MATH 2551 GT-E Midterm 2
VERSION B
Summer 2025
COVERS SECTIONS 14.3-14.8, 15.1-15.4

Full name: Key **GT ID:**

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	2
4	2
5	5
6	5
7	4
8	8
9	10
10	10
Total:	50

12:37

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) The function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$ has exactly two critical points.

○ **TRUE** ● **FALSE**

$\nabla f = \langle 3x^2 - 6x, 3y^2 - 6y \rangle = \langle 0, 0 \rangle$ So $x = 0, 2$ $y = 0, 2$

$\Rightarrow 3x^2 - 6x = 3x(x-2) = 0$ and $3y(y-2) = 0$ $(0,0), (0,2), (2,0), (2,2)$

four crit pt.

2. (2 points) Suppose S is a surface $F(x, y, z) = k$ and $P(x_P, y_P, z_P)$ is a point on S. If $\nabla F(P) = \langle a, b, c \rangle$ then what is the equation for the tangent plane of S at $P(x_P, y_P, z_P)$?

$$\boxed{a(x - x_P) + b(y - y_P) + c(z - z_P) = 0}$$

3. (2 points) Suppose $h = f(u, v, w)$ is a differentiable function and $u(t), v(t), w(t)$ are all differentiable. Give the formula for $\frac{dh}{dt}$. [AN]

$$\boxed{\frac{dh}{dt} = \frac{\partial h}{\partial u} \cdot \frac{du}{dt} + \frac{\partial h}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial h}{\partial w} \cdot \frac{dw}{dt}}$$

4. (2 points) Which of the following is FALSE about the function $f(x, y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$.

Select all that apply. [A]

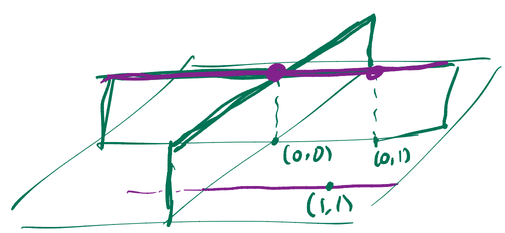
● **A)** f is differentiable at $(0, 0)$. ✗

○ **B)** f is differentiable at $(1, 1)$. ✓

○ **C)** f_y is continuous at $(0, 0)$. ✓

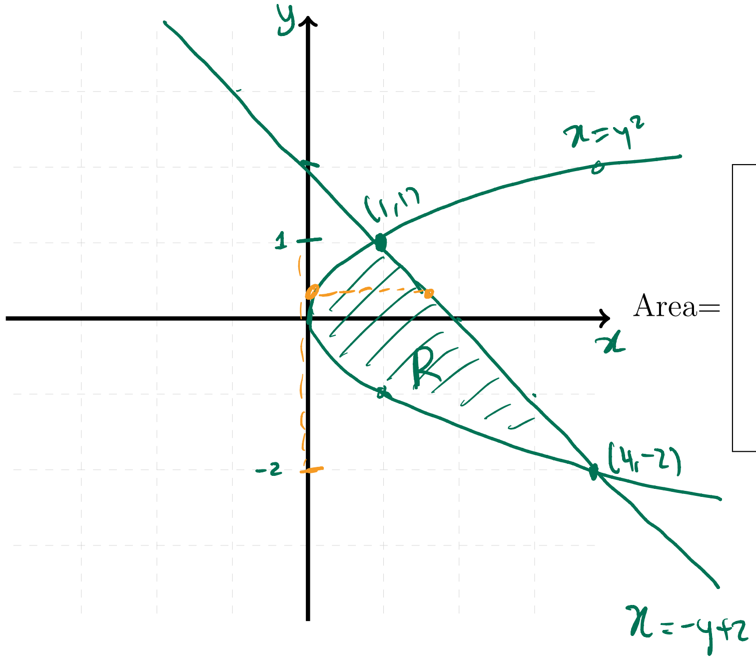
○ **D)** f_y is continuous at $(1, 1)$. ✓

○ **E)** f_y is continuous at $(0, 1)$. ✓



5. (5 points) (a) Sketch the region R on the axes provided which is bounded by $x = -y + 2$ and the curve $x = y^2$. Include labels for the axes, labels for the boundary curves or lines, and the points of intersection. [AN]

