MAIN IDEA $\frac{\partial}{\partial x}(xy^2) = y^2$

§14.3: Partial Derivatives

$\frac{\partial}{\partial y}(xy^2) = 2xy$

Goal: Describe how a function of two (or three, later) variables is changing a point (a, b).

Example 47. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Idea @ $(2, y)$ $\quad h(2,y) = 4 - \frac{1}{4}(2)^2 - \frac{1}{4}y^2$

$= 4 - 1 - \frac{1}{4}y^2$

$= 3 - \frac{1}{4}y^2$

So $\boxed{\frac{\partial}{\partial y} h(2,y) = -\frac{1}{2}y}$

@ $(2,1)$ $\quad \frac{\partial}{\partial y} h(2,1) = \boxed{-\frac{1}{2}}$

So the slope of line tangent $z = h(x,y)$ at @ $(2, 1, 11/4)$ in y-direction is $\boxed{m = -1/2}$

Now for x-direction

@ $(x, 1)$ $\quad h(x,1) = 4 - \frac{1}{4}x^2 - \frac{1}{4}(1)^2 = 3.75 - \frac{1}{4}x^2$

$\frac{\partial}{\partial y} h(x,1) = -\frac{1}{2}x$ $\quad$ @ $(2,1)$ $\quad h(2,1) = -\frac{1}{2}(2)$

$= \boxed{-1}$

Slope in x-direction is $m = -1$

Let's investigate graphically.

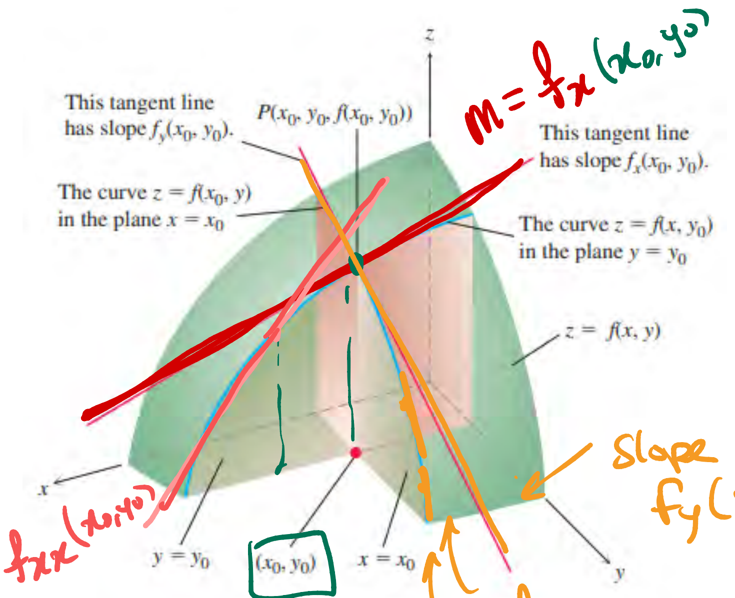
Definition 48. If f is a function of two variables x and y, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations: $f_x = \frac{\partial}{\partial x} f = \frac{\partial f}{\partial x}$ ← partial derivative of f in the x-direction

$f_y = \frac{\partial}{\partial y} f = \frac{\partial f}{\partial y}$ ← " " in the y-direction.

Interpretations:



Slope is $f_y(x_0,y_0)$ the slope at (x_0,y_0).

f_{yy} the concavity in y-direction

Example 49. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

a) $f(x,y) = \sqrt{5x - y}$

$f_x = \frac{\partial}{\partial x} f(x,y) = \frac{1}{2\sqrt{5x-y}} * 5 = \frac{5}{2\sqrt{5x-y}}$

at $(x,y) = (1,2)$ get $f_x(1,2) = \frac{5}{2\sqrt{5-2}}$

$= \boxed{\frac{5}{2\sqrt{3}}}$

$f_y = \frac{\partial}{\partial y} f(x,y) = \frac{1}{2\sqrt{5x-y}} * (-1) = \frac{-1}{2\sqrt{5x-y}}$

@ $(1,2)$ $f_x(1,2) = \frac{-1}{2\sqrt{5-2}} = \boxed{\frac{-1}{2\sqrt{3}}}$

b) $f(x,y) = \tan(xy)$

$(\tan x)' = \sec^2 x$

$f_x = \sec^2(xy) * y$

$f_y = \sec^2(xy) * x$

@ $(2,1)$

$$\boxed{\begin{aligned} f_x(2,1) &= 1 \sec^2(2) \\ f_y(2,1) &= 2\sec^2(2) \end{aligned}}$$

Question: How would you define the second partial derivatives?

