MATH 2551 GT-E Midterm 2
VERSION A
Summer 2025
COVERS SECTIONS 14.3-14.8, 15.1-15.4

Full name: Key **GT ID:**

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

( ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	2
4	2
5	5
6	5
7	4
8	8
9	10
10	10
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) The function $f(x, y) = e^{2x} \cos y$ has no critical points.

   ● **TRUE** ○ **FALSE**

   $\nabla f = \langle 2e^{2x}\cos y, -e^{2x}\sin y\rangle = \langle 0,0\rangle$ $\begin{cases} 2e^{2x}\cos y = 0 \\ -e^{2x}\sin y = 0 \end{cases}$

   would need $\cos y = \sin y = 0$ ✗ impossible!

2. (2 points) Suppose $f(x, y, z)$ is a differentiable function at $P(x_P, y_P, z_P)$. State an equation for the linearization $L(x, y, z)$ of f at P. [AN]

   $L(x,y,z) = z_P + f_x(P)(x - x_P) + f_y(P)(y - y_P) + f_z(P)(z - z_P)$

3. (2 points) Suppose $w = f(x, y, z)$ is a differentiable function and $x(u, v), y(u, v), z(u, v)$ are all differentiable. Give the formula for $\frac{\partial w}{\partial v}$. [AN]

   $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

4. (2 points) Which of the following is FALSE about the function $f(x, y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$.

   Select all that apply. [A]

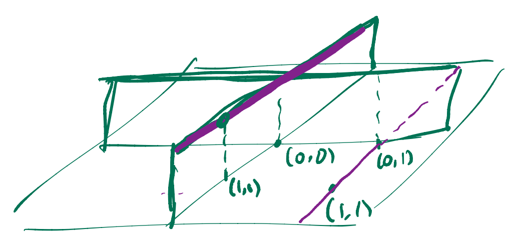
   ○ **A)** f_x is continuous at $(0, 0)$. ✓

   ○ **B)** f_x is continuous at $(1, 1)$. ✓

   ○ **C)** f_x is continuous at $(1, 0)$. ✓

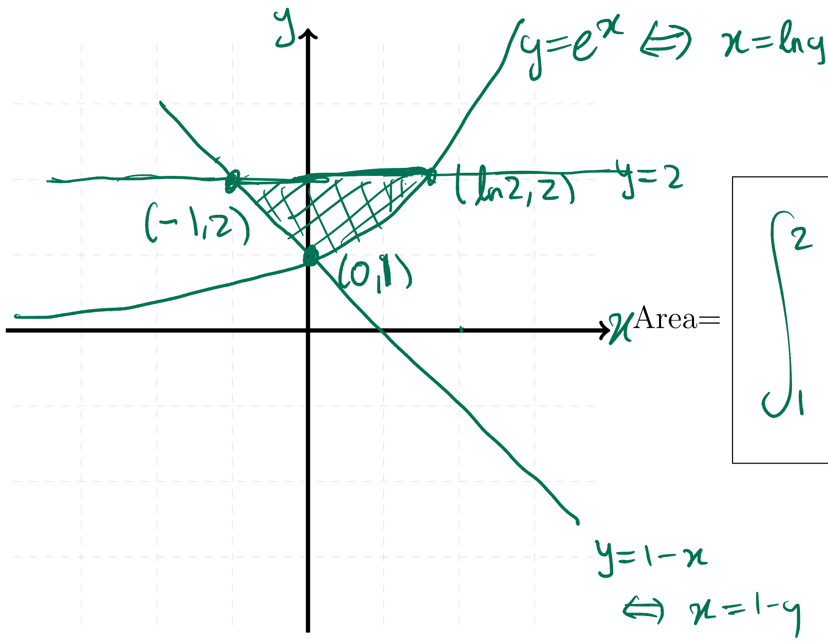
   ● **D)** f is differentiable at $(0, 0)$. ✗

   ○ **E)** f is differentiable at $(1, 1)$. ✓



5. (5 points) (a) Sketch the region R on the axes provided which is bounded by the curve $y = e^x$ and the lines $y = 2$ and $y = 1 - x$. Include labels for the axes, labels for the boundary curves or lines, and labels for the points of intersection. [AN]

