§16.4 Divergence, Curl, Green's Theorem

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R$

  so $\nabla \cdot F = P_x + Q_y + R_z$ (called $\mathrm{div}(F)$)

- $\nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$

"Formal"

Here $\cdot$ and $\times$ are being "slightly abused notation"

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q\right) - \hat{j}\left(\frac{\partial}{\partial x}R - \frac{\partial}{\partial z}P\right) + \hat{k}\left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \mathrm{curl}(F)$$

(From §16.3)

Ex. Find $\mathrm{div}(F)$ & $\mathrm{curl}(F)$, $F = \langle \underset{P}{xy}, \underset{Q}{2y^2}, \underset{R}{x+z} \rangle$

$\mathrm{div}\, F = P_x + Q_y + R_z = y + 4y + 1 = \boxed{5y+1}$

$$\mathrm{curl}\, F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2y^2 & x+z \end{vmatrix} = \langle 0 - 0, -(1 - 0), 0 - x \rangle$$

$$= \boxed{\langle 0, -1, -x \rangle}$$

How do we measure the change of a vector field?

Cross product only defined for vectors in $\mathbb{R}^3$

1. **Curl** (in $\mathbb{R}^3$)

   - Tells us Flow / Circulation density
   - Measures local circulation at a point
   - Is a vector
   - Direction gives axis of rotation (using right hand rule)
   - Magnitude gives rotation rate
   - $\operatorname{curl} \mathbf{F} = \nabla \times F$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle = \langle 0, 0, Q_x - P_y \rangle$

From §16.3 THM:
$\operatorname{Curl}(F) = \langle 0,0,0 \rangle \iff F$ is conservative
(ie. $F = \nabla f$ for some $f: \mathbb{R}^3 \to \mathbb{R}$)

Geometric meaning $\operatorname{Curl}(F) = \vec{0}$ is "F has no rotation"

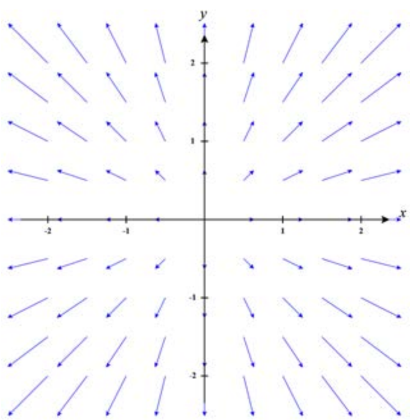
2. **Divergence** (in any $\mathbb{R}^n$)

   - Tells us Flux density
   - Measures Compression / expansion at a point
   - Is a Scalar
   - $\operatorname{div} \mathbf{F} = \nabla \cdot F$

$\boxed{\text{THM: } \operatorname{div} F = \vec{0} \Rightarrow F = \nabla \times G \text{ for some other vector field } G.}$

Geometric meaning to $\operatorname{div} F = \vec{0}$ is F is "incompressible" or "no expansion/compression"

Example 138. Let $\mathbf{F}(x, y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



lots of expansion, so div(F) should be positive at every point. (and Flux $= \oint_C \mathbf{F}\cdot\mathbf{n}\,ds > 0$)

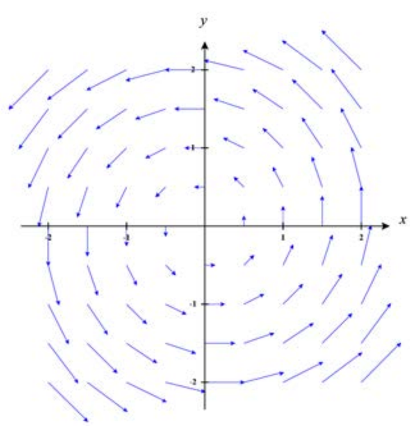
no rotation, so curl(F) should be the zero vector. (and $F = \nabla f$ & Flow $= \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = f(A) - f(A) = 0$)

Check: $\operatorname{div} F = P_x + Q_y = 1 + 1 = 2$ ✓

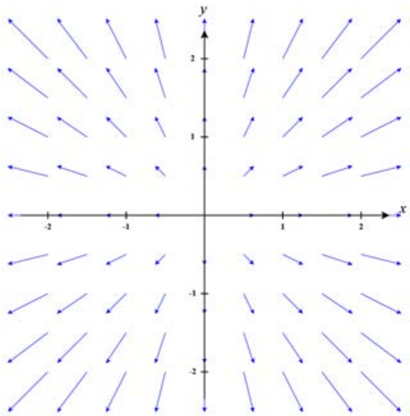
$\operatorname{curl} F = \langle 0, 0, Q_x - P_y\rangle = \langle 0, 0, 0 - 0\rangle$

$= \langle 0, 0, 0\rangle$ ✓

Example 139. *You try it!* Let $\mathbf{F}(x, y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Example 138. Let $\mathbf{F}(x, y) = \langle x, y \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.

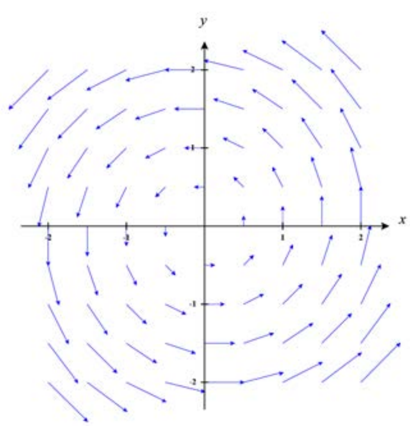


lots of expansion, so div(F) should be positive at every point (and Flux $= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds > 0$)

no rotation, so curl(F) should be the zero vector.
(and $\mathbf{F} = \nabla f$ & Flow $= \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = f(A) - f(A) = 0$)

Check: $\operatorname{div} \mathbf{F} = P_x + Q_y = 1 + 1 = 2$ ✓
$\operatorname{curl} \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, 0 - 0 \rangle$
$= \langle 0, 0, 0 \rangle$ ✓

Example 139. *You try it!* Let $\mathbf{F}(x, y) = \langle -y, x \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



lots of rotation so $\operatorname{curl}(\mathbf{F}) \neq \vec{0}$
and should be pointing "up" out of the page by RHR

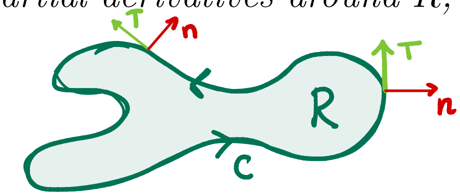
no compression, so $\operatorname{div}(\mathbf{F}) = 0$

Check $\operatorname{curl} \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle$
$= \langle 0, 0, 1 - (-1) \rangle = \langle 0, 0, 2 \rangle$ "up" ✓
$\operatorname{div} \mathbf{F} = P_x + Q_y = 0 + 0 = 0$ ✓

Question: How is this useful?

