0 \end{cases} \Rightarrow$ $s=1$ & $1+2t-1=0$ so $t=0$.

Check.

$\ell_1(0) = \langle 1,1,1\rangle$

$\ell_2(1) = \langle 1,1,1\rangle$ ✓ nice

② $\vec{v}_1 = \langle 2,1,0\rangle$ and $\vec{v}_2 = \langle 1,2,1\rangle$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = \hat{i}(1-0) - \hat{j}(2-0) + k(4-1)$$

$$= 1\hat{i} - 2\hat{j} + 3\hat{k} = \vec{n}$$

So eqn. of plane is

$$\boxed{(x-1) - 2(y-1) + 3(z-1) = 0}$$

or $\boxed{x - 2y + 3z = 2}$

5. Mei is looking for her sister Satsuki in the village of Sayama Hills, and her path up a hill is given by $\mathbf{r}(t) = \langle 2, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$, for $0 \leq t < 2$, where t is measured in hours and $\mathbf{r}(t)$ in meters. [AJN]

   (a) (2 points) What is Mei's position two hours into her journey?

   (b) (8 points) How far did Mei travel from time $t = 0$ to time $t = 2$?

(a) $\vec{r}(2) = \langle 2, \frac{1}{2}(2)^2, \frac{1}{3}(2)^3 \rangle$

$= \boxed{\langle 2, 2, 8/3 \rangle}$

(b) $L = \int_a^b |r'(t)|\, dt$ $\quad r'(t) = \langle 0, t, t^2 \rangle$ and

$|r'(t)| = \sqrt{t^2 + t^4} = \sqrt{t^2}\sqrt{1+t^2}$

$= t\sqrt{1+t^2}$ $(t>0)$

$L = \int_0^2 |r'(t)|\, dt = \int_0^2 t\sqrt{1+t^2}\, dt$

u-sub BOX

$u = 1+t^2$

$du = 2t\, dt$

$\frac{1}{2}du = dt$

@ $t=0$, $u=1$

@ $t=2$, $u=5$

$= \int_1^5 \frac{1}{2} u^{1/2}\, du = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} \Big|_1^5$

$= \frac{1}{3} 5^{3/2} - \frac{1}{3} 1^{3/2}$

$= \boxed{\left(\frac{1}{3}5^{3/2} - 1\right) m}$

6. In this problem, you will work with the spiral plane curve

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}$$

for $t > 0$. *Hint: product rule.* [AJN]

(a) (6 points) Compute the unit tangent vector $\mathbf{T}(t)$. Hint: product rule.

(b) (4 points) Compute the principal unit normal vector $\mathbf{N}(t)$.

(c) (2 points) Compute the curvature $\kappa(t)$.

(a) $T(t) = \dfrac{r'(t)}{|r'(t)|}$

$r'(t) = \langle \cancel{-\sin t} + \cancel{\sin t} + t\cos t, \cancel{\cos t} - \cancel{\cos t} + t\sin t, 0 \rangle$

$r'(t) = \langle t\cos t, t\sin t, 0 \rangle$

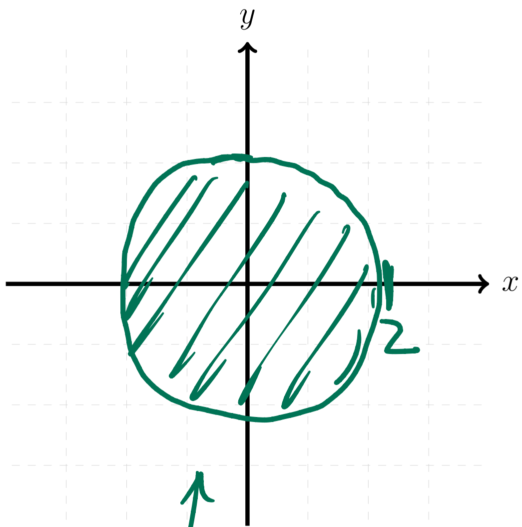
$|r'(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} \underset{(t>0)}{=} t$

So $T(t) = \frac{1}{t}\langle t\cos t, t\sin t \rangle = \boxed{\langle \cos t, \sin t, 0 \rangle}$

(b) $N(t) = \dfrac{T'(t)}{|T'(t)|} = \dfrac{\langle -\sin t, \cos t \rangle}{1} = \boxed{\langle -\sin t, \cos t, 0 \rangle}$

(c) $\kappa = \dfrac{|T'(t)|}{|r'(t)|} = \boxed{\dfrac{1}{t}}$

7. (6 points) Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. Graph the domain of f on the provided axes below. Be sure to show all your work in finding the domain. Indicate whether or not each part of the boundary of the domain is included. [AJN]



boundary included

Df need

$4 - x^2 - y^2 \geq 0$

$\Rightarrow \quad 4 \geq x^2 + y^2$

disk of $r = 2$ & center $(0,0)$

8. (6 points) Show that the limit does not exist. To receive full credit, you must show work supporting your answer, use proper limit notation, and mention the test that you are using. [AJN]

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Approach along x-axis, $y=0$

$$@\, y=0: \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2+0^2} = 0$$

Approach along line $y=x$,

$$@\, y=x: \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Since the limit depends on the path of approach,

by the TWO-PATH TEST the limit is DNE

FORMULA SHEET

- $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$
- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \displaystyle\int_a^b |\mathbf{r}'(t)|\ dt$
- $s(t) = \displaystyle\int_{t_0}^t |\mathbf{r}'(\tau)|\ d\tau$
- $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{d\mathbf{r}}{ds}$
- $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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