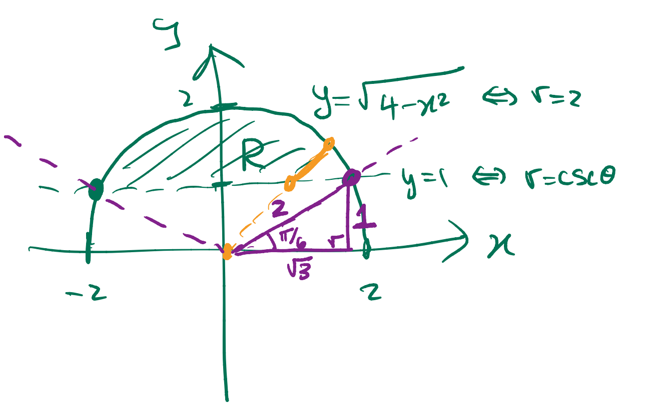
(b) Set up *but do not evaluate* an iterated integral which computes the area of the region. Put your answer in the box. [AN]



Area= $\boxed{\int_{-2}^{1}\int_{y^2}^{-y+2} 1 \, dx \, dy}$

6. (5 points) Convert the given integral to polar coordinates. *Do not evaluate!* [AN]

$$\int_{-2}^{2}\int_{1}^{\sqrt{4-x^2}} e^{x^2+y^2} \, dy \, dx$$



Volume= $\boxed{\int_{\pi/6}^{5\pi/6}\int_{\csc\theta}^{2} e^{r^2} * r \, dr \, d\theta}$