Answer: We can relate <u>rates of change of F</u> inside a region to the behavior of the vector field on the boundary of the region.

Theorem 140 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$. *If $\mathbf{F} = \langle P, Q \rangle$ has continuous partial derivatives around R, then*



a) Circulation form:

(a)
$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C P\,dx + Q\,dy = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA = \iint_R Q_x - P_y\,dA$$

b) Flux form:

notational equality

G's T

expand out

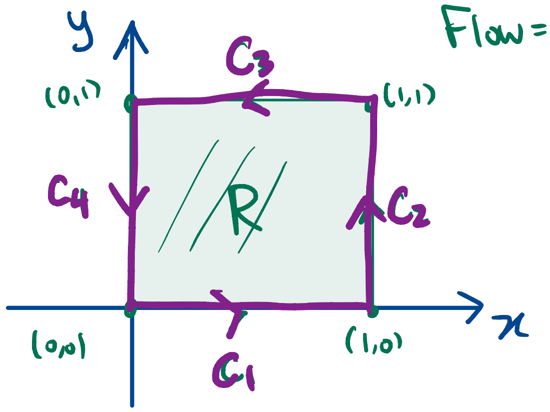
$(\nabla\times F)\cdot\hat{k}$ or $\nabla\cdot F$

(b)
$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C P\,dy - Q\,dx = \iint_R (\nabla\cdot\mathbf{F})\,dA = \iint_R P_x + Q_y\,dA$$

(a) Says: The $\boxed{\text{Flow}}$ across a closed simple loop C is the double-integral over the interior R of C of $\boxed{\text{the } \hat{k}\text{-component of Curl } F.}$

(b) Says: The $\boxed{\text{Flux}}$ across a Closed simple loop C is the double-integral of the interior R of C of $\boxed{\text{div } F}$

Example 141. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0$, and $y = 1$ oriented counterclockwise. $C = C_1 \cup C_2 \cup C_3 \cup C_4$ yuck!



$$\text{Flow} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$$

or use G's T

$$\oint \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \operatorname{curl} \mathbf{F} \cdot \hat{k}\, dA$$

$$= \iint_R Q_x - P_y\, dA = \iint_R y - (-2y)\, dA$$

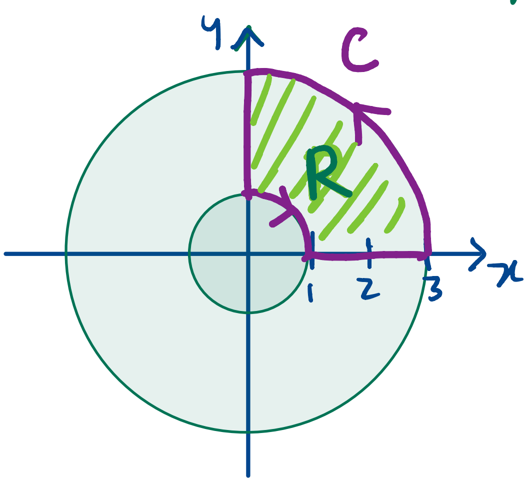
better!

$$= \int_0^1 \int_0^1 3y\, dy\, dx = \int_0^1 \frac{3}{2} y^2 \Big|_0^1 = \int_0^1 \frac{3}{2}\, dx$$

$$= \frac{3}{2} x \Big|_0^1 = \frac{3}{2}(1-0) = \boxed{\frac{3}{2}}$$

Example 142. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first ~~octant~~ quadrant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.

$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds \overset{\text{G's T}}{=} \iint_R P_x + Q_y\, dA$$



polar coord.
$x^2 + y^2 = r^2$

$$= \iint_R x^2 + y^2\, dA$$

polar coord

$$= \int_0^{\pi/2} \int_1^3 r^2 * r\, dr\, d\theta = \int_0^{\pi/2} \int_1^3 r^3\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} r^4 \Big|_1^3 d\theta = \int_0^{\pi/2} \frac{81}{4} - \frac{1}{4}\, d\theta = 20\theta \Big|_0^{\pi/2} = 20 * \frac{\pi}{2}$$

$$= \boxed{10\pi}$$

THM (G'ST Area Formula) for Flow $\boxed{\text{Area } R = \frac{1}{2}\oint_C -y\,dx + x\,dy}$

Example 143. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t)\rangle$ for $0 \le t \le \pi$. Find the area of R, using Green's Theorem applied to the vector field $\boxed{\mathbf{F} = \frac{1}{2}\langle x, y\rangle}$.

