MATH 2551 GT-E Midterm 3
VERSION B
Summer 2025
COVERS SECTIONS 15.5-15.8, 16.1-16.8

Full name: Key ______________________ **GT ID:** ____________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	2
3	2
4	8
5	6
6	10
7	8
8	12
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If $\mathbf{F}$ is any vector field and C is a closed simple loop in $\mathbb{R}^2$, then the circulation of $\mathbf{F}$ around C traversed in one orientation is the negative of the circulation of $\mathbf{F}$ around C traversed in the opposite orientation.

   ● **TRUE** ○ **FALSE**

2. (2 points) If C is a curve with parametrization $\mathbf{r}_1(t) = \langle x(t), y(t) \rangle$ with $t \in [a, b]$, then the parametrization $\mathbf{r}_2(t) = \mathbf{r}_1(-t)$ with $t \in [a, b]$ is also a parametrization of C with the opposite orientation. [A]

   ○ **TRUE** ● **FALSE**

3. (2 points) If $g(x, y)$ is strictly positive at all points in $\mathbb{R}^2$, then

$$\iint_{R_1} g(x,y)\, dA \geq \iint_{R_2} g(x,y)\, dA$$

   if the area of R_1 is larger than the area of R_2. [A]

   ○ **TRUE** ● **FALSE**

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If C_1 and C_2 are two curves with the same starting point and ending point, then $\int_{C_1} \nabla f \cdot \mathbf{T}\, ds = \int_{C_2} \nabla f \cdot \mathbf{T}\, ds$.

   ● **TRUE** ○ **FALSE**

2. (2 points) If C is a curve with parametrization $\mathbf{r}_1(t) = \langle x(t), y(t) \rangle$ with $t \in [a, b]$, then the parametrization $\mathbf{r}_2(t) = \mathbf{r}_1(-t)$ with $t \in [a, b]$ is also a parametrization of C with the opposite orientation. [A]

   ○ **TRUE** ● **FALSE**

3. (2 points) If $g(x, y)$ is strictly positive at all points in $\mathbb{R}^2$, then

$$\iint_{R_1} g(x, y)\, dA = \iint_{R_2} g(x, y)\, dA$$

   if the area of R_1 is equal to the area of R_2. [A]

   ○ **TRUE** ● **FALSE**

4. (8 points) In this problem you will compute the integral

$$M = \iint_R (x+y)e^{x^2-y^2}\,dx\,dy$$

for the region R in the first quadrant bounded by the lines $x - y = 0$, $x - y = 1$, $x + y = 0$, $x + y = 2$ using the transformation $u = x - y$, $v = x + y$.

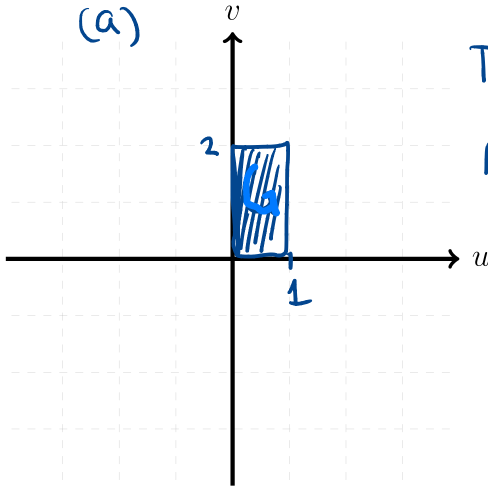
$u=0$ $u=1$ $v=0$ $v=2$

(a) On the axes provided, sketch the new region of integration G in the uv-plane. [A]

(b) Solve the transformation equations for x and y in terms of u and v and compute the Jacobian determinant, i.e., find $\mathbf{T}(u,v)$ and $|\det D\mathbf{T}(u,v)|$. [AJN]

(c) Evaluate the double integral using a change of variables and your results from parts (a) and (b) to find the value of M. *Hint: integrate with the ~~area element~~ order $du\,dv$.* [AJN]

(a)



(b)

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$T\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u+\frac{1}{2}v \\ -\frac{1}{2}u+\frac{1}{2}v \end{bmatrix}$

$\mathbf{T}(u,v) = \begin{bmatrix} \frac{1}{2}u + \frac{1}{2}v \\ -\frac{1}{2}u + \frac{1}{2}v \end{bmatrix}$

$|\det \mathbf{T}(u,v)| = \frac{1}{2}$

(c)

$$M = \iint_R (x+y)\,e^{x^2-y^2}\,\frac{1}{2}\,dx\,dy = \frac{1}{2}\int_0^2\int_0^1 v\,e^{uv}\,du\,dv$$