top curve : $r=2$

bot curve : $y=1$, and $y=r\sin\theta$

@ $y=1 \Rightarrow 1 = r\sin\theta$

$\Rightarrow r = \csc\theta$

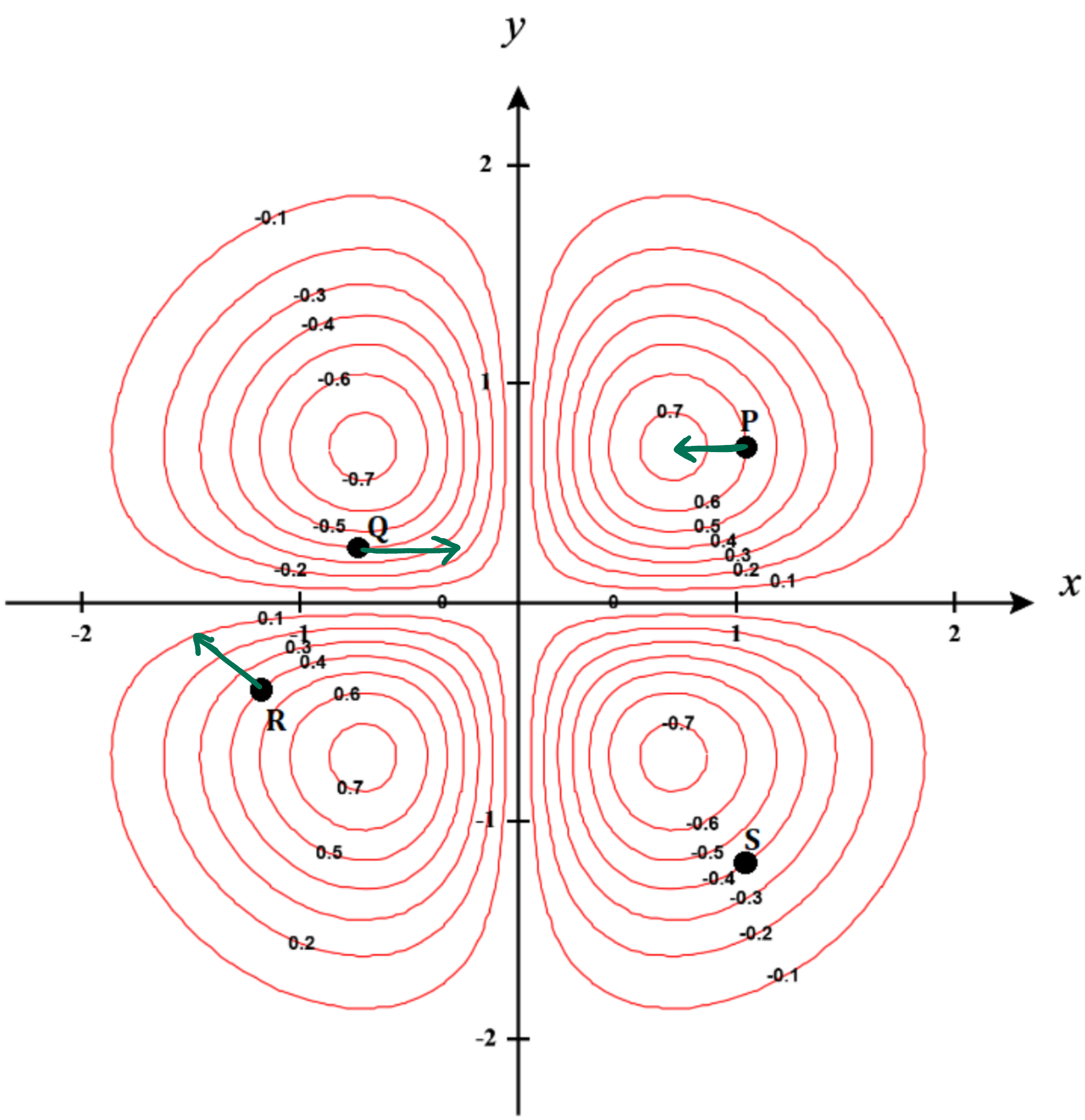
$\theta \in [\pi/6, \pi - \pi/6]$

7. In this problem, you will work with the function $f : \mathbb{R}^2 \to \mathbb{R}$ whose contour plot near the origin is shown below. [A N]

   (a) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative a the point Q in the direction $\langle 1, 0\rangle$. $\boxed{0}$

   (b) (1 point) Determine the sign $(+, -, 0)$ of the directional derivative a the point R in the direction $\langle -1, 1\rangle$. $\boxed{-}$

   (c) (2 points) Draw a vector which is a positive scalar multiple of gradient vector of f at the point P.



8. (8 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = (x - y)z$ and $\mathbf{r}(s, t) : \mathbb{R}^2 \to \mathbb{R}^3$ be the function

$$\mathbf{r}(s,t) = \begin{bmatrix} s+t \\ st \\ e^{s+t} \end{bmatrix}.$$

$$\begin{cases} x = s+t \\ y = st \\ z = e^{s+t} \end{cases}$$

(a) Find Df.

(b) Find $D\mathbf{r}$.

(c) Evaluate $\mathbf{r}(1, 0)$ and $D\mathbf{r}|_{(s,t)=(1,0)}$.

(d) Finally, use the Chain Rule to evaluate $D(f(\mathbf{r}(s, t)))|_{(s,t)=(1,0)}$.

[AJN]

(a) $Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} z & -z & x-y \end{bmatrix}$

(b) $D\vec{r} = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ t & s \\ e^{s+t} & e^{s+t} \end{bmatrix}$

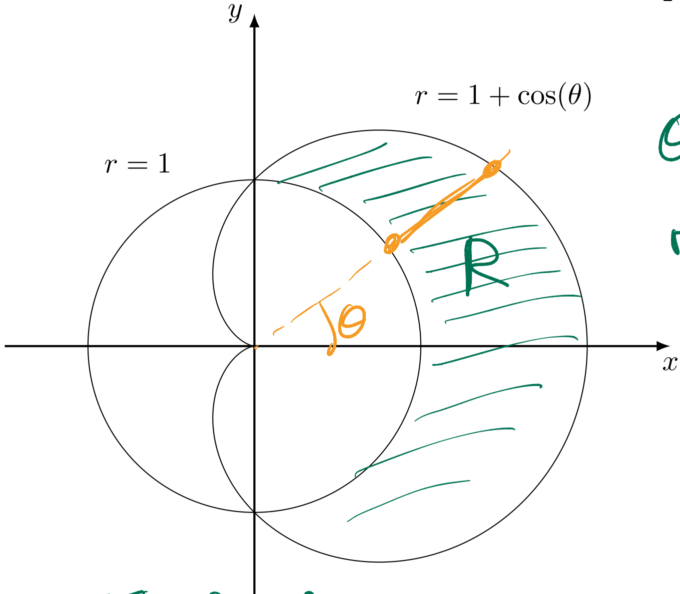
(c) $\vec{r}(1,0) = \begin{bmatrix} 1 \\ 0 \\ e \end{bmatrix}$ $\quad D\vec{r}\Big|_{e(1,0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ e & e \end{bmatrix}$