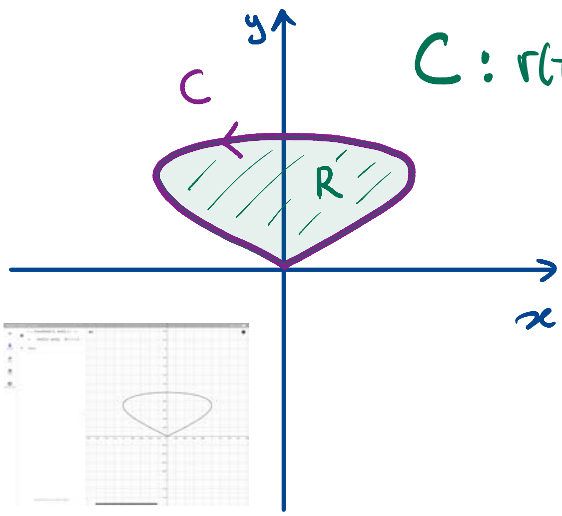
For Flux

or $F = \langle -\frac{1}{2}y, \frac{1}{2}x\rangle$ For Flow

Idea: $\text{Area } R = \iint_R 1\,dA = \iint \underbrace{\tfrac{1}{2}}_{Q_x} - \underbrace{(-\tfrac{1}{2})}_{P_y}\,dA$

G'ST (Flow)

$= \oint_C F \cdot T\,ds$ w/ $F = \langle -\frac{1}{2}y, \frac{1}{2}x\rangle$



$C: r(t) = \langle \sin 2t, \sin t\rangle$ $t \in [0, \pi]$ $r'(t) = \langle 2\cos 2t, \cos t\rangle$

$\text{Area } R = \oint_C \langle -\frac{1}{2}y, \frac{1}{2}x\rangle \cdot T\,ds$

$= \int_0^\pi \langle -\frac{1}{2}\sin t, \frac{1}{2}\sin 2t\rangle \cdot \langle 2\cos 2t, \cos t\rangle\,ds$

$= \int_0^\pi -\sin t\cos 2t + \frac{1}{2}\sin 2t\cos t\,dt$ unclear how to integrate?

$\cos 2t = 2\cos^2 t - 1$

$\sin 2t = 2\sin t\cos t$

$= \int_0^\pi -\sin t(2\cos^2 t - 1) + \frac{1}{2}(2\sin t\cos t)\cos t\,dt$

u-sub $u = \cos t$ $du = -\sin t\,dt$

$= \int_0^\pi -2\cos^2 t\sin t + \sin t + \sin t\cos^2 t\,dt$

Grame

$= \int_0^\pi -\cos^2 t\sin t + \sin t\,dt = \frac{1}{3}\cos^3 t - \cos t \Big|_0^\pi = \Big[\overbrace{\tfrac{1}{3}(\cos\pi)^3}^{-\frac{1}{3}} - \overbrace{\cos\pi}^{+1}\Big] - \Big[\overbrace{\tfrac{1}{3}(\cos(0))^3}^{\frac{1}{3}} - \overbrace{\cos(0)}^{-1}\Big]$

$= \frac{2}{3} + \frac{2}{3} = \boxed{4/3}$

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

THM (G'ST Area Formula) for Flux $\boxed{\text{Area } R = \frac{1}{2}\oint_C x\,dy - y\,dx}$

Example 143. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \le t \le \pi$. Find the area of R, using Green's Theorem applied to the vector field $\boxed{\mathbf{F} = \frac{1}{2}\langle x, y \rangle}$.

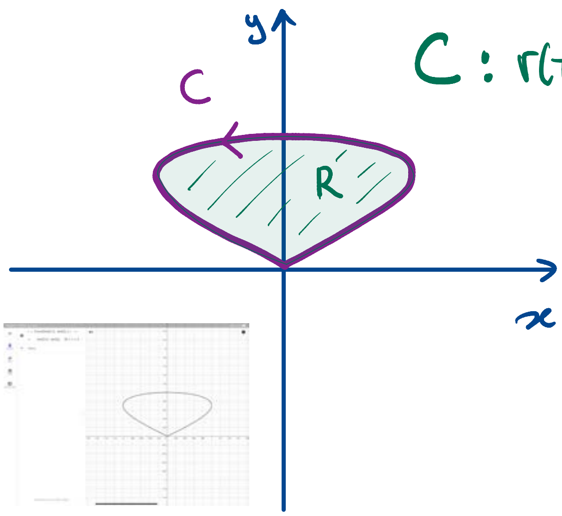
For Flux

or $F = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle$ for Flow

Idea: $\text{Area } R = \iint_R 1\, dA = \iint \underbrace{\tfrac{1}{2}}_{P_x} + \underbrace{\tfrac{1}{2}}_{Q_y}\, dA$

G'ST (Flux)

$$= \oint_C F \cdot n\, ds \quad \text{w/ } F = \langle \tfrac{1}{2}x, \tfrac{1}{2}y \rangle$$



$C: r(t) = \langle \sin 2t, \sin t \rangle$ $t \in [0, \pi]$

$r'(t) = \langle 2\cos 2t, \cos t \rangle$

$n \sim \langle \cos t, -2\cos 2t \rangle$

$$\text{Area } R = \oint_C \langle \tfrac{1}{2}x, \tfrac{1}{2}y \rangle \cdot n\, ds$$

$$= \int_0^\pi \langle \tfrac{1}{2}\sin 2t, \tfrac{1}{2}\sin t \rangle \cdot \langle \cos t, -2\cos 2t \rangle\, ds$$

$$= \int_0^\pi \tfrac{1}{2}\sin 2t \cos t - \sin t \cos 2t\, dt$$

unclear how to integrate?

$\cos 2t = 2\cos^2 t - 1$

$\sin 2t = 2\sin t \cos t$

$$= \int_0^\pi \tfrac{1}{2}(2\sin t\cos t)\cos t - \sin t\,(2\cos^2 t - 1)\, dt$$

$$= \int_0^\pi \cos^2 t \sin t - 2\cos^2 t \sin t + \sin t\, dt$$

$\boxed{\begin{array}{l} u\text{-sub} \\ u = \cos t \\ du = -\sin t\, dt \end{array}}$

same

$$= \int_0^\pi -\cos^2 t \sin t + \sin t\, dt = \tfrac{1}{3}\cos^3 t - \cos t \Big|_0^\pi = \Big[\underbrace{\tfrac{1}{3}(\cos\pi)^3}_{-\frac{1}{3}} \underbrace{- \cos\pi}_{+1}\Big] - \Big[\underbrace{\tfrac{1}{3}(\cos(0))^3}_{\frac{1}{3}} \underbrace{- \cos(0)}_{-1}\Big]$$

$$= \frac{2}{3} + \frac{2}{3} = \boxed{4/3}$$

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

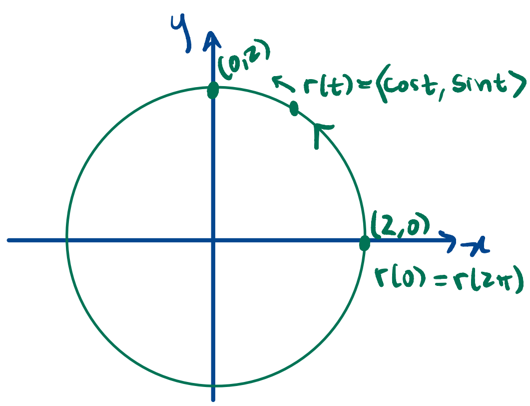
§16.5, 16.6 Surfaces & Surface Integrals

