

MATH 2551 GT-E Midterm 3

VERSION A

Summer 2025

COVERS SECTIONS 15.5-15.8, 16.1-16.8

Full name: _____ GT ID: _____

Key

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

() All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	2
4	8
5	6
6	10
7	10
8	10
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If $\mathbf{F}$ is any vector field and C is a closed simple loop in $\mathbb{R}^2$, then the circulation of $\mathbf{F}$ around C traversed in one orientation is the negative of the circulation of $\mathbf{F}$ around C traversed in the opposite orientation.



TRUE



FALSE

2. (2 points) If C is a curve with parametrization $\mathbf{r}_1(t) = \langle x(t), y(t) \rangle$ with $t \in [a, b]$, then the parametrization $\mathbf{r}_2(t) = \mathbf{r}_1(-t)$ with $t \in [-b, -a]$ is also a parametrization of C with the opposite orientation. [A]



TRUE



FALSE

3. (2 points) If $g(x, y)$ is strictly positive at all points in $\mathbb{R}^2$, then

$$\iint_{R_1} g(x, y) dA \geq \iint_{R_2} g(x, y) dA$$

if the area of R_1 is larger than the area of R_2 .

[A]



TRUE



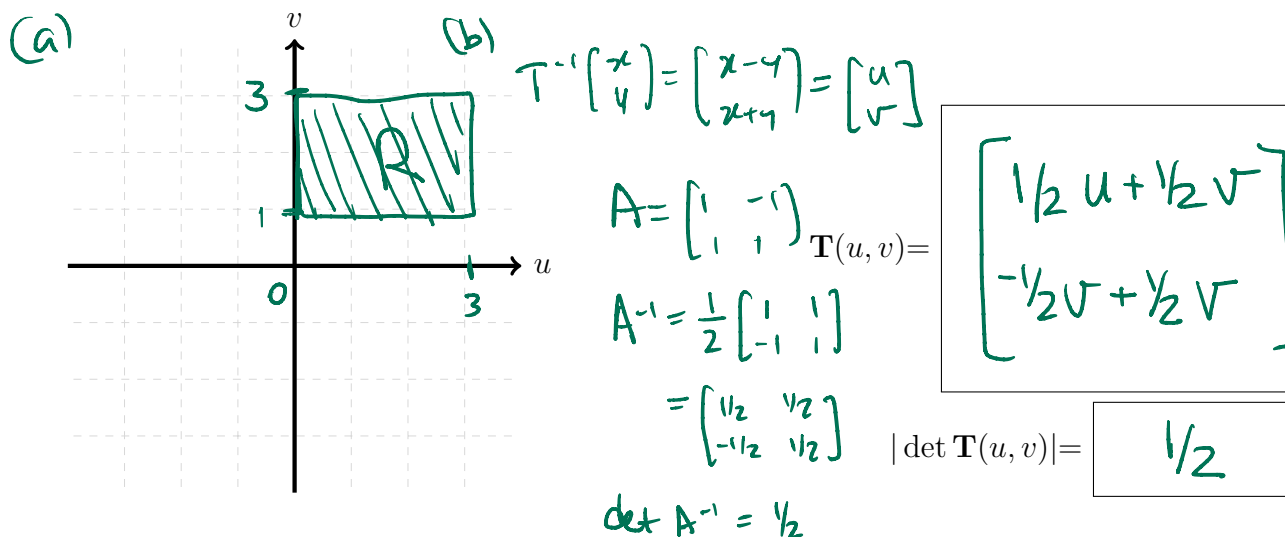
FALSE

4. (8 points) In this problem you will compute the integral

$$M = \iint_R (x+y)e^{x^2-y^2} dx dy$$

for the region R in the first quadrant bounded by the lines $x-y=0$, $x-y=3$, $x+y=1$, $x+y=3$ using the transformation $u = x-y$, $v = x+y$. u=0 u=3 v=1

- (a) On the axes provided, sketch the new region of integration G in the uv -plane. [A]
 (b) Solve the transformation equations for x and y in terms of u and v and compute the Jacobian determinant, i.e., find $\mathbf{T}(u, v)$ and $|\det D\mathbf{T}(u, v)|$. [AJN]
 (c) Evaluate the double integral using a change of variables and your results from parts (a) and (b) to find the value of M . *Hint: integrate with the order $du dv$.* [AJN]



(c)
$$M = \iint_R (x+y)e^{x^2-y^2} \frac{1}{2} du dv = \frac{1}{2} \int_1^3 \int_0^3 v e^{uv} du dv$$

$$= \frac{1}{2} \int_1^3 v \left[\frac{1}{v} e^{uv} \right]_0^3 dv = \frac{1}{2} \int_1^3 (e^{3v} - 1) dv$$

