MATH 2551 GT-E Midterm 2
VERSION B
Summer 2025
COVERS SECTIONS 14.3-14.8, 15.1-15.4

Full name: ______________________________ **GT ID:**______________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	2
4	2
5	5
6	5
7	4
8	8
9	10
10	10
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) The function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$ has exactly two critical points.

   ○ **TRUE** ○ **FALSE**

2. (2 points) Suppose S is a surface $F(x, y, z) = k$ and $P(x_P, y_P, z_P)$ is a point on S. If $\nabla F(P) = \langle a, b, c \rangle$ then what is the equation for the tangent plane of S at $P(x_P, y_P, z_P)$?

3. (2 points) Suppose $h = f(u, v, w)$ is a differentiable function and $u(t), v(t), w(t)$ are all differentiable. Give the formula for $\frac{dh}{dt}$. [AN]

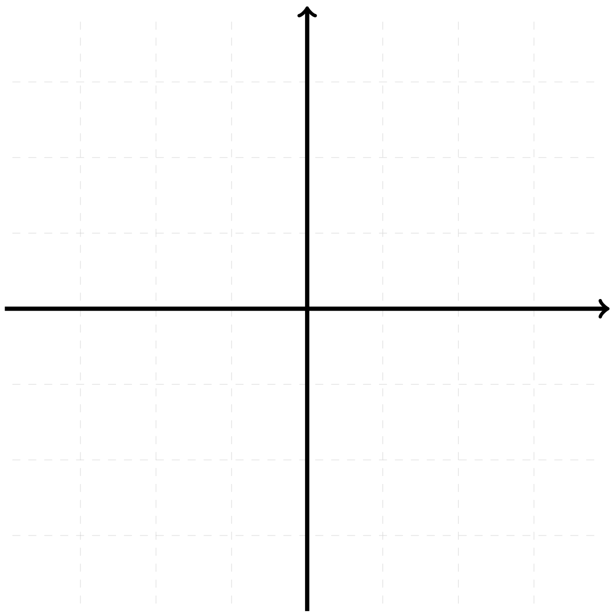
4. (2 points) Which of the following is FALSE about the function $f(x, y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$.

   Select all that apply. [A]

   ○ **A)** f is differentiable at $(0, 0)$.
   ○ **B)** f is differentiable at $(1, 1)$.
   ○ **C)** f_y is continuous at $(0, 0)$.
   ○ **D)** f_y is continuous at $(1, 1)$.
   ○ **E)** f_y is continuous at $(0, 1)$.

5. (5 points) (a) Sketch the region R on the axes provided which is bounded by $x = -y + 2$ and the curve $x = y^2$. Include labels for the axes, labels for the boundary curves or lines, and the points of intersection. [AN]

(b) Set up *but do not evaluate* an iterated integral which computes the area of the region. Put your answer in the box. [AN]



Area=

6. (5 points) Convert the given integral to polar coordinates. *Do not evaluate!* [AN]

