§16.1 Line Integrals of Scalar Functions

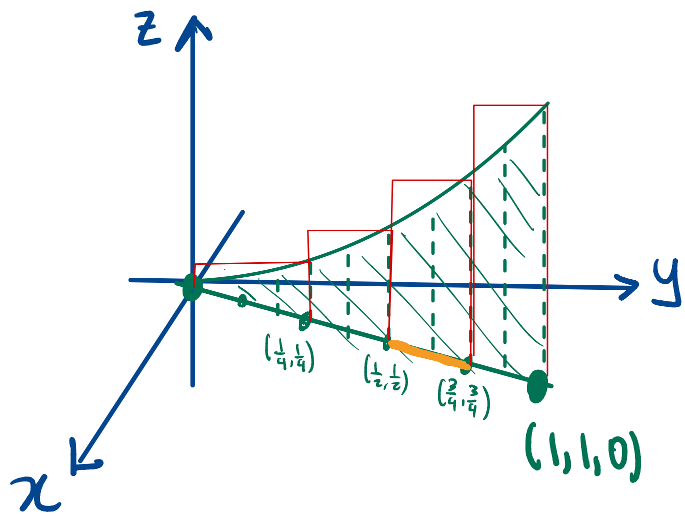
Chapter 16: Vector Calculus

Goals:

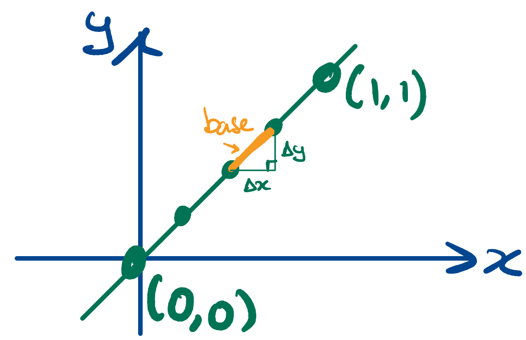
- Extend 1D/2D integrals to 1D/2D objects living in higher-dimensional space e.g curves in $\mathbb{R}^3$, surfaces in $\mathbb{R}^3$
- Extend the Fundamental Thm of Calc. in new ways

We will use tools from everything we have covered so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Example 121. Suppose we build a wall whose base is the straight line from $(0,0)$ to $(1,1)$ in the xy-plane and whose height at each point is given by $h(x,y) = 2x + y^2$ meters. What is the area of this wall?



$$\text{Area} \approx \sum_{i=1}^{k} h(x_i, y_i) * \text{base}$$



$$\text{base} = \sqrt{\Delta x^2 + \Delta y^2}$$

So $\text{Area} \approx \sum_{i=1}^{k} h(x_i, y_i)\sqrt{\Delta x^2 + \Delta y^2}$

and $\text{Area} = \lim_{k \to \infty} \sum_{i=1}^{k} h(x_i, y_i)\sqrt{\Delta x^2 + \Delta y^2}$

$$= \int_a^b h(\vec{r}(t)) * |r'(t)|\, dt$$

So Solve by

① Parametrize Curve

$r(t) = \langle t, t \rangle, \; 0 \le t \le 1$

$r'(t) = \langle 1, 1 \rangle, \; |r'(t)| = \sqrt{2}$

② Substitute

$$\int_C h(x,y)\, ds = \int_a^b h(r(t))\, |r'|\, dt = \int_0^1 (2t + t^2) * \sqrt{2}\, dt$$

$$= \left(t^2 + \tfrac{1}{3}t^3\right)\sqrt{2}\Big|_0^1 = \sqrt{2}\left[\left(1 + \tfrac{1}{3}\right) - (0 - 0)\right] = \boxed{\sqrt{2} * 4/3}$$

Definition 122. The **line integral** of a scalar function $f(x, y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x,y)\, ds = \int_a^b f(r(t)) * |v|\, dt$$

$ds = |v|\, dt$

where $v = r'(t)$ and

$C: r(t),\ t \in [a,b]$

What things can we compute with this?

- If $f = 1$: $\int_C 1\, ds$ is the arc length of the curve C
- If $f = \delta$ is a density function: $\int_C f\, ds$ tells mass of object C
- If f is a height: $\int_C f\, ds$ gives area over C and under f.

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \leq t \leq b$
2. Compute $ds = \|\mathbf{r}'(t)\|\ dt$
3. Substitute: $\int_C f(x, y, z)\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt$
4. Integrate

Example 123. *You try it!* Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \leq t \leq \frac{1}{10}$.

Strategy for computing line integrals:

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4. Integrate

Wait! Same f!!

Example 123. *You try it!* Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \le t \le \frac{1}{10}$.

Same path, new SPEED!!