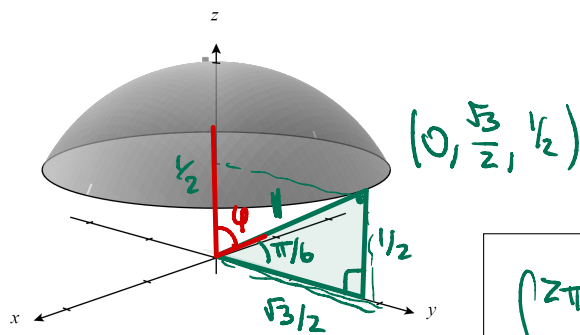
$$= \frac{1}{2} \left(\frac{1}{3} e^{3v} - v \right) \Big|_1^3 = \frac{1}{2} \left[\left(\frac{1}{3} e^9 - 3 \right) - \left(\frac{1}{3} e^3 - 1 \right) \right]$$

$$= \frac{1}{6} (e^9 - e^3) - 1$$

5. (6 points) Let D be the smaller cap cut from a solid ball of radius 1 units by the plane $z = 1/2$. The point $P(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$ is on D at the intersection of the ball and the plane.

(a) Find the ~~cylindrical~~ ^{spherical} coordinates (ρ, φ, θ) of the point $P(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$. [AN]

(b) Express the volume of D as an iterated triple integral in spherical coordinates. *Do not evaluate!* [AN]



Volume =

$$(0, \frac{\sqrt{3}}{2}, \frac{1}{2})_{cyl} = (1, \pi/3, \pi/2)$$

$$\int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{1}{2}\sec\varphi}^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\varphi = \pi/2 - \pi/6 = \pi/3$$

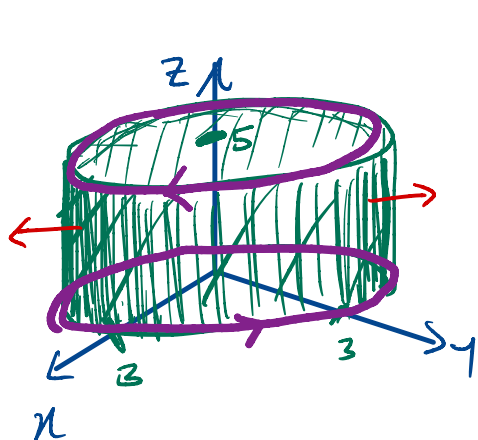
$$\rho = 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$$

$$\theta = \pi/2 \text{ (on y-axis)}$$

$$z = 1/2 = \rho \cos \varphi \Rightarrow \rho = \frac{1}{2 \cos \varphi} = \frac{1}{2} \sec \varphi$$

$$\text{so } D: \theta \in [0, 2\pi], \varphi \in [0, \pi/3], \rho \in [\frac{1}{2} \sec \varphi, 1]$$

6. (10 points) This problem will have you compute the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$ through S the open ended circular cylinder of radius 3 and height 5 with its base on the xy -plane and centered about the z -axis, oriented away from the z -axis. [AJN]
- (a) Set up and evaluate a surface integral which computes the flux of $\mathbf{F}$ through S .
- (b) Why can we NOT evaluate a line integral over the boundary C to obtain the result of part (b) using Stokes' Theorem? Use at least one or two complete sentences to answer.



cylindrical coord

$$S: \mathbf{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$$

$$\theta \in [0, 2\pi], z \in [0, 5]$$

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

$$\mathbf{r}_\theta = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$$

$$\mathbf{r}_z = \langle 0, 0, 1 \rangle$$

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 3\cos\theta, 3\sin\theta, 0 \rangle \sim \hat{n}$$

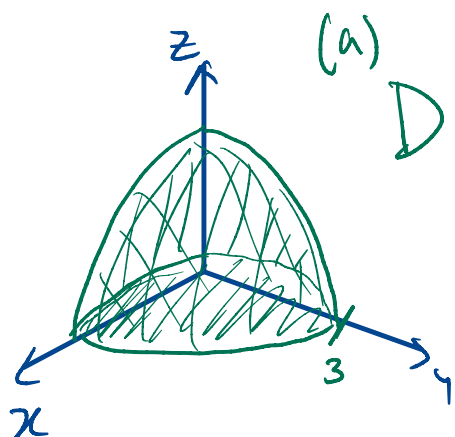
$$\text{So Flux} = \int_0^{2\pi} \int_0^5 \langle 3\cos\theta, 3\sin\theta, 0 \rangle \cdot \langle 3\cos\theta, 3\sin\theta, 0 \rangle \, dz \, d\theta$$

$$= \int_0^{2\pi} \int_0^5 \underbrace{9\cos^2\theta + 9\sin^2\theta}_9 \, dz \, d\theta = \int_0^{2\pi} \int_0^5 9 \, dz \, d\theta$$

$$= \int_0^{2\pi} 9z \Big|_0^5 \, d\theta = \int_0^{2\pi} 45 \, d\theta = 45\theta \Big|_0^{2\pi} = \boxed{90\pi}$$

- (b) Stokes' Theorem helps to evaluate $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$, the Flux of the curl of $\mathbf{F}$. Note: Even though the boundary C of S is two disjoint curves, this is actually not an issue (although it makes evaluation of $\int_C \mathbf{F} \cdot d\mathbf{r}$ more cumbersome).