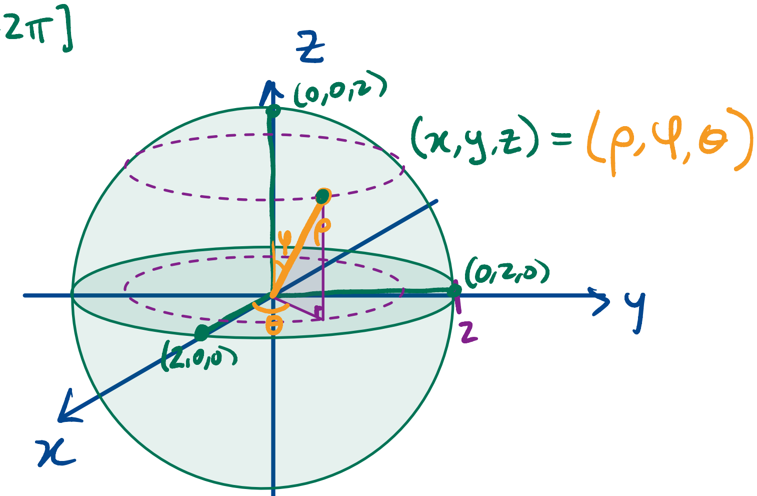
Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$ $y = \sqrt{4 - x^2}$	$z = f(x, y)$ $z = \sqrt{4 - x^2 - y^2}$
Implicit:	$F(x, y) = 0$ $x^2 + y^2 = 4$	$F(x, y, z) = 0$ $x^2 + y^2 + z^2 = 4$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$	$\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

$r(t) = \langle 2\cos t, 2\sin t \rangle$

$t \in [0, 2\pi]$





Sphere of radius $\rho = 2$

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

We've already done a few surface parametrizations.

e.g.

⊛ plane through the origin

$r(s,t) = s\vec{v}_1 + t\vec{v}_2$

⊛ spheres w/ fixed radius ρ using spherical coords.

So $\vec{r}(s,t) = \langle 2\sin(s)\cos(t), 2\sin(s)\cos(t), 2\cos(s) \rangle$

or can just call parameters φ, θ, so

$$\vec{r}(\varphi, \theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\cos\theta, 2\cos\varphi \rangle$$

GOAL: $r: \mathbb{R}^2 \to \mathbb{R}^3$ such that range of r is surface.

Example 144. Give parameteric representations for the surfaces below.

given x as a function of y & z so set $y=s$ & $z=t$ the parameters.

a) $x = y^2 + \frac{1}{2}z^2 - 2$

① $\boxed{\vec{r}(s,t) = \left\langle s^2 + \frac{1}{2}t^2 - 2, s, t \right\rangle \quad s \in \mathbb{R},\ t \in \mathbb{R}}$

Can also swap roles of s and t

② $\boxed{\vec{r}(s,t) = \left\langle t^2 + \frac{1}{2}s^2 - 2, t, s \right\rangle \quad s,t \in \mathbb{R}}$

Or can try something like

$y = r\cos\theta$, $z = \sqrt{2}\, r\sin\theta$ } Then $y^2 + \frac{1}{2}z^2 = r^2$ so $x = r^2 - 2$

③ $\boxed{\vec{r}(r,\theta) = \left\langle r^2 - 2, r\cos\theta, \sqrt{2}\, r\sin\theta \right\rangle \quad r \geq 0,\ \theta \in [0, 2\pi]}$

①, ②, ③ all are parametrizations of the surface for (a).

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

Same $\vec{r}$ w/ new ranges for s, t. (x,y,z) is behind yz-plane if $x \leq 0$

So need $x = y^2 + \frac{1}{2}z^2 - 2 \leq 0$

$\Rightarrow y^2 + \frac{1}{2}z^2 \leq 2$ (ellipse) $\Rightarrow \frac{y^2}{2} + \frac{z^2}{4} \leq 1$, so

for ① & ② $\boxed{y \in [-\sqrt{2}, \sqrt{2}] \text{ and } z \in \left[-\sqrt{4-2y^2}, \sqrt{4-2y^2}\right]}$

Or just need $r^2 - 2 \leq 0$ so $r \leq \sqrt{2}$

for ③ $\boxed{r \in [0, \sqrt{2}] \quad \theta \in [0, 2\pi]}$

c) $x^2 + y^2 + z^2 = 9$

Sphere of radius $\rho = 9$.

Spherical coords.

@ $\rho = 3$ $\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

$\boxed{\vec{r}(\varphi,\theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi \rangle \quad \varphi \in [0,\pi],\ \theta \in [0, 2\pi]}$

Cartesian coords

$x \in [-3, 3]$

$y \in \left[-\sqrt{9-x^2}, \sqrt{9-x^2}\right]$

$z = \sqrt{9 - x^2 - y^2}$ (top half only!)

$\boxed{\vec{r}(s,t) = \left\langle s, t, \sqrt{9 - s^2 - t^2} \right\rangle \quad s \in [-3,3],\ t \in \left[-\sqrt{9-s^2}, \sqrt{9-s^2}\right]}$

d) $x^2 + y^2 = 25$

cylinder w/ horizontal cross-sections of radius $r = 5$.

Cylindrical coords

@ $r = 5$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$\boxed{\vec{r}(\theta,t) = \langle 5\cos\theta, r\sin\theta, t \rangle \quad \theta \in [0, 2\pi],\ t \in \mathbb{R}}$

Can try cartesian?