$C: r(t) = \langle 10t, 10t \rangle,\ t \in [0, 1/10]$

$r'(t) = \langle 10, 10 \rangle$ so $|v|^2 = 100 + 100 = 200$ & $|v| = \sqrt{200} = 10\sqrt{2}$

So $A = \int_C 2x + y^2\ ds = \int_0^{1/10} \left(2(10t) + (10t)^2\right) 10\sqrt{2}\, ds$

$= \int_0^{1/10} 10\sqrt{2}(20t + 100t^2)\ dt$

$= 10\sqrt{2}\left(10t^2 + \frac{100}{3}t^3\right)\Big|_0^{1/10}$

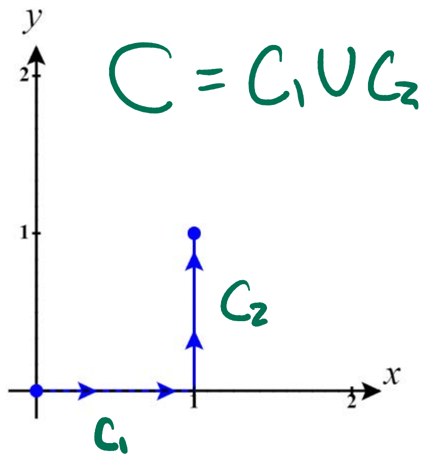
$= 10\sqrt{2}\left(10 * \frac{1}{10^2} + \frac{100}{3} * \frac{1}{10^3}\right) = \cancel{10}\sqrt{2}\left(\cancel{\frac{1}{10}}\right)\left(1 + \frac{1}{3}\right)$

$= \boxed{\sqrt{2} * \frac{4}{3}}$

Uhhh... That's an odd coincidence!

HA! You won't fool me again!!

Example 124. Compute $\int_C 2x + y^2 \, ds$ along the curve C pictured below.



$C = C_1 \cup C_2$

$C_1: r_1(t) = \langle t, 0 \rangle \quad t \in [0,1] \quad |v| = 1$

$C_2: r_2(t) = \langle 0, t \rangle \quad t \in [0,1] \quad |v| = 1$

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds$$

$$= \int_0^1 2t \cdot 1 \, dt + \int_0^1 t^2 \cdot 1 \, dt$$

$$= \int_0^1 2t + t^2 \, dt$$

$$= t^2 + \tfrac{1}{3}t^3 \Big|_0^1 = 1 + \tfrac{1}{3} = \boxed{4/3}$$

Wait... what!? I GIVE UP! jk.

Punchline: It doesn't matter where you go as much as THE PATH you take to get there!

Observation 1: Ex. 121 & 123 same answer since SAME CURVE was parametrized in two different ways (just changing SPEED along the curve)

Observation 2: Ex. 124 DIFFERENT answer since we used a different curve to get from (0,0) to (1,1).

Example 125. *You try it!* Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \leq t \leq b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \leq t \leq b$ is also a parameterization of C with the same orientation

b) $\mathbf{r}(2t)$ for $a/2 \leq t \leq b/2$ is also a parameterization of C with the same orientation

c) $\mathbf{r}(-t)$ for $a \leq t \leq b$ is also a parameterization of C with the opposite orientation

d) $\mathbf{r}(-t)$ for $-b \leq t \leq -a$ is also a parameterization of C with the opposite orientation

e) $\mathbf{r}(b-t)$ for $0 \leq t \leq b-a$ is also a parameterization of C with the opposite orientation

Example 125. *You try it!* Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \le t \le b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \le t \le b$ is also a parameterization of C with the same orientation

False

e.g. From Ex 121-124
$r(t) = \langle t, t \rangle \quad 0 \le t \le 1$
different start/ending point if
$\hat{r}(t) = \langle t+4, t+4 \rangle = r(t+4), \ 0 \le t \le 1.$

b) $\mathbf{r}(2t)$ for $a/2 \le t \le b/2$ is also a parameterization of C with the same orientation

True

just increasing speed by $*2$.

e.g. $r(t) = \langle 2t, 2t \rangle \quad 0 \le t \le 1/2$
get there @ $t = 1/2$ (twice as fast)

c) $\mathbf{r}(-t)$ for $a \le t \le b$ is also a parameterization of C with the opposite orientation

False

in general. (actually works for Ex. 121-124)

would have been TRUE if
$\hat{r}(t) = r(-t)$ and $-b \le t \le -a$ Though!

d) $\mathbf{r}(-t)$ for $-b \le t \le -a$ is also a parameterization of C with the opposite orientation

like This one ↑

True

e) $\mathbf{r}(b-t)$ for $0 \le t \le b-a$ is also a parameterization of C with the opposite orientation

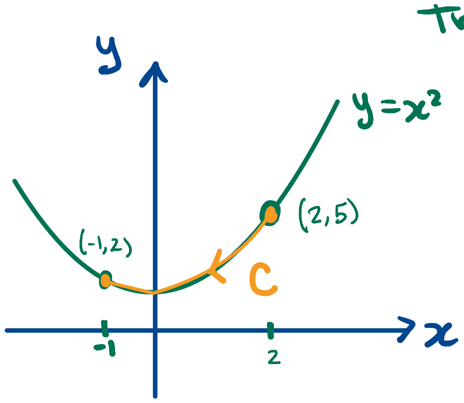
True

$\hat{r}(t) = r(b-t)$

$\hat{r}(0) = r(b)$ ✓
$\hat{r}(b-a) = r(a)$

Dr. H says!
Need to know how to parametrize
① line segments between two points
② circles & ellipses
③ graph portions $x = f(y)$ or $y = f(x)$.

Example 126. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2+1$ from $(2,5)$ to $(-1,2)$ and use it to write the integral $\int_C x^2+y^2\,ds$ as an integral with respect to your parameter.



