22min
+4 26min

MATH 2551 GT-E Midterm 1
VERSION B
Summer 2025
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-14.2

Full name: Key **GT ID:**

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	4
4	10
5	10
6	10
7	6
8	6
Total:	50

12:00

2 min

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

○ **TRUE** ● **FALSE**

2. (2 points) The sphere $(x+1)^2 + (y-2)^2 + z^2 = 9$ is centered at $(1, -2, 0)$ and has radius $r = 3$.

○ **TRUE** ● **FALSE**

3. (4 points) Which of the following vectors could be the principal unit normal vector at time $t = 2$ to a curve whose tangent line at $t = 2$ is given by

$$\ell(t) = \langle -1, 0, 0 \rangle + t\langle 1, -1, 1 \rangle.$$

You must justify your answer in the space provided to receive full credit. [AJ]

○ **A)** $\langle 1, 2, 1 \rangle$
○ **B)** $\langle 0, 0, 1 \rangle$
○ **C)** $\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \rangle$
● **D)** $\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$
○ **E)** $\left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$

$\vec{v} = \langle 1, -1, 1 \rangle$ direction of line ℓ

$\vec{n} = \langle 1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6} \rangle$ has

① unit length?

$|\vec{n}| = \sqrt{\frac{1}{6} + \frac{4}{6} + \frac{1}{6}} = 1$ ✓

② $\vec{n} \cdot \vec{v} = 0$?

$\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle \cdot \langle 1, -1, 1 \rangle$

$= \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} = 0$ ✓

12:02

7min

4. Let P_1 be the plane $x - 2y + 2z = 6$ and P_2 be the plane $2x - y + z = 6$. [AJN]

(a) (2 points) Show that P_1 and P_2 are intersecting planes by showing that the point $(2, 2, 4)$ lies on both planes.

P_1 @ $(2,2,4)$: $2 - 2(2) + 2(4) = 2 - 4 + 8 = 6$ ✓

P_2 @ $(2,2,4)$: $2(2) - 2 + 4 = 4 - 2 + 4 = 6$ ✓

(b) (2 points) Find normal vectors to both planes.

$ax + by + cz = d$ has normal vector $\vec{n} = \langle a, b, c \rangle$

So $\vec{n}_1 = \langle 1, -2, 2 \rangle$ $\quad \vec{n}_2 = \langle 2, -1, 2 \rangle$

(c) (6 points) Find a parameterization of the line of intersection of the planes.
Hint: $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$.

$l(t) = \vec{OP} + t\vec{v}$ where $P(2,2,4)$ & $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}(-4+2) - \hat{j}(2-4) + \hat{k}(-1+4)$$

$$\vec{v} = -2\hat{i} + 2\hat{j} + 3\hat{k}$$

check $\vec{n}_1 \cdot \vec{v} = -2 - 4 + 6 = 0$ ✓

$\vec{n}_2 \cdot \vec{v} = -4 - 2 + 6 = 0$ ✓

So

$$\boxed{l(t) = \langle 2, 2, 4 \rangle + t\langle -2, 2, 3 \rangle}$$

12:09 6 min w/ picture

5. Satsuki is riding the Cat-Bus up a large hill in the village of Sayama Hills, and the path the bus takes up the hill is given by $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$, for $0 \le t < 2\pi$, where t is measured in minutes and $\mathbf{r}(t)$ in kilometers. [AJN]

   (a) (2 points) What was Satsuki's speed halfway through her journey, when $t = \pi$?

   (b) (2 points) What is Satsuki's **position** at the end of her journey, when $t = 2\pi$?

   (c) (6 points) How far did Satsuki travel in total from time $t = 0$ to time $t = 2\pi$?

(a) $r'(t) = \langle -2\sin 2t, 2\cos 2t, 1 \rangle$

$|r'(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 1}$

$= \sqrt{4+1} = \boxed{\sqrt{5} \text{ km/min}}$ (for all t)

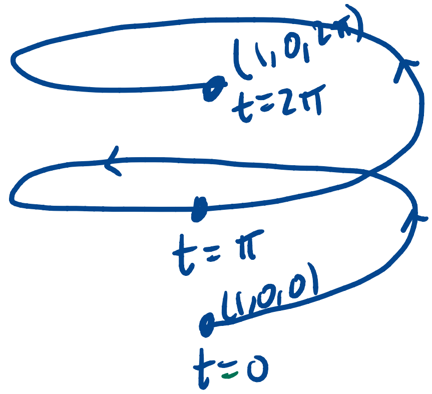
(b)

$\vec{r}(2\pi) = \langle \cos(2*2\pi), \sin(2*2\pi), 2\pi \rangle$

$= \langle \cos 4\pi, \sin 4\pi, 2\pi \rangle = \boxed{\langle 1, 0, 2\pi \rangle}$

(c) $L = \int_0^{2\pi} |r'(t)|\, dt = \int_0^{2\pi} \sqrt{5}\, dt = \sqrt{5}t \Big|_0^{2\pi}$

$= \boxed{\sqrt{5} * 2\pi \text{ km}}$



total distance covered $2\sqrt{5}\pi$

6. In this problem, you will work with the curve

$$\mathbf{r}(t) = t\mathbf{i} + (\ln \cos t)\mathbf{j}$$

for $-\pi/2 < t < \pi \cancel{/} 2$. [AJN]

(a) (5 points) Compute the unit tangent vector $\mathbf{T}(t)$. Hint: Chain rule.