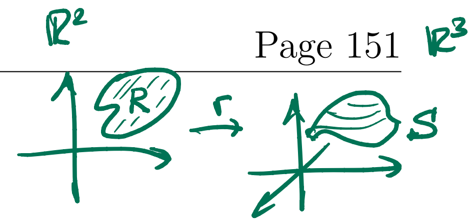
$x \in [-5, 5]$

$y = \sqrt{25 - x^2}$ (right half only)

$z = z$

$\boxed{r(s,t) = \left\langle s, \sqrt{25 - s^2}, t \right\rangle \quad s \in [-5, 5],\ t \in \mathbb{R}}$

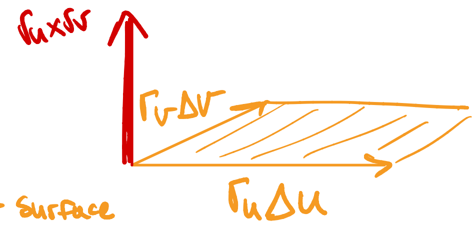
What can we do with this? Surface area



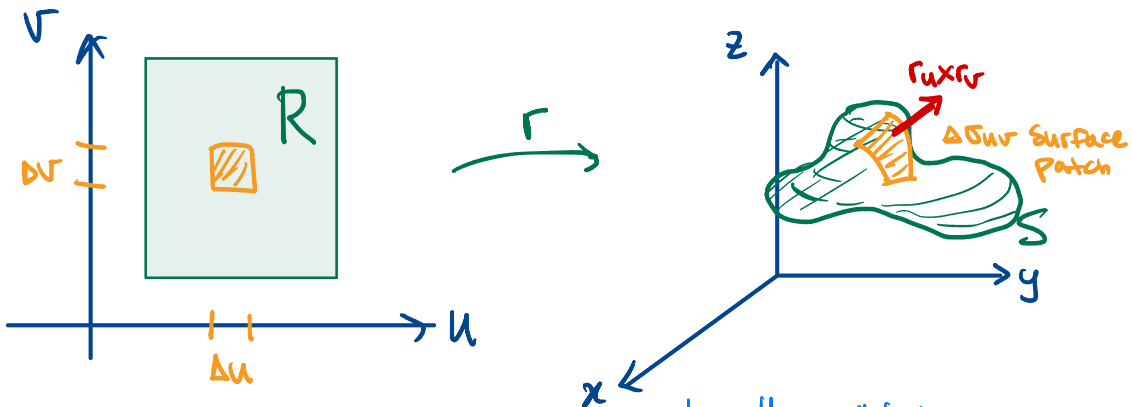
If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

S: $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
$u, v \in R$

- $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a rectangle of size $\|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v$ on the surface in $\mathbb{R}^3$.



Area parallelogram
$= \| r_u \Delta u \times r_v \Delta v \|$
$= \| r_u \times r_v \| \Delta u \Delta v$
$\approx \Delta \sigma_{uv}$



- Thus, Area(S) = $\iint_S 1 \, d\sigma = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$

$d\sigma = \|r_u \times r_v\| dx dy$

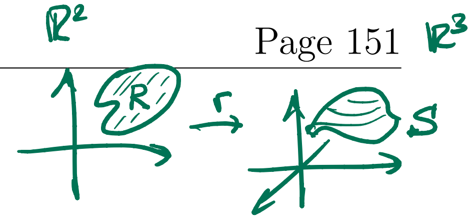
$dA = dx dy = r dr d\theta$ etc

Surface integral w/ surface measure $d\sigma$

double integral w/ Area measure

Example 145. *You try it!* Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

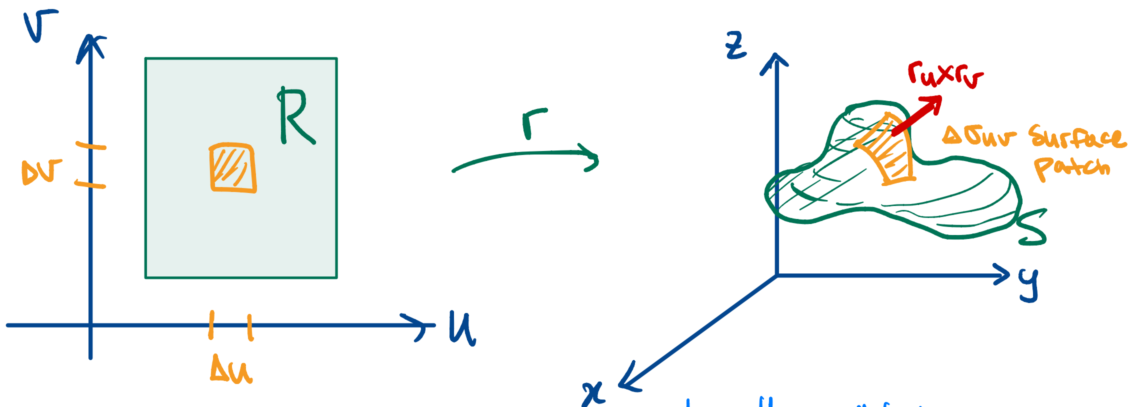
What can we do with this? Surface area

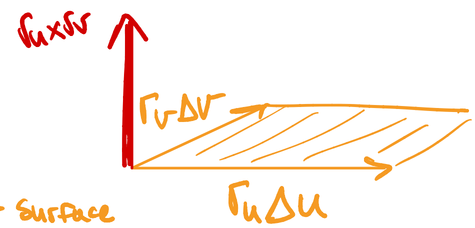


If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

$S: r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
$u, v \in R$

- $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a rectangle of size $\|r_u \times r_v\| \Delta u \Delta v$ on the surface in $\mathbb{R}^3$.