Try (?) $C: r(t) = \langle t, t^2+1 \rangle \quad t \in [-1,2]$?

goes wrong direction ☹

$\boxed{\hat{r}(t) = r(-t), \quad t \in [-b,-a]}$

$C: \hat{r}(t) = r(-t) = \langle -t, t^2+1 \rangle$

$t \in [-2,1]$

$r'(t) = \langle -1, 2t \rangle$

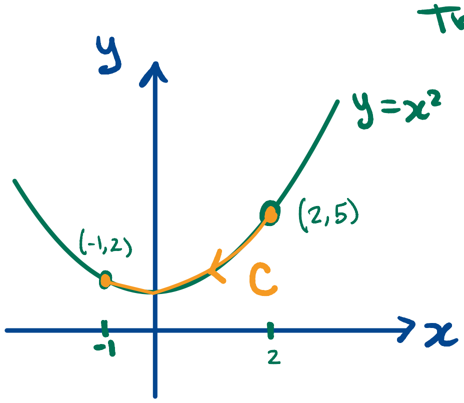
$|v|^2 = 1 + (2t)^2$

$|v| = \sqrt{1+4t^2}$

So $\int_C x^2+y^2\,ds$

$$= \boxed{\int_{-2}^{1} \left[t^2 + (t^2+1)^2 \right] * \sqrt{1+4t^2}\,dt}$$

Example 126. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2+1$ from $(2, 5)$ to $(-1, 2)$ and use it to write the integral $\int_C x^2+y^2\,ds$ as an integral with respect to your parameter.



Try (?) C: $r(t) = \langle t, t^2+1\rangle$ $t\in[-1,2]$?

goes wrong direction

$$\boxed{\hat{r}(t) = r(b-t),\quad t\in[0, b-a]}$$

C: $\hat{r}(t) = r(2-t) = \langle 2-t, (2-t)^2+1\rangle$

$t\in[0,3]$

$r'(t) = \langle -1, 2(2-t)*-1\rangle$

$= \langle -1, -4+2t\rangle$

$|v|^2 = 1+(4+2t)^2$

$= 1+16+16t+4t^2$

$= 1+17+4t^2$

So $\int_C x^2+y^2\,ds$

$$= \int_0^3 \left[(2-t)^2+\left[(2-t)^2+1\right]^2\right] * \sqrt{1+17+4t^2}\,dt$$

§16.2 Vector Fields & Vector Line Integrals

Vector Fields:

Definition 127. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain. e.g. $F : \mathbb{R}^2 \to \mathbb{R}^2 \quad F(x,y) = \langle P(x,y), Q(x,y) \rangle$

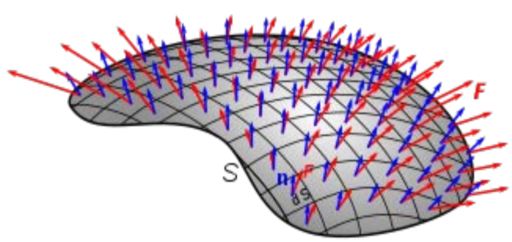
$F : \mathbb{R}^3 \to \mathbb{R}^3 \quad F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

Examples:

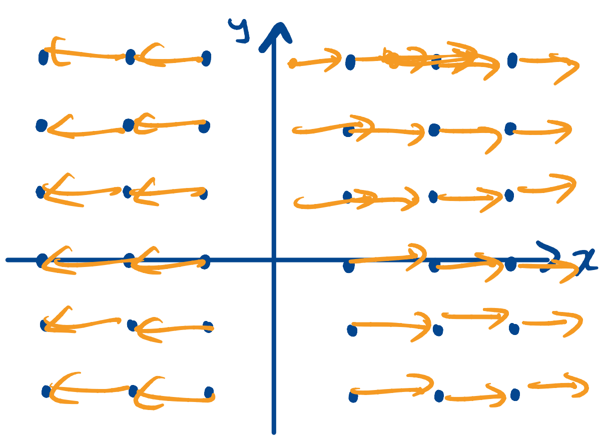
- magnetic fields
- flowing fluids
- differential equation slope field
- just plot ∇f at each point?
- tangent vectors on a curve
- normal vectors on a surface

Graphically: For each point (a, b) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b)$ with its base at (a, b).

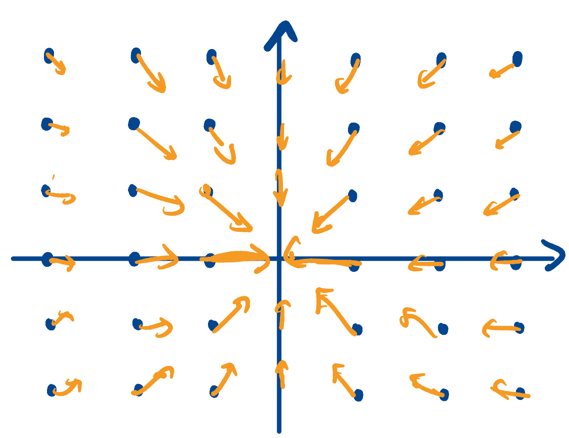
Tools: CalcPlot3d
Field Play



e.g. $F(x,y) = \left\langle \frac{x}{|x|}, 0 \right\rangle$

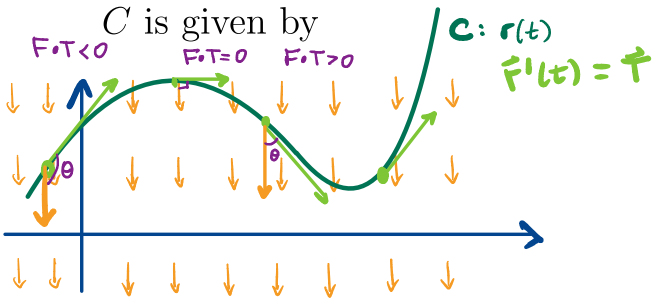


e.g. $F(x,y) = \left\langle \frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right\rangle$



Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The Work done by a field **F** on an object moving along a curve C is given by



$\vec{r}'(t) = \vec{T}$

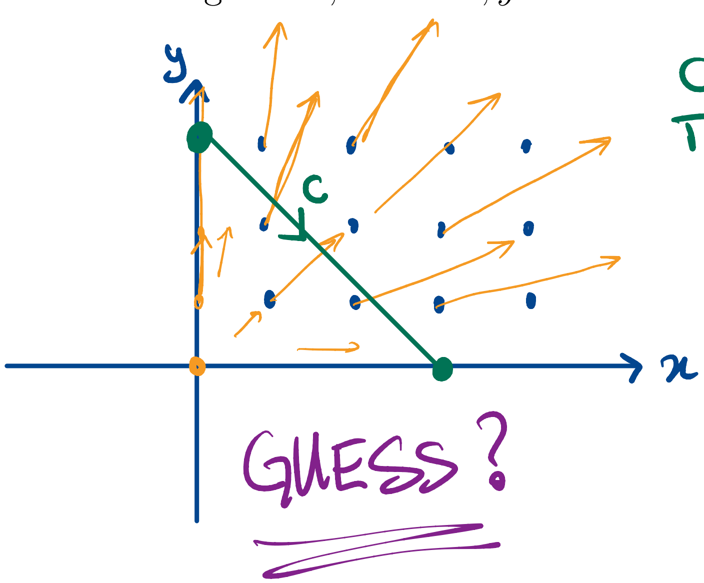
Component of F along C: $\vec{F} \cdot \vec{T}$

Work done: $\int_C \vec{F} \cdot \vec{T} \, ds$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{\cancel{\|\vec{r}'(t)\|}} * \cancel{\|\vec{r}'(t)\|} \, dt$$

$$W = \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt}$$

Example 128. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x, y) = \langle x, y \rangle$ N. Find the work done by **F** on a moving object from $(0, 3)$ to $(3, 0)$ in a straight line, where x, y are measured in meters.