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{(\partial x)^2}$$

∂ "del" mean partial deriv.

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

Example 50. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function below.

$$\frac{5}{\sqrt{2x-1}} = 5(2x-1)^{-1/2}$$

a) $f(x,y) = \sqrt{5x-y}$

rewrite.

$$f_x = \frac{5}{2\sqrt{5x-y}} \quad \& \quad f_y = \frac{-1}{2\sqrt{5x-y}}$$

$$f_x = \frac{5}{2}(5x-y)^{-1/2} \qquad f_y = \frac{-1}{2}(5x-y)^{-1/2}$$

pure partials.

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{5}{2} * \underbrace{\frac{-1}{2}(5x-y)^{-3/2}}_{} * 5 = \boxed{\frac{-25}{4}(5x-y)^{-3/2}}$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{5}{2} * \frac{-1}{2}(5x-y)^{-3/2} * (-1) = \boxed{\frac{5}{4}(5x-y)^{-3/2}}$$

$= f_{yx}$ by C.THM.

mixed partials

$$f_{yx} = \frac{\partial}{\partial x} f_y = \frac{-1}{2} * \frac{-1}{2}(5x-y)^{-3/2} * 5 = \boxed{\frac{5}{4}(5x-y)^{-3/2}}$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{-1}{2} * \frac{-1}{2}(5x-y)^{-3/2} * (-1) = \boxed{\frac{-1}{4}(5x-y)^{-3/2}}$$

pure partials

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 51 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

$$\boxed{f_{xy} = f_{yx}}$$

corollary: $f_{xyx} = f_{xxy} = f_{yxx}$ ~~Same~~?? why?

also $f_{yxy} = f_{yxy} = f_{xyy}$

Example 52. *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

$$f_x = 2yz - 0 = \boxed{2yz}$$

$$f_y = \boxed{2xz - z^2}$$

$$f_z = \boxed{2xy - 2zy}$$

$f: \mathbb{R}^2 \to \mathbb{R}^3$

Example 53. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them. Idea. $f(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

$x(s,t) = s^2+t$
$y(s,t) = 2s-t$
$z(s,t) = st$

We've seen how to deal with

$\frac{\partial}{\partial s} x(s,t) = 2s$

$\frac{\partial}{\partial t} x(s,t) = 1$

What about y?

$\frac{\partial}{\partial s} y = 2$

$\frac{\partial}{\partial t} y = -1$

What about z?

$\frac{\partial}{\partial s} z = t$

$\frac{\partial}{\partial t} z = s$

$$\begin{array}{c|cc} & \frac{\partial}{\partial s} & \frac{\partial}{\partial t} \\ \hline x \to & & \\ y \to & & \\ z \to & & \end{array} \quad \begin{bmatrix} \boxed{2s} & 1 \\ 2 & -1 \\ t & s \end{bmatrix} = Df$$

the total derivative.

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f\underbrace{(x_1, \dots, x_n)}_{n \text{ inputs}} = \left.\begin{bmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{bmatrix}\right\} m \text{ output values}$

we have <u>n</u> inputs, <u>m</u> output, and <u>$m \times n$</u> partial derivatives, which we can use to form the **total derivative**.

This is a <u>linear</u> map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an <u>$m \times n$ matrix</u>, with one column per input and one row per output.