Area parallelogram
$= \| r_u \Delta u \times r_v \Delta v \|$
$= \| r_u \times r_v \| \Delta u \Delta v$
$\approx \Delta \sigma_{uv}$

- Thus, Area(S) = $\iint_S 1 \, d\sigma = \iint_R \|r_u \times r_v\| \, dA$

$d\sigma = \|r_u \times r_v\| dx dy$

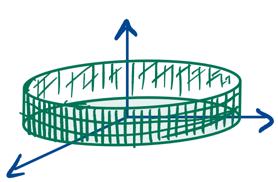
$dA = dx dy = r\, dr\, d\theta$ etc

Surface integral w/ surface measure $d\sigma$

double integral w/ Area measure

Example 145. *You try it!* Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

$\vec{r}(\theta, t) = \langle 5\cos\theta, 5\sin\theta, t \rangle, \quad \theta \in [0, 2\pi], \; z \in [0,1]$



$\vec{r}_\theta = \langle -5\sin\theta, 5\cos\theta, 0 \rangle$

$\vec{r}_t = \langle 0, 0, 1 \rangle$

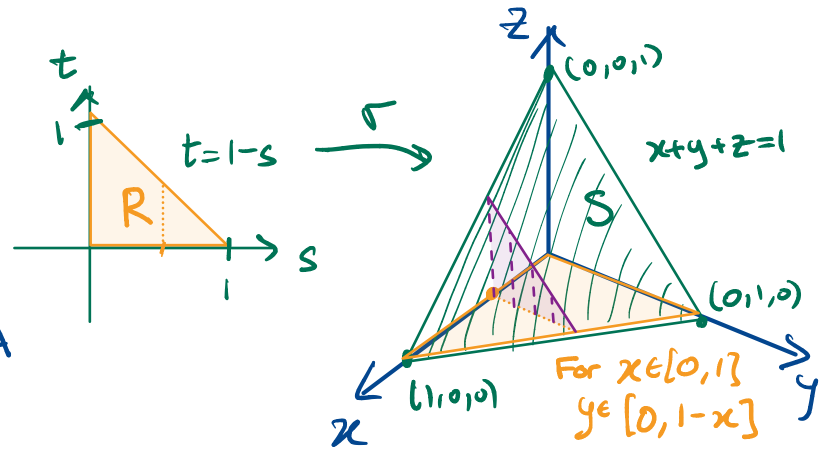
So $n = r_\theta \times r_t = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -5\sin\theta & 5\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 5\cos\theta, -(-5\sin\theta), 0 \rangle$

So $\|\vec{n}\|^2 = 25\cos^2\theta + 25\sin^2\theta + 0 = 25, \quad \|n\| = 5.$

$\text{Area } S = \iint_R 5 \, dA = \int_0^{2\pi}\int_0^1 5 \, dt \, d\theta = \int_0^{2\pi} 5t \Big|_0^1 d\theta = \int_0^{2\pi} 5 \, d\theta = 5 * 2\pi = \boxed{10\pi}$

Example 146. Suppose the density of a thin plate S in the shape of the portion of the plane $x+y+z=1$ in the first octant is $\delta(x,y,z)=6xy$. Find the mass of the plate.

$$M=\iint_S \delta(x,y,z)\,d\sigma$$



Step 1: parametrize S

Step 2: Compute $d\sigma=\|r_u\times r_v\|\,dA$

Step 3: Substitute

$x\in[0,1]$

$y\in[0,1-x]$

Then $z=1-x-y$

Step 1:

$$r(s,t)=\langle s,t,1-s-t\rangle$$

R: $s\in[0,1]$, $t\in[0,1-s]$

Step 2: $r_s=\langle 1,0,-1\rangle$

$r_t=\langle 0,1,-1\rangle$

$$r_s\times r_t=\begin{vmatrix} i & j & k\\ 1 & 0 & -1\\ 0 & 1 & -1\end{vmatrix}=\langle 1,-(-1),1\rangle=\langle 1,1,1\rangle$$

So $\|r_s\times r_t\|=\sqrt{3}$

Step 3:

$$\text{Mass}=\iint_S \delta\,d\sigma=\iint_R 6xy\sqrt{3}\,dA=\int_0^1\int_0^{1-s}6\sqrt{3}\,st\,dt\,ds$$

$$=\int_0^1 3\sqrt{3}st^2\Big|_0^{1-s}ds=\int_0^1 3\sqrt{3}s(1-s)^2\,ds=\int_0^1 3\sqrt{3}s(s^2-2s+1)\,ds$$

$$=\int_0^1 3\sqrt{3}(s^3-2s^2+s)\,ds=3\sqrt{3}\left(\tfrac{1}{4}s^4-\tfrac{2}{3}s^3+\tfrac{1}{2}s^2\right)\Big|_0^1=3\sqrt{3}\left[\left(\tfrac{1}{4}-\tfrac{2}{3}+\tfrac{1}{2}\right)-0\right]$$

$$=3\sqrt{3}\left(\tfrac{3}{4}-\tfrac{2}{3}\right)=3\sqrt{3}\left(\tfrac{9-8}{12}\right)=\tfrac{3\sqrt{3}}{12}=\boxed{\tfrac{\sqrt{3}}{4}}$$

§16.6, 16.7 Flux Surface Integrals, Stokes' Theorem

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of the outward pointing normal vector of S
(as chosen in the orientation of S)

If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

unit normal plug in parametrization into F

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \underbrace{(\mathbf{r}_u \times \mathbf{r}_v)}\, dA.$$

$\vec{n} = \frac{r_u \times r_v}{\|r_u \times r_v\|}$ so ↑ is $n * \|r_u \times r_v\|$.

Example 147. Find $\mathbf{r}_u \times \mathbf{r}_v$ and $\|\mathbf{r}_u \times \mathbf{r}_v\|$ when $z = f(x, y)$ so that S is the graph of a scalar function with domain in $\mathbb{R}^2$.

Example 148. Find $\mathbf{r}_u \times \mathbf{r}_v$ and $\|\mathbf{r}_u \times \mathbf{r}_v\|$ when S is a portion of a sphere of radius $\rho = a$, for some fixed constant a, using the standard spherical coordinates for your parametrization.