$C: r(t) = \langle 3t, (1-t)3 \rangle \quad t \in [0, 1]$

$T = r'(t) = \langle 3, -3 \rangle$

$F(x,y) = \langle x, y \rangle$

So $W = \int_0^1 \langle 3t, (1-t)3 \rangle \cdot \langle 3, -3 \rangle \, dt$

$= \int_0^1 9t - 9(1-t) \, dt$

$= \int_0^1 18t - 9 \, dt = 9t^2 - 9t \Big|_0^1$

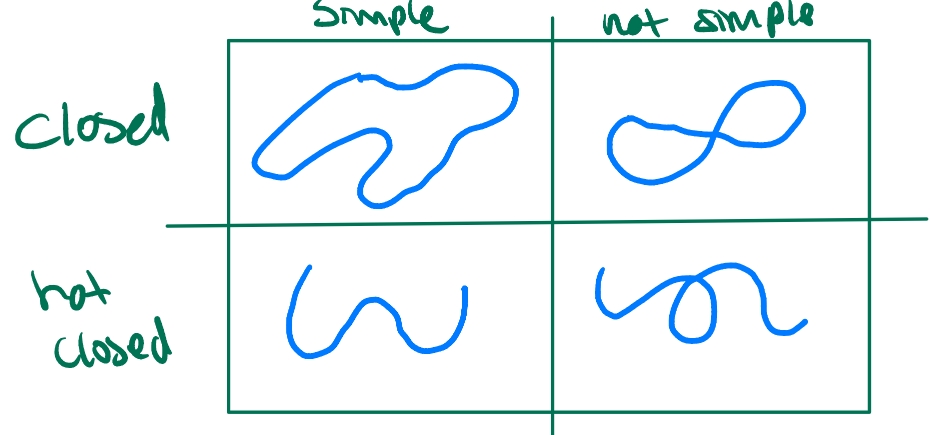
$= (9-9) - (0-0) = \boxed{0}$

doesn't feel like no work was done

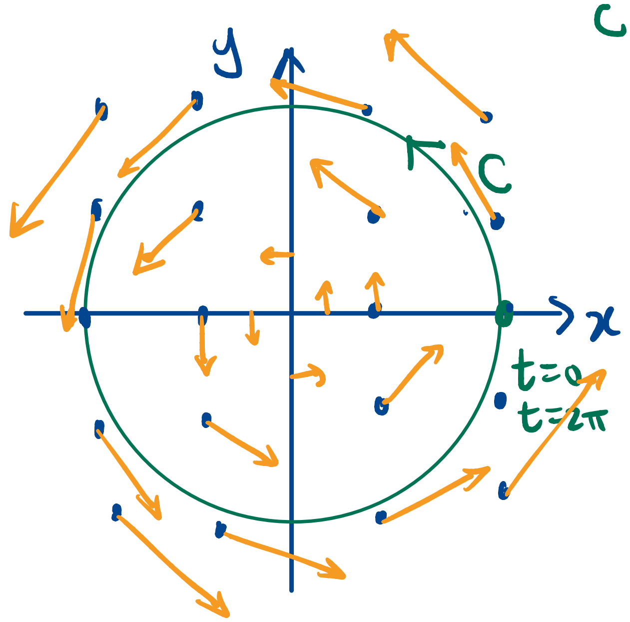
same formula?

2. The <u>Flow</u> along a curve C of a velocity field $\mathbf{F}$ for a fluid in motion is given by $\int_C \vec{F} \cdot \vec{T}\, ds$

When C is <u>Closed</u>, this is called <u>circulation</u>. C is called <u>Simple</u> if it does not intersect itself.



Example 129. Flow of a Velocity Field. Find the circulation of the velocity field $\boxed{\mathbf{F}(x, y) = \langle -y, x\rangle}$ cm/s around the unit circle, parameterized counterclockwise.



$C: r(t) = \langle \cos t, \sin t\rangle \quad t \in [0, 2\pi]$

$r'(t) = \langle -\sin t, \cos t\rangle \quad |r'(t)| = 1$

$$\text{Flow} = \int_a^b F(\vec{r}(t)) \cdot r'(t)\, dt$$

$$\text{Flow} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$$

$$= \int_0^{2\pi} \underbrace{\sin^2 t + \cos^2 t}_{1}\, dt$$

$$= \int_0^{2\pi} 1\, dt = t \Big|_0^{2\pi} = \boxed{2\pi}$$

If we reverse orientation of C, then what would flow value be?

(pick) $0, 2\pi, \boxed{-2\pi}$?

Example 130. *You try it!* What is the circulation of $\mathbf{F}(x, y) = \langle x, y \rangle$ around the unit circle, parameterized counterclockwise?

Strategy for computing tangential component line integrals

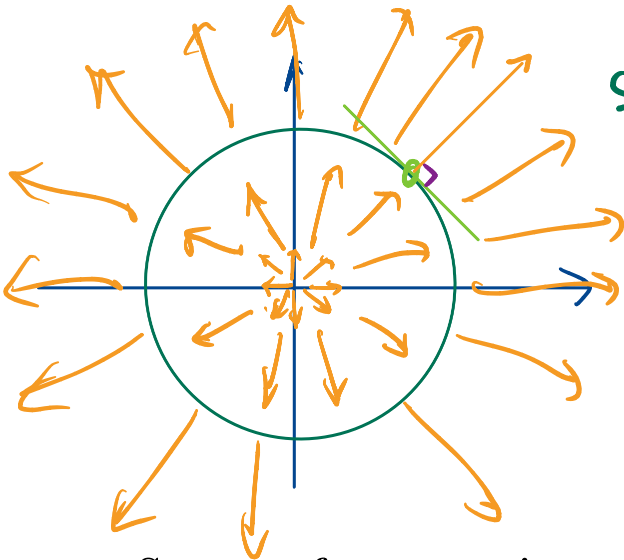
e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt$
4. Integrate

Example 130. *You try it!* What is the circulation of $\mathbf{F}(x, y) = \langle x, y \rangle$ around the unit circle, parameterized counterclockwise?

$r(t) = \langle \cos t, \sin t \rangle \quad t \in [0, 2\pi]$

$r'(t) = \langle -\sin t, \cos t \rangle \quad |r| = 1.$



So

$$\text{Flow} = \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle * 1 \, dt$$

$$= \int_0^{2\pi} -\cos t \sin t + \sin t \cos t \, dt$$

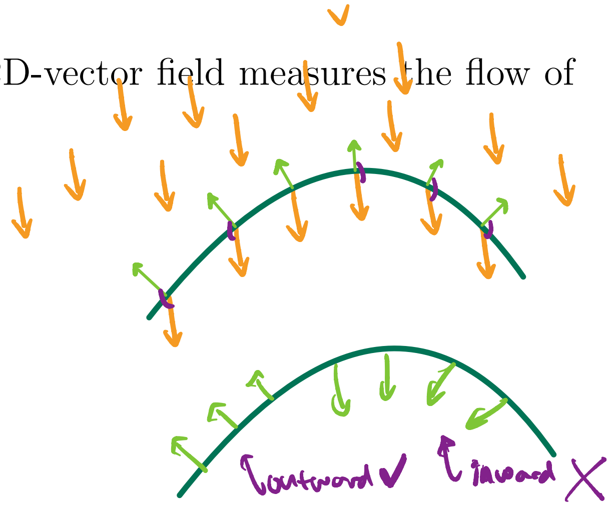
$$= \int_0^{2\pi} 0 \, dt = \boxed{0}$$

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$
4. Integrate

Idea: Flux across a plane curve of a 2D-vector field measures the flow of the field across that curve (instead of along it).