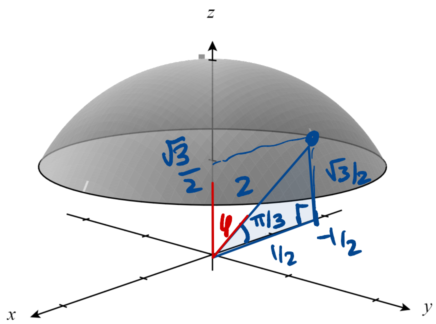
$$= \frac{1}{2}\int_0^2 v * \frac{1}{v} e^{uv}\Big|_0^1 dv = \frac{1}{2}\int_0^2 e^v - 1\,dv = e^v - v\Big|_0^2$$

$$= \frac{1}{2}\left[(e^2-2)-(1-0)\right] = \boxed{\frac{1}{2}(e^2-3)}$$

5. (6 points) Let D be the smaller cap cut from a solid ball of radius 1 units by the plane $z = \sqrt{3}/2$. The point $P(-\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$ is on D at the intersection of the ball and the plane.

   (a) Find the spherical coordinates (ρ, φ, θ) of the point $P(-\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$. [AN]

   (b) Express the volume of D as an iterated triple integral in spherical coordinates. *Do not evaluate!* [AN]



$(-\frac{1}{2}, 0, \frac{\sqrt{3}}{2})_S =$ $(1, \pi/6, \pi)$

Volume= $\displaystyle\int_0^{2\pi}\int_0^{\pi/6}\int_{\frac{\sqrt{3}}{2}\sec\varphi}^{1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\varphi = \pi/2 - \pi/3 = \pi/6$

$\theta = \pi$ (on neg. x-axis)

$\rho = (\frac{-1}{2})^2 + 0^2 + (\frac{\sqrt{3}}{2})^2 = 1.$

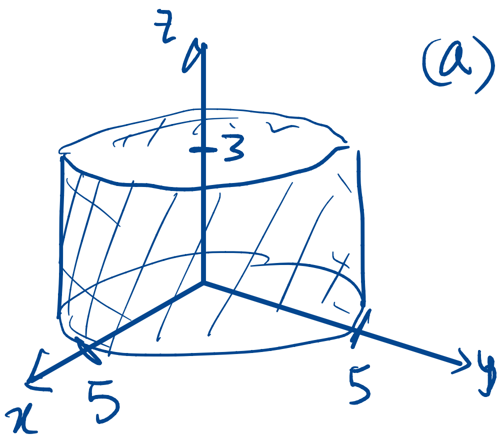
$z = \frac{\sqrt{3}}{2} = \rho\cos\varphi \Rightarrow \rho = \frac{\sqrt{3}}{2}\sec\varphi$

$D: \theta \in [0, 2\pi], \rho \in [\frac{\sqrt{3}}{2}\sec\varphi, 1], \varphi \in [0, \pi/6]$

6. (10 points) This problem will have you compute the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$ through S the open ended circular cylinder of radius 5 and height 3 with its base on the xy-plane and centered about the z-axis, oriented away from the z-axis. [AJN]

(a) Set up and evaluate a surface integral which computes the flux of $\mathbf{F}$ through S.

(b) Why can we NOT evaluate a line integral over the boundary C to obtain the result of part (a) using Stokes' Theorem? Use at least one or two complete sentences to answer.



(a) $\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$

$S: \mathbf{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z\rangle$
$\theta \in [0, 2\pi],\ z \in [0, 3]$

$\mathbf{r}_\theta = \langle -5\sin\theta, 5\cos\theta, 0\rangle$
$\mathbf{r}_z = \langle 0, 0, 1\rangle$

$$\mathbf{n} \sim \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5\sin\theta & 5\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 5\cos\theta, 5\sin\theta, 0\rangle$$

So $$\text{Flux} = \int_0^{2\pi}\int_0^3 \langle 5\cos\theta, 5\sin\theta, 0\rangle \cdot \langle 5\cos\theta, 5\sin\theta, 0\rangle\, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 25\cos^2\theta + 25\sin^2\theta\, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 25\, dz\, d\theta = \int_0^{2\pi} 25z\Big|_0^3 d\theta = \int_0^{2\pi} 75\pi\, d\theta$$

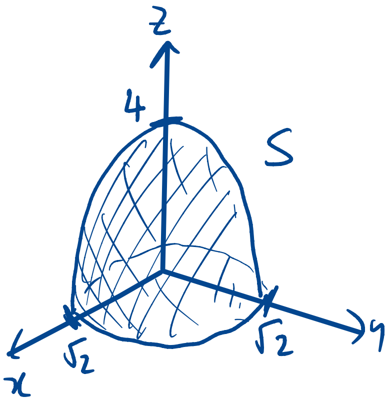
$$= 75\pi\theta\Big|_0^{2\pi} = \boxed{150\pi}$$

(b) Stokes' Theorem helps to calculate the flux across S of the curl of F, not F. Also, C the boundary of S is two disjoint curves.