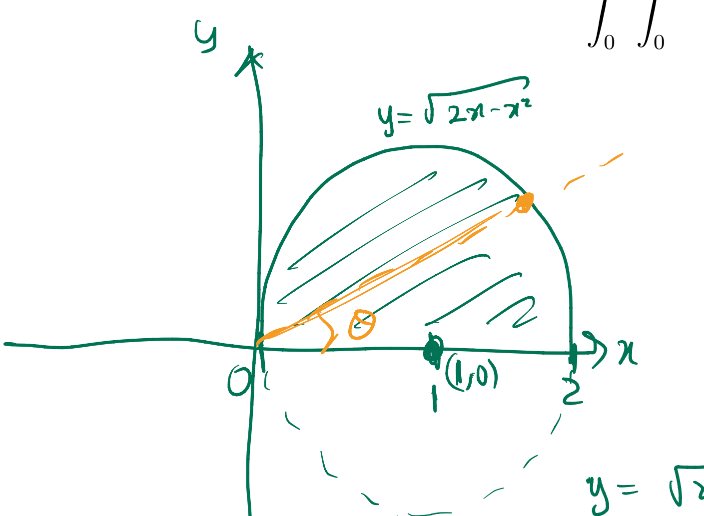
(b) Set up *but do not evaluate* a single iterated integral which computes the area of the region. Put your answer in the box. [AN]



Area= $\displaystyle\int_1^2\int_{1-y}^{\ln y} 1 \, dx\, dy$

6. (5 points) Convert the given integral to polar coordinates. *Do not evaluate!* [AN]

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\, dx$$



Volume= $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{1}{r^4} * r \, dr\, d\theta$

$\theta \in [0, \pi/2]$

$r \in [0, 2\cos\theta]$ * $\longrightarrow$

$y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2$

$\Rightarrow x^2 - 2x + 1 + y^2 = 1$

$\Rightarrow (x-1)^2 + y^2 = 1$

circle w/ r=1
center @ (1,0)