$$\int_{-2}^{2} \int_{1}^{\sqrt{4-x^2}} e^{x^2+y^2} \, dy \, dx$$

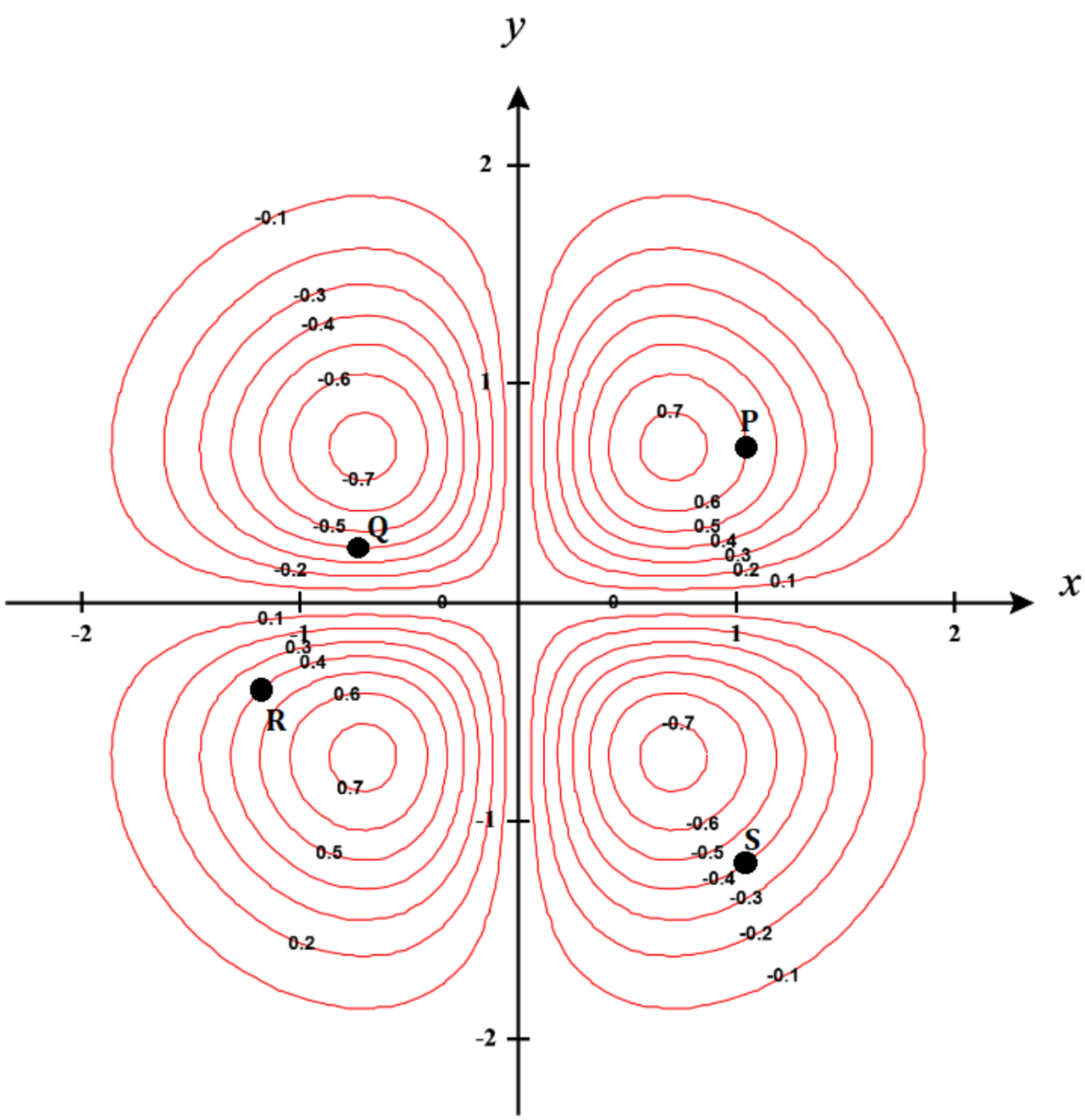
Volume=

7. In this problem, you will work with the function $f : \mathbb{R}^2 \to \mathbb{R}$ whose contour plot near the origin is shown below. [AN]

   (a) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative a the point Q in the direction $\langle 1, 0 \rangle$. ☐

   (b) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative a the point R in the direction $\langle -1, 1 \rangle$. ☐

   (c) (2 points) Draw a vector which is a positive scalar multiple of gradient vector of f at the point P.



8. (8 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = (x - y)z$ and $\mathbf{r}(s, t) : \mathbb{R}^2 \to \mathbb{R}^3$ be the function

$$\mathbf{r}(s,t) = \begin{bmatrix} s+t \\ st \\ e^{s+t} \end{bmatrix}.$$

(a) Find Df.

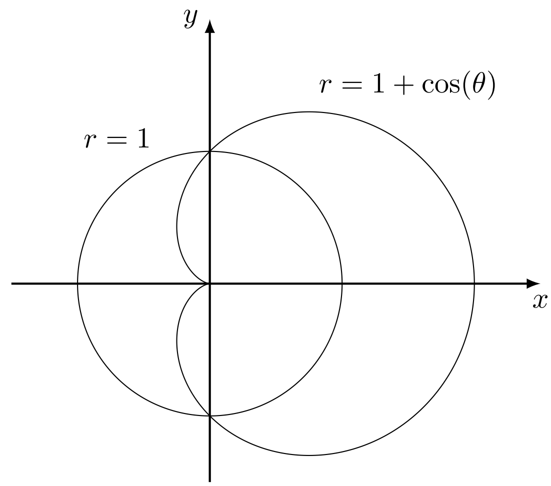
(b) Find $D\mathbf{r}$.

(c) Evaluate $\mathbf{r}(1, 0)$ and $D\mathbf{r}|_{(s,t)=(1,0)}$.

(d) Finally, use the Chain Rule to evaluate $D(f(\mathbf{r}(s, t)))|_{(s,t)=(1,0)}$.

[AJN]

9. (10 points) (a) Write and evaluate the integral for the area of the region R between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$ which is completely contained in Quadrants I and IV. (b) Then, use the fact that $\iint_R f(x, y)\, dA = 5\pi/8$ to find the average value of $f(x, y) = x$ over the region R. *Hint:* $\cos^2 t = \frac{1+\cos 2t}{2}$. [AJN]



10. (10 points) Find all critical points and use the Hessian matrix to classify the critical points of the function. [AJN]

$$f(x, y) = x^3 - 3x + y^3 - 3y^2$$

FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y) dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $\quad dA = r\ dr\ d\theta$

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