7. (8 points) Consider the vector field $\mathbf{F} = \langle zx, zy, z \rangle$ and the surface S consisting of the portion of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$ together with the circular disk $x^2 + y^2 \leq 4$ in the xy-plane, oriented with normal vectors away from the origin. [AJN]

   (a) Use the Divergence Theorem to set up a triple iterated integral over the region D which is enclosed by S, which evaluates to the same value as $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$. *Do not evaluate!*

   (b) Is there a vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$? Explain in one or two complete sentences how you know.



cylindrical coords

$D: \theta \in [0, 2\pi],\ r \in [0, 2],\ z \in [0, 4 - r^2]$

$\text{div}\, F = P_x + Q_y + R_z$

$= z + z + 1 = 2z + 1$

(a) $\text{Flux} = \iint_S F \cdot n \, d\sigma \overset{DT}{=} \iiint_D \text{div} F \, dV$

$$= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (2z+1) * r \, dz \, dr \, d\theta$$

(b) Stokes' Theorem helps to evaluate $\iint_S \nabla \times F \cdot n \, d\sigma$, the flux of the curl of F. **Note:** Even though the boundary C of S is two disjoint curves, this is actually not an issue (although it makes evaluation of $\int_C F \cdot dr$ more cumbersome).

8. (12 points) Consider the vector field $\mathbf{F} = \langle x+2, y \rangle$ and the curve C which is the circle $x^2 + y^2 = 25$ oriented counterclockwise with outward pointing normal vector. [AJN]

(a) Compute the flow of F around T, which is $\int_C F \cdot \mathbf{T}\, ds$, **using** Green's Theorem.

(b) Compute the flux of F around T, which is $\int_C F \cdot \mathbf{n}\, ds$, **with or without** using Green's Theorem. *Use either method, but you can use the other integral to check your answer.*

(c) Is F conservative? Explain in at least one or two complete sentences how you know.

If you need more room, please use the next page and leave a note at the bottom of this page.

(a) $\text{Flow} = \int_C F \cdot T\, ds = \iint_R \text{curl} F \cdot \hat{k}\, dA = \iint_R 0 - 0\, dA = \boxed{0}$

since $\text{curl} F = \langle 0, 0, Q_x - P_y \rangle$ for 2D vector fields.

(b) $\text{Flux} = \int_C F \cdot n\, ds$ $\quad r(t) = \langle 5\cos t, 5\sin t \rangle \quad t \in [0, 2\pi]$

$r'(t) = \langle -5\sin t, 5\cos t \rangle \quad n \sim \langle 5\cos t, 5\sin t \rangle$

$$\text{Flux} = \int_0^{2\pi} \langle 5\cos t + 2, 5\sin t \rangle \cdot \langle 5\cos t, 5\sin t \rangle\, dt$$

$$= \int_0^{2\pi} 25\cos^2 t + 10\cos t + 25\sin^2 t\, dt = \int_0^{2\pi} 25 + 10\cos t\, dt$$

$$= 25t + 10\sin t \Big|_0^{2\pi} = 50\pi + 0 - (0 + 0) = \boxed{50\pi}$$

Check w/ GT.

$$\text{Flux} \overset{GT}{=} \iint_R \text{div} F\, dA = \int_0^{2\pi} \int_0^5 (1+1)\, r * dr\, d\theta$$

$$= \int_0^{2\pi} r^2 \Big|_0^5 d\theta = \int_0^{2\pi} 25\, d\theta = 25\theta \Big|_0^{2\pi} = \boxed{50\pi}$$

✓

(c) F is conservative since $\text{curl} F = 0$. Also, $f = \frac{1}{2}x^2 + 2x + \frac{1}{2}y^2$ is an easy potential function for F.

If you need more room on the previous problem please use this page and indicate it at the bottom of the previous page.

FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$ or $\dfrac{\int_C f(x,y,z)\ ds}{\text{length of } C}$,
  Mass: $M = \iiint_D \delta\ dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$,
  $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Scalar line integral: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- Surface Area$=\iint_S 1\ d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_D \nabla\cdot\mathbf{F}\ dV.$$

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