It has the formula $Df_{ij} = \frac{\partial}{\partial j} f_i(x_1, \dots, x_n)$ $\quad 1 \le j \le n$ ($n = \#$ cols) $\quad 1 \le i \le m$ ($m = \#$ rows)

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

$f: \mathbb{R} \to \mathbb{R}$

Df has size 1×1

$\begin{bmatrix} 2x \end{bmatrix}$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$r: \mathbb{R} \to \mathbb{R}^3$

Df has size 3×1

$r_1(t) = \cos t$

$r_2(t) = \sin t$

$r_3(t) = t$

$$Df = \begin{bmatrix} r_1'(t) \\ r_2'(t) \\ r_3'(t) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$$

c) $f(x, y) = \sqrt{5x - y}$

$f: \mathbb{R}^2 \to \mathbb{R}$

Df is 1×2

$$Df = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{bmatrix}$$

d) $f(x, y, z) = 2xyz - z^2 y$

$f: \mathbb{R}^3 \to \mathbb{R}$

Df is 1×3

$$Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} 2yz & 2xz - z^2 & 2xy - 2yz \end{bmatrix}$$

e) $\mathbf{f}(s, t) = \langle \underbrace{s^2 + t}_{x}, \underbrace{2s - t}_{y}, \underbrace{st}_{z} \rangle$

$F: \mathbb{R}^2 \to \mathbb{R}^3$

Df size 3×2

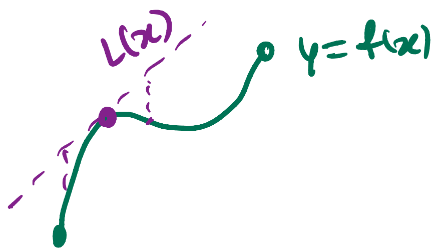
$$Df = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

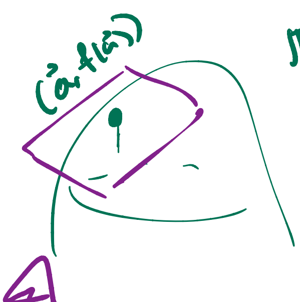
Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) \approx f(a) + f'(a)(x-a) = L(x)$



Definition 55. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is



$\mathbb{R}^{m\times n}$

$$L(\mathbf{x}) = \underbrace{f(\vec{a})}_{\text{vector}} + \underbrace{Df(\vec{a})}_{\text{matrix}}\underbrace{(\vec{x} - \vec{a})}_{\text{vector}}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix}$$

Example 56. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

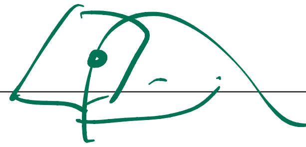
$$Df = \begin{bmatrix} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{bmatrix} \quad @\ \vec{a} = (1,1) \quad Df(1,1) = \begin{bmatrix} \frac{5}{2\sqrt{4}} & \frac{-1}{2\sqrt{4}} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{-1}{4} \end{bmatrix}$$

$$L(\vec{x}) = \underbrace{2}_{f(1,1)} + \begin{pmatrix} \frac{5}{4} & \frac{-1}{4} \end{pmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

So $\sqrt{5(1.1) - 1.1} \approx 2.1$

$$L\left(\begin{bmatrix} 1.1 \\ 1.1 \end{bmatrix}\right) = 2 + \begin{bmatrix} 1.25 & -.25 \end{bmatrix}\begin{bmatrix} .1 \\ .1 \end{bmatrix} = 2 + (.125 - .025) = 2 + (.1) = \boxed{2.1}$$

Question: What do you notice about the equation of the linearization?

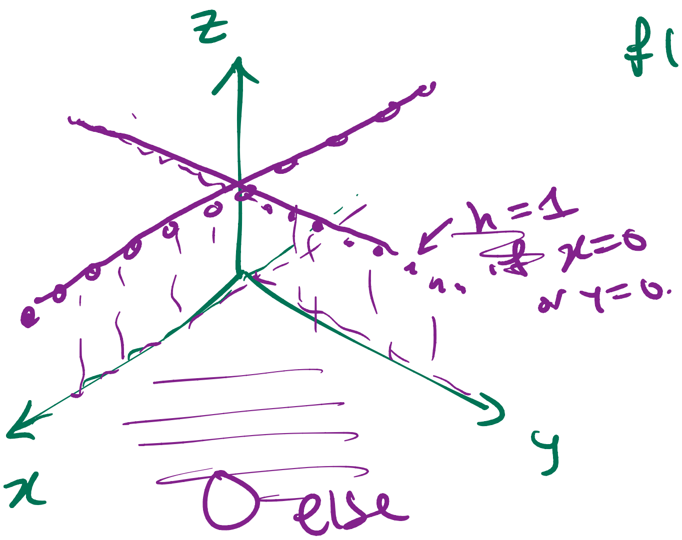


We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at **a** if its linearization is a good approximation of f near **a**.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y) - (a,b)\|} = 0.$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) its graph has a unique tangent plane at $(a,b,f(a,b))$.