(d) $Df\Big|_{(s,t)=(1,0)} = Df\Big|_{(x,y,z)=(1,0,e)} = \begin{bmatrix} e & -e & 1 \end{bmatrix}$

So $D(f \circ g)\Big|_{e(1,0)} = \begin{bmatrix} e & -e & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ e & e \end{bmatrix}$

$= \begin{bmatrix} e+e & e-e+e \end{bmatrix} = \boxed{\begin{bmatrix} 2e & e \end{bmatrix}}$

9. (10 points) (a) Write and evaluate the integral for the area of the region R between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$ which is completely contained in Quadrants I and IV. (b) Then, use the fact that $\iint_R f(x,y)\,dA = 5\pi/8$ to find the average value of $f(x,y) = x$ over the region R. *Hint:* $\cos^2 t = \frac{1+\cos 2t}{2}$.



$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$r \in [1, 1+\cos\theta]$

(a) $\text{Area} = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \Big|_1^{1+\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1+\cos\theta)^2 - 1\,d\theta$

using hint

$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cancel{1} + 2\cos\theta + \cos^2\theta - \cancel{1}\,d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta + \int_{-\pi/2}^{\pi/2} \frac{1}{2}\cdot\frac{1+\cos 2\theta}{2}\,d\theta$

$= 2\sin\theta \Big|_0^{\pi/2} + \frac{1}{4}*2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = 2(1-0) + \frac{1}{2}\left(\left(\frac{\pi}{2} + \frac{1}{2}(0)\right) - (0+0)\right)$

y=cosθ is EVEN

1+cos2θ is EVEN.

$= \boxed{2 + \pi/4}$ ANS for (a).

So (b) $\text{Avg}_R f = \frac{1}{\text{Area}} * \text{Vol} = \frac{1}{2+\pi/4} * \frac{5\pi}{8} = \boxed{\frac{5\pi}{16+2\pi}}$ Ans for (b)

10. (10 points) Find all critical points and use the Hessian matrix to classify the critical points of the function. [AJN]

$$f(x,y) = x^3 - 3x + y^{\cancel{4}\,3} - \cancel{4}\,3y^2$$

Set up $Df = \begin{bmatrix} 3x^2-3 & 3y^2-6y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

Solve $\begin{cases} 3x^2-3=0 \\ 3y^2-6y=0 \end{cases} \Rightarrow \begin{cases} x^2=1 \\ y(y-2)=0 \end{cases} \Rightarrow \begin{array}{l} x=\pm 1 \\ y=0,2 \end{array}$

So get $(1,0)$, $(-1,0)$, $(1,2)$, $(-1,2)$.

$$Hf = \begin{bmatrix} 6x & 0 \\ 0 & 6y-6 \end{bmatrix}$$

@ $(1,0)$ $|Hf| = \begin{vmatrix} 6 & 0 \\ 0 & -6 \end{vmatrix} = -36 < 0$

@ $(-1,0)$ $|Hf| = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 > 0$ and $f_{xx} < 0$

@ $(1,2)$ $|Hf| = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0$ and $f_{xx} > 0$

@ $(-1,2)$ $|Hf| = \begin{vmatrix} -6 & 0 \\ 0 & 6 \end{vmatrix} = -36 < 0$

Saddle @ $(1,0)$
MAX @ $(-1,0)$
MIN @ $(1,2)$
Saddle @ $(-1,2)$

FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b\rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta), \quad y = r\sin(\theta), \quad dA = r\ dr\ d\theta$

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