We compute this with the integral

$\mathbf{r}(t) = \langle x(t), y(t)\rangle$
$\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$

$$\text{Flux} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds.$$

The sign of the flux integral tells us whether the net flow of the field across the curve is in the direction of $\vec{n}$ or in the opposite direction.

OUTWARD pointing!!

We can choose $\mathbf{n}$ to be either of

① $\vec{n} = \dfrac{\langle y'(t), -x'(t)\rangle}{\|\vec{r}\,'(t)\|}$ or ② $\dfrac{\langle -y'(t), x'(t)\rangle}{\|\vec{r}\,'(t)\|}$

(often this one)

If ①

$$\text{Flux} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds$$
$$= \int_a^b F(r(t))\cdot n(t) * \|r'(t)\|\,dt$$
$$= \int_a^b F(r(t))\cdot\langle y'(t), x(t)\rangle\,dt$$

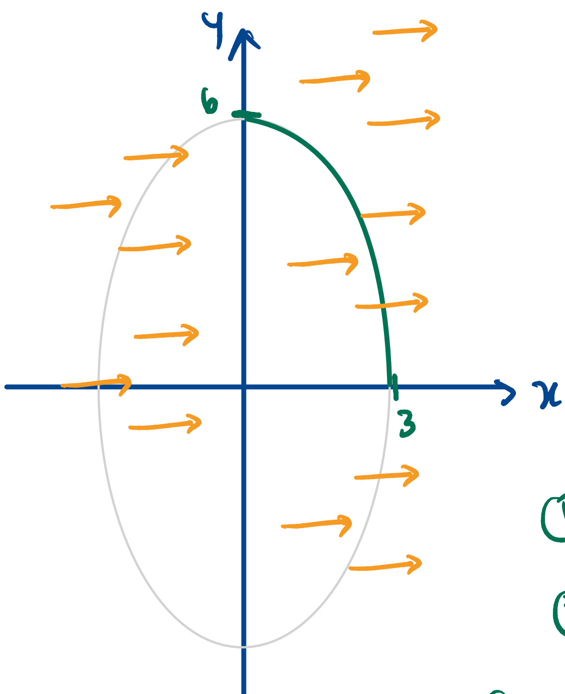
Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.
2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.
3. Substitute: $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \pm\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\,dt$ (sign based on choice of normal)
4. Integrate

Example 131. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0\rangle$ cm/s across the quarter of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ in the first quadrant, oriented away from the origin.

y(t)=0 so v is horizontal



$$\text{Flux} = \int_C F \cdot \vec{n}\, ds$$

Step 1: Parametrize C: $r(t) = \langle x(t), y(t)\rangle$, $t \in [a,b]$

Step 2: Find $r'(t) = \langle x', y'\rangle$ and determine $\vec{n} = \pm\langle y', -x'\rangle$

Step 3: Evaluate $\text{Flux} = \int_a^b F(r(t)) \cdot n(t) * \|r'(t)\|\, dt$

① $r(t) = \langle 3\cos t, 6\sin t\rangle$, $t \in [0, \pi/2]$

② $r'(t) = \langle -3\sin t, 6\cos t\rangle$

Guess $n(t) = \dfrac{\langle 6\cos t, -3\sin t\rangle}{\|r'(t)\|}$ try $t=0$ $n(0) \sim \langle 6, 0\rangle$ ✓

will cancel w/ $ds = \|r'(t)\|\, dt$

So ③

$$\text{Flux} = \int_0^{\pi/2} \langle 3 + 2(6\sin t) - (6\sin t)^2/3, 0\rangle \cdot \langle 6\cos t, -3\sin t\rangle\, dt$$

$$= \int_0^{\pi/2} \left(3 + 12\sin t - \frac{36\sin^2 t}{3}\right) * 6\cos t\, dt$$

u-sub Box: $u = \sin t$, $du = \cos t$

$t=0 \Rightarrow u=0$

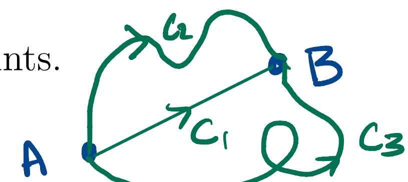
$t=\pi/2 \Rightarrow u=1$

$$= 6\int_0^1 3 + 12u - 12u^2\, du = 6\left(3t + 6u^2 - 4u^3\Big|_0^1\right)$$

$$= 6([3+6-4] - [0+0-0]) = 6 * 5 = \boxed{30}$$

§16.3 Conservative Vector Fields & Fundamental Theorem

Definition 132. A vector field **F** is **path independent** on an open region D if $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is the sa for all paths C in the region that have the same endpoints.



$$\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = \dots \text{ etc.}$$

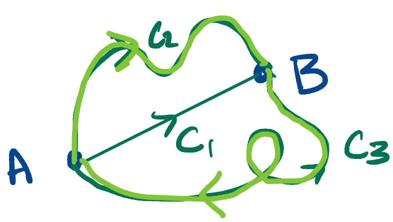
When **F** is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F}\cdot\mathbf{T}\;ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F}\cdot d\mathbf{r}.$$

Example 133. If C is any closed path and **F** is path independent on a region containing C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_A^A \mathbf{F}\cdot\mathbf{T}\,ds = 0$$

Since



$$= \int_A^B \mathbf{F}\cdot\mathbf{T}\,ds + \int_B^A \mathbf{F}\cdot\mathbf{T}\,ds$$

$$= \int_A^B \mathbf{F}\cdot\mathbf{T}\,ds - \int_A^B \mathbf{F}\cdot\mathbf{T}\,ds$$

$$= 0 \;\checkmark$$

Question: Given $\mathbf{F}$, how do we tell if it is path independent on a particular region?