@ $x = r\cos\theta$ & $y = r\sin\theta$

$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$

$\Rightarrow r^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1}$

$\Rightarrow r^2 - 2r\cos\theta = 0$

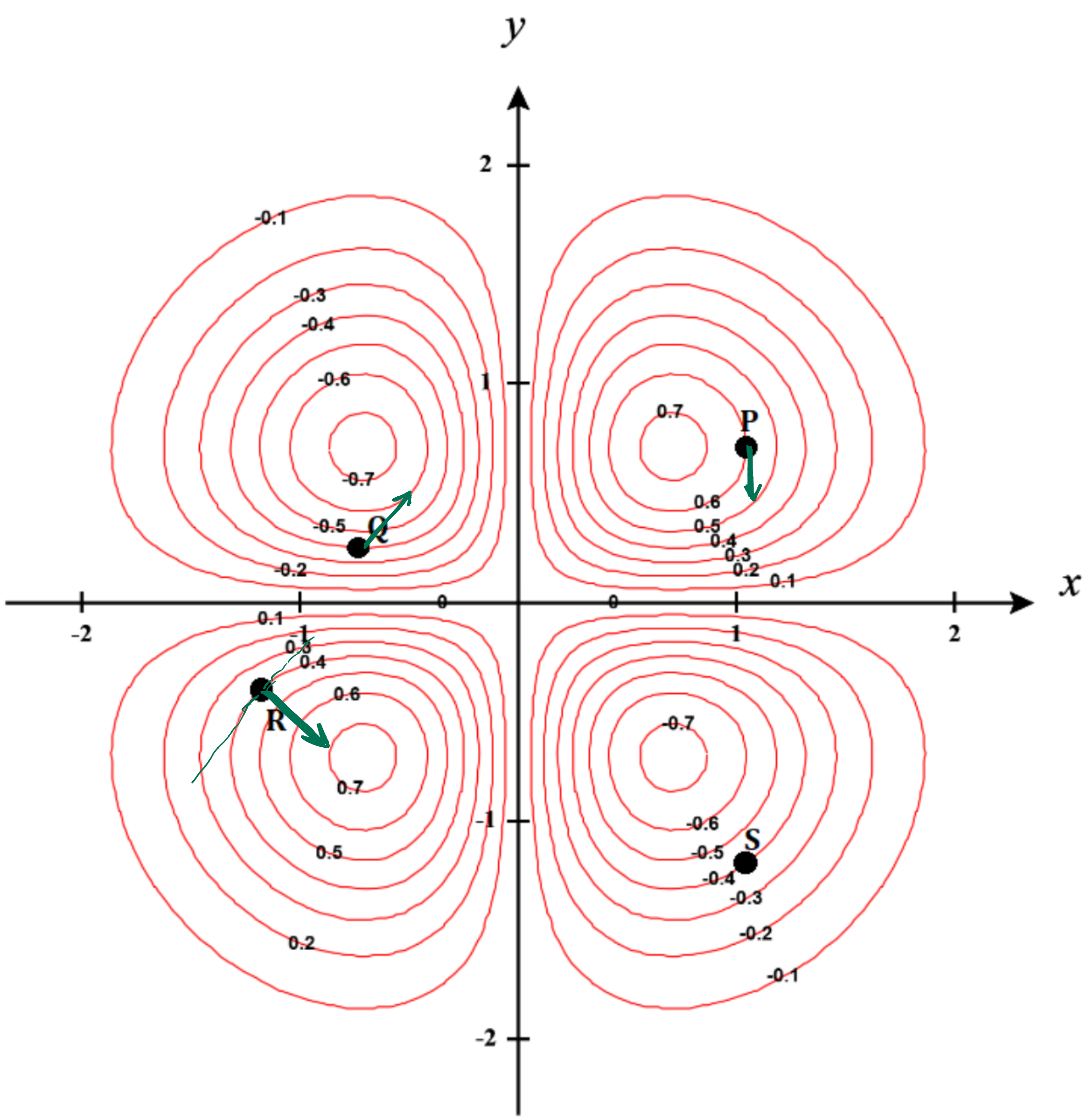
$\Rightarrow \boxed{r = 2\cos\theta}$ *

7. In this problem, you will work with the function $f : \mathbb{R}^2 \to \mathbb{R}$ whose contour plot near the origin is shown below. [A N]

   (a) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative a the point Q in the direction $\langle 1, 1\rangle$. $\boxed{-}$

   (b) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative a the point P in the direction $\langle 0, -1\rangle$. $\boxed{0}$

   (c) (2 points) Draw a vector which is a positive scalar multiple of the gradient vector of f at the point R.



8. (8 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = (x - z)y$ and $\mathbf{r}(s, t) : \mathbb{R}^2 \to \mathbb{R}^3$ be the function

$$\mathbf{r}(s,t) = \begin{bmatrix} s-2 \\ st \\ e^{st} \end{bmatrix}.$$

$x = s-2$

$y = st$

$z = e^{st}$

(a) Find Df.

(b) Find $D\mathbf{r}$.

(c) Evaluate $\mathbf{r}(3, 1)$ and $D\mathbf{r}|_{(s,t)=(3,1)}$.

(d) Finally, use the Chain Rule to evaluate $D(f(\mathbf{r}(s, t)))|_{(s,t)=(3,1)}$.

[AJN]

(a) $Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} y & x-z & -y \end{bmatrix}$

(b) $D\vec{r} = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & s \\ te^{st} & se^{st} \end{bmatrix}$

(c) $\vec{r}(3,1) = \begin{bmatrix} 3-2 \\ 3 \\ e^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ e^3 \end{bmatrix}$ $\quad Dr\big|_{@(3,1)} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ e^3 & 3e^3 \end{bmatrix}$