7. (8 points) Consider the vector field $\mathbf{F} = \langle zx, zy, z \rangle$ and the surface S consisting of the portion of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$ together with the circular disk $x^2 + y^2 \leq 3$ in the xy -plane, oriented with normal vectors away from the origin. [AJN]
- (a) Use the Divergence Theorem to set up a triple iterated integral over the region D which is enclosed by S , which evaluates to the same value as $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$. *Do not evaluate!*
- (b) Is there a vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$? Explain in one or two complete sentences how you know.



(a) $D: \theta \in [0, 2\pi], r \in [0, \sqrt{3}], z \in [0, 9 - r^2]$

$$\text{Flux} = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{9-r^2} (2z+1) r \, dz \, dr \, d\theta$$

$$\text{div } \mathbf{F} = P_x + Q_y + R_z$$

$$= z + z + 1 = 2z + 1$$

- (b) No such $\mathbf{G}$ exists since $\text{div } \mathbf{F} \neq 0$, which is a necessary condition. Since $\text{div}(\text{Curl } \mathbf{G}) = 0$ for any $\mathbf{G}$.

8. (12 points) Consider the vector field $\mathbf{F} = \langle -y, 2x \rangle$ and the curve C which is the circle $x^2 + y^2 = 25$ oriented counterclockwise with outward pointing normal vector. [AJN]

(a) Compute the flow of F around T , which is $\int_C \mathbf{F} \cdot \mathbf{T} ds$, ^{with or} **without** using Green's Theorem.

Hint: check answer with other method.

(b) Compute the flux of F around T , which is $\int_C \mathbf{F} \cdot \mathbf{n} ds$, **using** Green's Theorem.

(c) Is F conservative? Explain in at least one or two complete sentences how you know.

If you need more room, please use the next page and leave a note at the bottom of this page.

$$C: \mathbf{r}(t) = \langle 5\cos t, 5\sin t \rangle \quad t \in [0, 2\pi]$$

$$\mathbf{r}'(t) = \langle -5\sin t, 5\cos t \rangle$$

$$(a) \quad \text{Flow} = \int_0^{2\pi} \langle -5\sin t, 5\cos t \rangle \cdot \langle -5\sin t, 5\cos t \rangle dt$$

$$= \int_0^{2\pi} 25\sin^2 t + 25\cos^2 t dt = 25 \int_0^{2\pi} 1 dt$$

$$= 25 \int_0^{2\pi} 1 dt = 25 \int_0^{2\pi} \frac{3}{2} + \frac{1}{2} \cos 2t dt$$

$$= 25 \left(\frac{3}{2}t + \frac{1}{4}\sin 2t \right) \Big|_0^{2\pi} = 25 \left[(3\pi + 0) - (0 + 0) \right] = \boxed{75\pi}$$

Check w/ G's T

$$\text{Flow} = \overset{\text{G's T}}{\iint_R} Q_x - P_y dA = \iint_R 3 dA$$

$$Q_x = 2, P_y = -1 \\ Q_x - P_y = 3$$

$$= 3 * \text{Area} = 3 * \pi(5)^2 = \boxed{75\pi} \quad \checkmark$$

$$(b) \quad \text{Flux} = \iint_R P_x + Q_y dA = \iint_R 0 + 0 dA = \boxed{0} \quad \checkmark$$

(c) F NOT conservative b/c $\text{curl } F = 3 \neq 0$. Also, if it was then Flow around C must be 0 (not 75π).

If you need more room on the previous problem please use this page and indicate it at the bottom of the previous page.

FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Volume(D) = $\iiint_D dV$, $f_{avg} = \frac{\iiint_D f(x, y, z) dV}{\text{volume of } D}$ or $\frac{\int_C f(x, y, z) ds}{\text{length of } C}$,
Mass: $M = \iiint_D \delta dV$
- Cylindrical coordinates: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $z = z$, $dV = r dz dr d\theta$
- Spherical coordinates: $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, $z = \rho \cos(\phi)$,
 $dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u, v) = \langle x(u, v), y(u, v) \rangle$ then

$$\iint_R f(x, y) dx dy = \iint_G f(\mathbf{T}(u, v)) |\det D\mathbf{T}(u, v)| du dv.$$

- Scalar line integral: $\int_C f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$
- Tangential vector line integral: $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$
- Normal vector line integral: $\int_C \mathbf{F}(x, y) \cdot \mathbf{n} ds = \int_C P dy - Q dx = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if $\text{curl } \mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$ $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \quad \int_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R (\nabla \cdot \mathbf{F}) dA.$$

- Surface Area = $\iint_S 1 d\sigma$
- Scalar surface integral: $\iint_S f(x, y, z) d\sigma = \iint_R f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$
- Flux surface integral: $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S , then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV.$$

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- (a) Compute the flow of F around T , which is $\int_C F \cdot \mathbf{T} \, ds$, **without** using Green's Theorem.
 - (b) Check your answer to part (a) by **using** Green's Theorem.
 - (c) Compute the flux of F around T , which is $\int_C F \cdot \mathbf{n} \, ds$, **using** Green's Theorem.
 - (d) Is F conservative? Explain in at least one or two complete sentences how you know.

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