Soln. $r_\varphi = \langle a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi \rangle$

$r_\theta = \langle -a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0 \rangle$

So $r_\varphi \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$

$= \langle a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi\cos^2\theta + \overbrace{a^2\sin\varphi\cos\varphi\sin^2\theta}^{1} \rangle$

$= \langle a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi \rangle$ ✓ (a)

AND (b) $\|r_\varphi \times r_\theta\|^2 = a^4\sin^4\varphi\cos^2\theta \overbrace{+ a^4\sin^4\varphi\sin^2\theta}^{1} + a^4\sin^2\varphi\cos^2\varphi$

$= a^4\sin^4\varphi + a^4\sin^2\varphi\cos^2\varphi = a^4\sin^2\varphi(\sin^2\varphi + \cos^2\varphi) = a^4\sin^2\varphi$

So $\|r_\varphi \times r_\theta\| = a^2\sin\varphi$ (the spherical coord measure element)

Example 149. Find the flux of $\mathbf{F} = \langle x, -y, z\rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.

Want to compute $\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$

① parametrize S

② Compute partials & cross product r_u, r_v, $r_u \times r_v$

③ substitute & integrate.

① S: upper-half sphere w/ $\rho = 2$

so $r(\varphi, \theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle$

$\varphi \in [0, \pi/2]$, $\theta \in [0, 2\pi]$

② From previous page: (w/ $a = 2$)

$$r_\varphi \times r_\theta = \langle 4\sin^2\varphi\cos\theta, -4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\rangle$$

Flux =

③ $$\int_0^{2\pi}\int_0^{\pi/2} \langle 2\sin\varphi\cos\theta, -2\sin\varphi\sin\theta, 2\cos\varphi\rangle \cdot \langle 4\sin^2\varphi\cos\theta, -4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\rangle\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin^3\varphi\cos^2\theta + 8\sin^3\varphi\sin^2\theta + 8\sin\varphi\cos^2\varphi\; d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 8\sin^3\varphi + 8\sin\varphi\cos^2\varphi\; d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 8\sin\varphi(\overbrace{\sin^2\varphi + \cos^2\varphi}^{1})\, d\varphi\, d\theta = \int_0^{2\pi} -8\cos\varphi\,\Big|_0^{\pi/2}\, d\theta$$

$$= \int_0^{2\pi} -8\cos(\pi/2) - (-8\cos 0)\; d\theta = \int_0^{2\pi} 8\, d\theta = 8\theta\,\Big|_0^{2\pi}$$

$$= \boxed{16\pi}$$

Example 150. *You try it!* Compute $\iint_S G \cdot \mathbf{n}\, d\sigma$ the flux of G across the surface S.

$$G(x, y, z) = x^2, \quad S: \; x^2 + y^2 + z^2 = 1$$

Example 150. *You try it!* Compute $\iint_S G \cdot \mathbf{n}\, d\sigma$ the flux of G across the surface S.

$$G(x,y,z) = x^2, \quad S:\ x^2+y^2+z^2 = 1$$

S: unit sphere $\rho = 1$

$r(\varphi,\theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$

R: $\varphi \in [0,\pi]$, $\theta \in [0,2\pi]$

Then surface measure is the standard spherical element $d\sigma = \|\vec{r}_\varphi \times \vec{r}_\theta\| = \rho^2 \sin\varphi\, d\varphi\, d\theta$

So $M = \iint_S G\, d\sigma = \int_0^{2\pi}\int_0^{\pi} (\sin\varphi\cos\theta)^2 \sin\varphi\, d\varphi\, d\theta$

u-sub
$\boxed{u = \cos\varphi \quad du = -\sin\varphi}$

$\varphi = 0 \Rightarrow u =$
$\varphi = \pi \Rightarrow u$

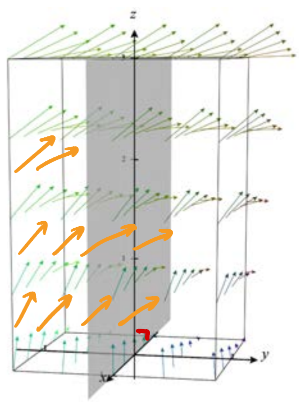
$= \int_0^{2\pi}\int_0^{\pi} (1-\cos^2\varphi)\cos^2\theta \sin\varphi\, d\varphi = \int_0^{2\pi}\int_1^{-1} -(1-u^2)\cos^2\theta\, du\, d\theta = \int_0^{2\pi}\int_{-1}^{1} (1-u^2)\cos^2\theta\, du\, d\theta$

$= \int_0^{2\pi} \cos^2\theta \left(u - \frac{1}{3}u^3 \Big|_{-1}^{1}\right) d\theta = \int_0^{2\pi} 2\cos^2\theta\left(1-\frac{1}{3}\right) d\theta = \frac{4}{3}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{2}{3}\left(\theta - \frac{1}{2}\cos 2\theta \Big|_0^{2\pi}\right)$

$= \frac{2}{3}\left[\left(2\pi - \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right)\right] = \boxed{\frac{4\pi}{3}}$

Example 151. *You try it!* Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

Example 152. *You try it!* Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ positive, negative, or zero?

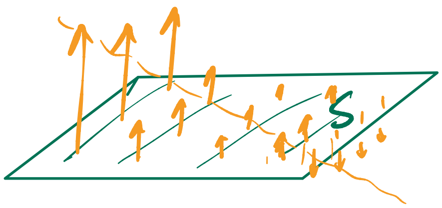


Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$

2. *curl:* $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

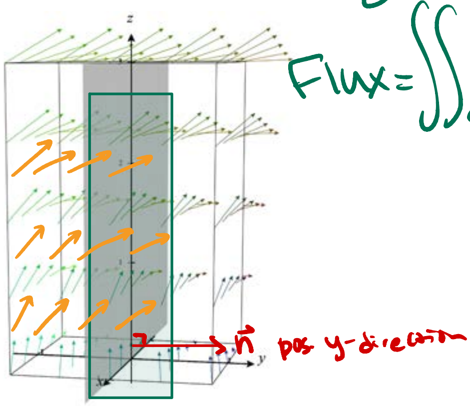
Example 151. *You try it!* Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.



False just need "more work done" in the direction of $\vec{n}$ as opposed to the opposite direction.

Example 152. *You try it!* Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ positive, negative, or zero?

guess



Flux $= \iint_S F\cdot n\, ds$ is positive

Since vectors are going in same direction as $\vec{n}$.