Example 57. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.



$f(x,y) = \begin{cases} 1 & \text{if } x=0 \text{ or } y=0 \\ 0 & \text{otherwise.} \end{cases}$

Is $f(x,y)$ diff'ble at $(0,0)$?

Idea f_x & f_y at $(0,0)$.

Notice $f(0,0)=1$ @ $(x,0)$ $f_x(x,0)=0$

@ $(0,y)$ $f_y(0,y)=0$

f_x & f_y both zero.

$Df = [0 \quad 0]$

Ans No. $L(\vec{x})=0$ doesn't do a good job approx f @ $(0,0)$.

§14.4 The Chain Rule

Recall the Chain Rule from single variable calculus:

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx} f(g(x)) * \frac{d}{dx} g(x)$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$\underbrace{D(f(g(\mathbf{x})))}_{m \times n} = \overbrace{Df(g(\mathbf{x}))Dg(\mathbf{x})}^{(m \times p) * (p \times n)}.$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

"f after g"

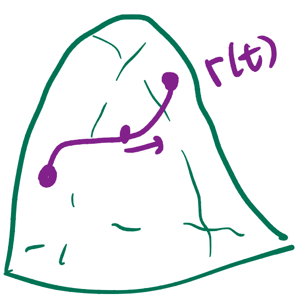
Df size $m \times p$

Dg size $p \times n$

Example 58. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t + 1, 2 - t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

$h(t) = h(\vec{r}(t)) = h(t+1, 2-t^2)$

$Dh(t) = Dh(\vec{r}(t)) * D\vec{r}(t)$



So $Dh(1) = Dh\Big|_{(x,y)=(2,1)} * \begin{bmatrix} 1 \\ -2t \end{bmatrix}\Big|_{t=1}$

$= \begin{bmatrix} -\frac{1}{2}(2) & -\frac{1}{2}(1) \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1 + 1 = \boxed{0}$

$Dh(x,y) = \begin{bmatrix} h_x & h_y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}$

$Dr(t) = \begin{pmatrix} r_1' \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 \\ -2t \end{pmatrix}$

@ $t = 1$
$r(1) = \langle 2, 1 \rangle$

$g(s,t) = \langle u(s,t), v(s,t)\rangle$

Example 59. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$
$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$
$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$
$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

$DW = [W_s \; W_t]$

$D(f(g(\vec{x}))) = Df(g(\vec{x}))\, Dg(\vec{x})$

Find $W_s(1,0)$ and $W_t(1,0)$.

$f = F$
$g = \langle u, v\rangle$
$x = \langle s, t\rangle$

$$DW = DF(g(s,t)) * Dg(s,t)$$
$$= [F_u \; F_v]\begin{bmatrix} u_s & u_t \\ v_s & v_t \end{bmatrix}$$

e.g.
@ $s=1,\ t=0$

$u_s(1,0) = -2$

$u(1,0) = 2$
$v(1,0) = 3$

$$DW\Big|_{(s,t)=(1,0)}$$
$$= [-1 \;\; 10] * \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$
$$= [52 \;\; 34]$$

$W_s|_{@(1,0)}$ $\quad$ $W_t|_{@(1,0)}$

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

Formula $\left\{ \begin{aligned} \frac{\partial z}{\partial x} &= \frac{-F_x}{F_z} \\ \frac{\partial z}{\partial y} &= \frac{-F_y}{F_z} \end{aligned} \right.$

Example 60. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

Get

$F(x,y,z) = x^2+y^2+z^2$

$F_x = 2x$

$F_y = 2y$

$F_z = 2z$

So $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-2x}{2z} = \boxed{\frac{-x}{z}}$

and

$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-2y}{2z} = \boxed{\frac{-y}{z}}$