(b) (3 points) Compute the principal unit normal vector $\mathbf{N}(t)$.

(c) (2 points) Compute the curvature $\kappa(t)$.

(a) $T(t) = \dfrac{r'(t)}{|r'(t)|}$ $\quad r'(t) = \langle 1, \dfrac{1}{\cos t} * -\sin t \rangle = \langle 1, -\tan t \rangle$

So $|r'(t)| = \sqrt{1 + \tan^2 t}$

$= \sqrt{\sec^2 t}$

$= \sec t$

$\sin^2 t + \cos^2 t = 1$

$\Rightarrow \tan^2 + 1 = \sec^2 t$

$T(t) = \left\langle \dfrac{1}{\sec t}, \dfrac{-\tan t}{\sec t} \right\rangle$ $\quad \dfrac{-\sin t/\cos t}{1/\cos t} = -\sin t$

$= \boxed{\langle \cos t, -\sin t \rangle}$

(b) $N(t) = \dfrac{T'(t)}{|T'(t)|} = \dfrac{\langle -\sin t, -\cos t \rangle}{1} = \boxed{\langle -\sin t, -\cos t \rangle}$

(c) $\kappa = \dfrac{|T'(t)|}{|r'(t)|} = \dfrac{1}{\sec t} = \boxed{\cos t}$

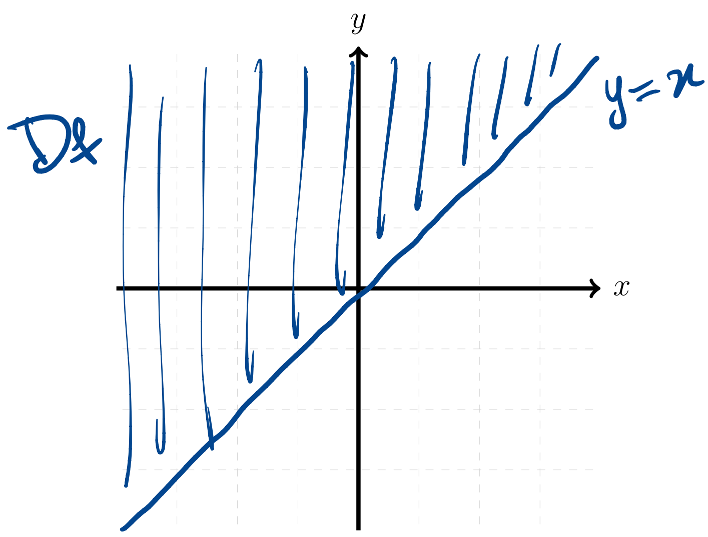
12:18

7. (6 points) Let $f(x, y) = \sqrt{y - x}$. Graph the domain of f on the provided axes below. Be sure to show all your work in finding the domain. Indicate whether or not each part of the boundary of the domain is included. [AJN]

D_f needs

$y - x \geq 0$

$\Rightarrow y \geq x$



line included.

12:20 12:22

8. (6 points) The limit below exists. Evaluate the limit. To receive full credit, you must show work supporting your answer and use proper notation. *Hint: use algebra.* [AJN]

$$\lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$$

$$\lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{(x,y)\to(4,3)} \frac{\overbrace{\sqrt{x}}^{a}-\overbrace{\sqrt{y+1}}^{b}}{x-y-1} * \frac{\overbrace{\sqrt{x}}^{a}+\overbrace{\sqrt{y+1}}^{b}}{\sqrt{x}+\sqrt{y+1}}$$

$$= \lim_{\text{"}} \frac{\cancel{x-(y+1)}}{\cancel{(x-y-1)}(\sqrt{x}+\sqrt{y+1})}$$

$$= \lim_{(x,y)\to(4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{\sqrt{4}+\sqrt{4}} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

FORMULA SHEET

- $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \displaystyle\int_a^b |\mathbf{r}'(t)|\ dt$
- $s(t) = \displaystyle\int_{t_0}^t |\mathbf{r}'(\tau)|\ d\tau$
- $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{d\mathbf{r}}{ds}$
- $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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