Several options!

① Check some paths C_1, C_2 if $\int_{C_1} F \cdot T\,ds \neq \int_{C_2} F \cdot T\,ds$, then F is NOT path independent.

② Check some CLOSED LOOPS C to check if $\int_C F \cdot T\,ds = 0$.

For example, is $\mathbf{F}(x, y) = \langle x, y \rangle$ a path independent vector field on its domain?

Try ② w/ $C: r(t) = \langle \cos t, \sin t \rangle$, $t \in [0, 2\pi]$

Then $\int_C F \cdot T\,ds = \int_0^{2\pi} \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\,dt$

$= \int_0^{2\pi} -\sin t \cos t + \sin t \cos t\,dt = \int_0^{2\pi} 0\,dt$

$= 0$. Ok but other loops?

ANS: Not sure yet!

Example 134. *You try it!* Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x \rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

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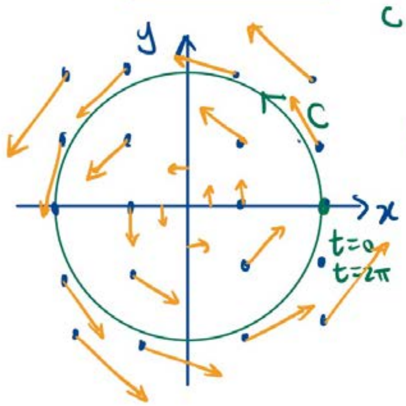
Example 129. Flow of a Velocity Field. Find the circulation of the velocity field $\boxed{\mathbf{F}(x, y) = \langle -y, x\rangle}$ cm/s around the unit circle, parameterized counterclockwise.



$C: r(t) = \langle \cos t, \sin t\rangle$ $t \in [0, 2\pi]$

$r'(t) = \langle -\sin t, \cos t\rangle$ $|r'(t)| = 1$

$\text{Flow} = \int_a^b F(\vec{r}(t)) \cdot r'(t)\, dt$

$\text{Flow} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$

$= \int_0^{2\pi} \underbrace{\sin^2 t + \cos^2 t}_{1}\, dt$

$= \int_0^{2\pi} 1\, dt = t \Big|_0^{2\pi} = \boxed{2\pi}$

If we reverse orientation of C, then what would Flow value be?

(pick) $0, 2\pi, \boxed{-2\pi}$?

So $F(x, y) = \langle -y, x\rangle$ is <u>NOT</u> path independent.

A different idea: Suppose $\mathbf{F}$ is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a potential function for $\mathbf{F}$. In this case we also say that $\mathbf{F}$ is **conservative.**

Is $\mathbf{F}(x,y) = \langle x, y\rangle$ conservative? Need to find $f(x,y)$ s.t. $\nabla f = F$.

But that means $\nabla f = \langle f_x, f_y \rangle = \langle x, y \rangle$

can be any expression w/ only y's!!

Step 1: Integrate (wrt x) $\frac{\partial}{\partial x} f = x \Rightarrow f = \int x\,dx = \frac{1}{2}x^2 + C(y)$

Step 2: Partial derv. (wrt y) $\frac{\partial}{\partial y} f = y \Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + C(y)\right) = 0 + C'(y) = y$.

So $C = \int y\,dy = \frac{1}{2}y^2 + c$ and $\boxed{f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C}$

Whoa.

So $\nabla f = \langle x, y\rangle = F$ for this choice of f. Cool.

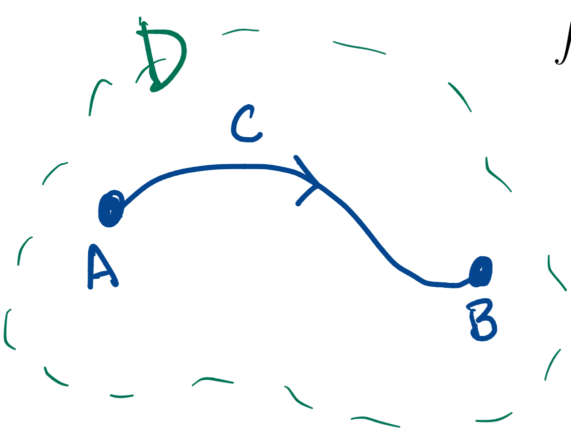
Theorem 135 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

$$\int_C \nabla f \cdot \mathbf{T}\, ds = f(B) - f(A)$$

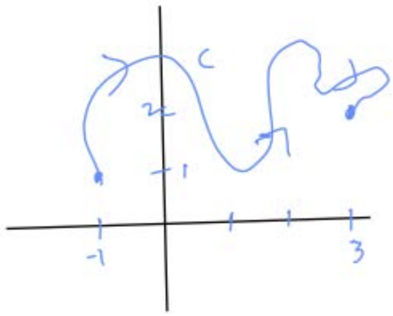
This doesn't depend on choice of C, so if $F = \nabla f$ is conservative, then F is path independent!