So $F\cdot\vec{n} \geq 0$

Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla\cdot\mathbf{F} = P_x + Q_y + R_z$

2. *curl:* $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

$\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

Theorem 154 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

- If S is a region R in the xy-plane, then we get:

- An **oriented surface** is one where ______________________________

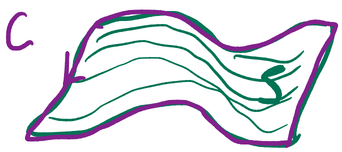
- S and C are oriented compatibly if:

Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

$$\nabla \cdot (\nabla \times F) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle R_y - Q_z, P_z - R_x, Q_x - P_y \right\rangle$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = \boxed{0}$$ by Fubini's Thm!

Theorem 154 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

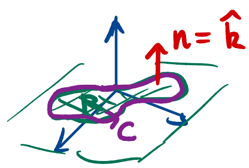


$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

Flux of Curl(F) across surface S

Circulation (Flow) around closed loop C which is the boundary of S

- If S is a region R in the xy-plane, then we get:



$$\iint_R \text{Curl}\, F \cdot \hat{k} \, dA = \oint_C F \cdot T \, ds$$ Green's Theorem!

- An **oriented surface** is one where normal vector stays consistent as you move along surface

* mobius strip is <u>NOT</u> oriented.

- S and C are oriented compatibly if:

  ↑ surface ↑ boundary

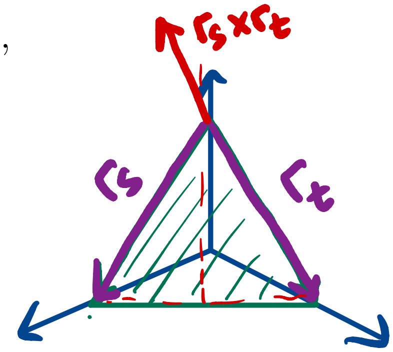
  Walking along C keeps S to your LEFT

  (ie walking "counter clockwise" relative to S)

Example 155. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by calculating the flux across the interior of C.

$$\mathbf{F} = \langle y, xz, x^2 \rangle$$

C : boundary of $x + y + z + 1$ in first octant, oriented counter-clockwise from above.



C: boundary of $x+y+z=1$ in first octant $(x\geq 0, y\geq 0, z\geq 0)$

S: $\vec{r}(s,t) = \langle s, t, 1-s-t \rangle$, R: $s \in [0,1]$, $t \in [0, 1-s]$

$r_s = \langle 1, 0, -1 \rangle \quad r_t = \langle 0, 1, -1 \rangle \quad r_s \times r_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$

$$\begin{array}{c} \quad P \quad\; Q \quad\; R \\ F = \langle y, xz, x^2 \rangle \end{array}$$

outward pointing ✓

$\nabla \times F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle 0 - x, 0 - 2x, z - 1 \rangle = \langle -x, -2x, z-1 \rangle$

$\text{Flow} = \oint_C F \cdot dr \overset{\text{SST}}{=} \iint_S \nabla \times F \cdot n \, d\sigma$

$= \int_0^1 \int_0^{1-s} \langle -s, -2s, (1-s-t)-1 \rangle \cdot \langle 1, 1, 1 \rangle \, dt \, ds$

$= \int_0^1 \int_0^{1-s} -s - 2s - s - t \, dt \, ds = -\int_0^1 \int_0^{1-s} +4s + t \, dt \, ds$

$= -\int_0^1 4st + \frac{1}{2}t^2 \Big|_0^{1-s} ds = -\int_0^1 4s(1-s) + \frac{1}{2}(1-s)^2 \, ds$

$= -\int_0^1 4s - 4s^2 + \frac{1}{2}s^2 - s + \frac{1}{2} \, ds = \int_0^1 \frac{7}{2}s^2 - 3s - \frac{1}{2} \, ds = \frac{7}{6}s^3 - \frac{3}{2}s^2 - \frac{1}{2}s \Big|_0^1$

$= \frac{7}{6} - \frac{3}{2} - \frac{1}{2} = \frac{7}{6} - \frac{12}{6} = \boxed{\frac{-5}{6}}$

Example 156. *You try it!* Use Stokes' Theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ the flux of $\mathbf{F}$ across S by calculating the circulation line integral around the boundary curve C of S.

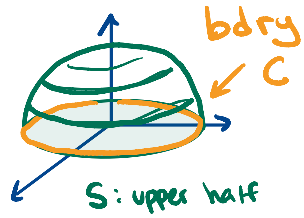
$$\begin{aligned} \mathbf{F} &= \langle 2z, 3x, 5y\rangle \\ S :&\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, (4-r^2)\rangle \\ R :&r \in [0,2],\ \theta\in[0,2\pi] \end{aligned}$$

Example 156. *You try it!* Use Stokes' Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ the flux of $\mathbf{F}$ across S by calculating the circulation line integral around the boundary curve C of S.

$$\mathbf{F} = \langle 2z, 3x, 5y \rangle$$
$$S : \mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, (4 - r^2) \rangle$$
$$R : r \in [0, 2],\ \theta \in [0, 2\pi]$$

S: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 4 - r^2 \rangle$ R: $r \in [0,2]$, $\theta \in [0, 2\pi]$

C: $\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$, $\theta \in [0, 2\pi]$



$\vec{r}\,'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

$F = \langle 2z, 3x, 5y \rangle$

bdry circle of radius 2 in xy-plane

So Flux thru $S = \iint_S \nabla \times F \cdot n\, d\sigma = \oint_C F \cdot dr$

$$= \int_0^{2\pi} \langle 0, 6\cos\theta, 10\sin\theta \rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0 \rangle\, d\theta$$

$$= \int_0^{2\pi} 12\cos^2\theta\, d\theta = \int_0^{2\pi} 6(1 + \cos 2\theta)\, d\theta$$

$$= 6\theta + 3\sin 2\theta \Big|_0^{2\pi} = \left(6(2\pi) + 3\sin 4\pi\right) - \left(6(0) + 3\sin 0\right)$$

$$= \boxed{12\pi}$$