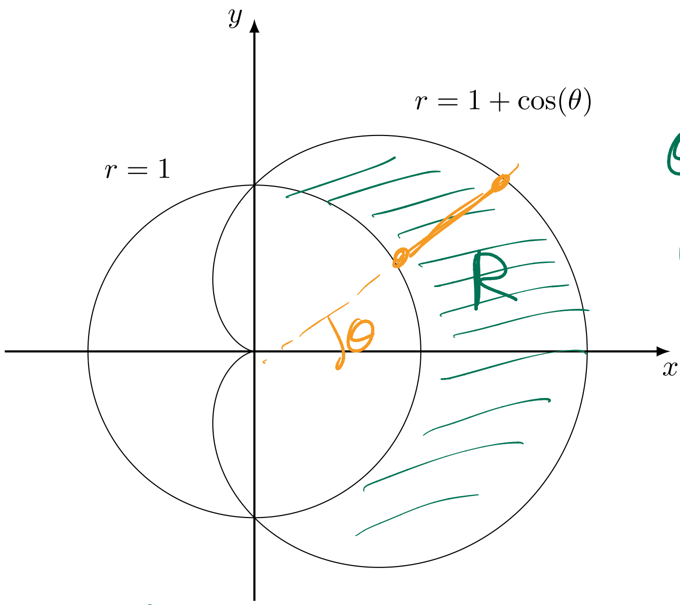
(d) $D(f\circ r)\big|_{(s,t)=(3,1)} = D(f\circ r)\big|_{(x,y,z)=(1,3,e^3)}$

$= \begin{bmatrix} 3 & 1-e^3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ e^3 & 3e^3 \end{bmatrix}$

$= \begin{bmatrix} 3+1-e^3-3e^3 & 3-3e^3-9e^3 \end{bmatrix} = \boxed{\begin{bmatrix} 4-4e^3 & 3-12e^3 \end{bmatrix}}$

9. (10 points) (a) Write and evaluate the integral for the area of the region R between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$ which is completely contained in Quadrants I and IV. (b) Then, use the fact that $\iint_R f(x, y)\, dA = 5\pi/8$ to find the average value of $f(x, y) = x$ over the region R. *Hint:* $\cos^2 t = \frac{1+\cos 2t}{2}$.

[AJN]



$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$r \in [1, 1+\cos\theta]$

(a) $\text{Area} = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \Big|_1^{1+\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1+\cos\theta)^2 - 1\, d\theta$

using hint

$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cancel{1} + 2\cos\theta + \cos^2\theta - \cancel{1}\, d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta + \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cdot \frac{1+\cos 2\theta}{2} d\theta$

$= 2\sin\theta \Big|_0^{\pi/2} + \frac{1}{4} * 2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = 2(1-0) + \frac{1}{2}\left(\left(\frac{\pi}{2} + \frac{1}{2}(0)\right) - (0+0)\right)$

$y = \cos\theta$ is EVEN

$1 + \cos 2\theta$ is EVEN.

$= \boxed{2 + \pi/4}$ ANS for (a).

So (b) $\text{Avg}_R f = \frac{1}{\text{Area}} * \text{Vol} = \frac{1}{2+\pi/4} * \frac{5\pi}{8} = \boxed{\frac{5\pi}{16+2\pi}}$ Ans for (b)

10. (10 points) Find all critical points and use the Hessian matrix to classify the critical points of the function. [AJN]

$$f(x, y) = x^2 + y^3 - 3y$$

Set up $Df = [f_x \;\; f_y] = [2x \;\; 3y^2-3] = [0 \;\; 0]$

Solve $\begin{cases} 2x = 0 \\ 3y^2 - 3 = 0 \end{cases} \Rightarrow x=0 \;\&\; y = \pm 1$ get $(0,1)$, $(0,-1)$.

And $Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6y \end{bmatrix}$

@ $(0,1)$

$|Hf| = \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix} = 12 > 0$ and $f_{xx} = 2 > 0$

So $\boxed{(0,1) \text{ is a MIN}}$

@ $(0,-1)$

$|Hf| = \begin{vmatrix} 2 & 0 \\ 0 & -6 \end{vmatrix} = -12$ So $\boxed{(0,-1) \text{ is a saddle}}$

FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y) dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta), \quad y = r\sin(\theta), \quad dA = r\ dr\ d\theta$

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