Compare to FTC

$\int_a^b f(x)\,dx = F(b) - F(a)$



Example 136. Compute $\int_C \langle x, y\rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1,1)$ to $(3,2)$.



$F = \langle x, y\rangle$ is conservative w/ $\nabla f = F$ where

$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C.$

path independence

So $\int_C F \cdot dr = \int_{(-1,1)}^{(3,2)} F \cdot dr$

FToLI

$$= f(3,2) - f(-1,1)$$

$$= \left(\tfrac{1}{2}(3)^2 + \tfrac{1}{2}(2)^2 + C\right) - \left(\tfrac{1}{2}(-1)^2 + \tfrac{1}{2}(1)^2 + C\right)$$

$$= \frac{9}{2} + 2 + \cancel{C} - \frac{1}{2} - \frac{1}{2} - \cancel{C}$$

$$= \boxed{11/2}$$

NOTE: trying the same trick to find f w/ $\nabla f = \langle -y, x\rangle$ MUST FAIL (why?)

Let's try anyways.

① $\frac{\partial}{\partial x} f = -y \Rightarrow f = -yx + C_1(x)$

② $\frac{\partial}{\partial y} f = x \Rightarrow f = yx + C_2(y)$

no way to choose C_1 & C_2 so that you get f here.

(this is a strategy for showing F is not conservative, so also not path independent)

It follows, from THM 135, that **every conservative field is path independent.**

Does the implication go the other way, too? (Iff?)

In fact, by carefully constructing a potential function, we can show the converse is also true: every path independent F is conservative.

Idea: $f(x,y) = \int_{(a,b)}^{(x,y)} F \cdot T \, ds$

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0,0,0\rangle$, then $\mathbf{F}$ is conservative.

ie $F(x,y) = \langle P(x,y), Q(x,y), 0\rangle$

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} = \langle 0-0, 0-0, Q_x - P_y\rangle = \vec{0}$ $\Rightarrow \boxed{Q_x = P_y}$
- This is also called the **mixed-partials test**, ~~because~~

Note: Ex 1 $F(x,y) = \langle x,y\rangle = \langle P,Q\rangle$

then $\left.\begin{array}{l} Q_x = \frac{\partial}{\partial x} y = 0 \\ P_y = \frac{\partial}{\partial y} x = 0 \end{array}\right\} \checkmark$

Ex 2 $F(x,y) = \langle -y, x\rangle = \langle P,Q\rangle$

$\left.\begin{array}{l} Q_x = 1 \\ P_y = -1 \end{array}\right\} \times$

Cool.

Simply connected means "no holes"

loops have to not be able to go around parts of the domain that aren't included.

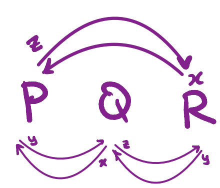
not simply connected ✗

simply connected ✓

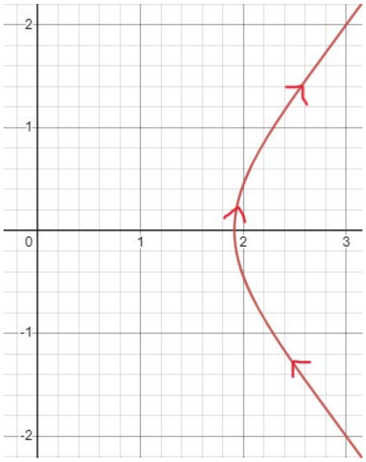
textbook calles the curl test the component test/mixed partials

$F = \langle P, Q, R\rangle$ conservative on simply connected D

iff $R_y = Q_z$
$P_z = R_x$
$Q_x = R_y$



Example 137. Evaluate $\int_C (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3,-2)$ to $(3,2)$.



Let's check the Curl Test for $F=\langle P,Q\rangle$.

$Q(x,y) = -3x^2y^2$

$Q_x = -6xy^2$

$P(x,y) = 10x^4 - 2xy^3$

$P_y = 0 - 6xy^2$.

So $\text{Curl } F = \langle 0,0,Q_x - P_y\rangle = \langle 0,0,-6xy^2-(-6xy^2)\rangle$
$= \langle 0,0,0\rangle$ ✓

Need $f(x,y)$ s.t. $\nabla f = F$ so we can use FTofLI.

① $f_x = 10x^4 - 2xy^3 \Rightarrow f = 2x^5 - x^2y^3 + C(y)$

② $f_y = \frac{\partial}{\partial y}\left(2x^5 - x^2y^3 + C(y)\right) = 0 - \cancel{3x^2y^2} + C'(y) \stackrel{?}{=} Q = -\cancel{3x^2y^2}$

So $C'(y) = 0 \Rightarrow C(y) = C$.

So $f(x,y) = 2x^5 - x^2y^3 + C$

And $\int_C P\,dx + Q\,dy = f(3,-2) - f(3,2)$
$= \cancel{2*3^5} - 3^2(-2)^3 - (\cancel{2*3^5} - 3^2 * 2^3) = \boxed{2^4 3^2}$

or $9*16 = \boxed{144}$