MATH 2551 GT-E w/ Dr. Sal Barone

- Dr. Barone, Prof. Sal, or just Sal, as you prefer

Office Hours

Daily Announcements & Reminders:

* Gradescope safety-Quiz 5/14
* WeBWorK
* Quiz 0 (Practice Quiz) not for a grade
* PA §12.1 (First peer assessment) CPs
* PLEASE read the syllabus!
* Piazza
* HWHW CPs

Goals for Today: Sections 12.1, 12.4, 12.5

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
- Academic honesty
- Learn new things.

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus	Integral Calculus
limits/continuity	Integrals
derivatives/partial derivatives	Riemann sums
tangent lines	Improper integrals
Optimization	Integration techniques
unit circle	Area under the curve
graphing/max/min/inflection	Series
Fund. thm. of Calc.	Taylor series/Approx
	Convergence of series.

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

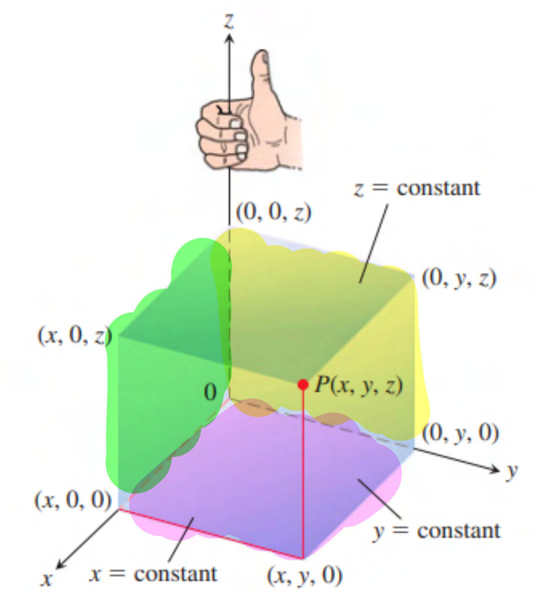
Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

e.g. $f(x,y,z) = (2x-3y,\ 4z,\ 21x-4y)$ ← linear! use matrices!

or

$f(x,y) = (x^2-1,\ \cos x e^y,\ \ln|x+y|)$ ← yea, ... no.

§12.1: Three-Dimensional Coordinate Systems

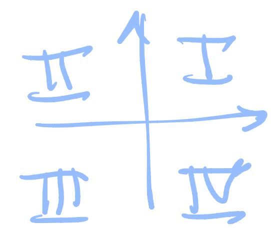


* $(x, y, z) \in \mathbb{R}^3$

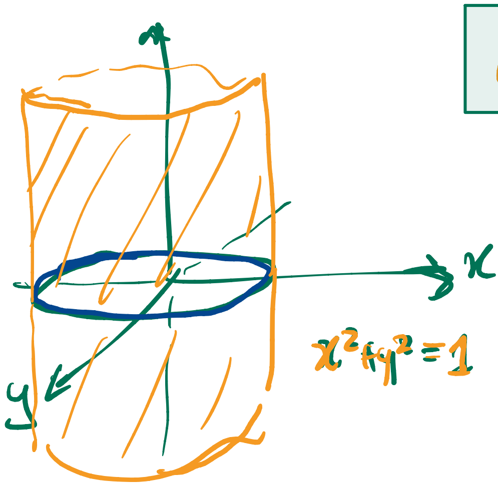
* coordinate planes
  - $x=0$ "back wall"
  - $z=0$ "Floor"
  - $y=0$ "side wall"

* Octants (vs. quadrants)

cf. $\mathbb{R}^2$



Question: What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x^2+y^2 = 1$?



cylinder!

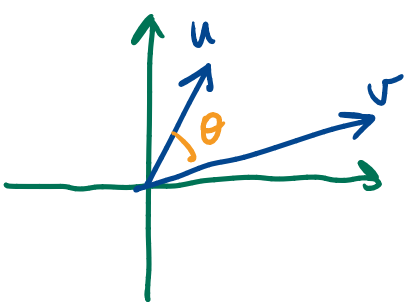
$x^2+y^2=1$? & $z=0$

§12.3, 12.4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n = \sum_{i=1}^{n} u_iv_i$$

This product tells us about the angle between the vectors u, v.



Then $u \cdot v = |u||v| \cos\theta$

and $u \cdot u = u_1^2 + u_2^2 + \ldots + u_n^2$

So $\sqrt{u \cdot u} = \|u\|$

In particular, two vectors are **orthogonal** if and only if their dot product is 0.

Example 2. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

$\langle 1, 1, 4 \rangle \cdot \langle -3, -1, 1 \rangle = -3 - 1 + 4 = 0$

$\boxed{\text{Yes}}$

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

we want the method to play nice with vector addition & scalar mult. That is:

1. $u \times (v+w) = u \times v + u \times w$

2. $cu \times v = c(u \times v) = u \times cv$

Definition 3. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where $\hat{i}, \hat{j}, \hat{k}$ are just the standard basis vectors in $\mathbb{R}^3$

$\hat{i} = \langle 1, 0, 0 \rangle$

$\hat{j} = \langle 0, 1, 0 \rangle$

$k = \langle 0, 0, 1 \rangle$

Example 4. Find $\langle 1,2,0\rangle \times \langle 3,-1,0\rangle$.

$$\langle 1,2,0\rangle \times \langle 3,-1,0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= \hat{k}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= (-1-6)\hat{k}$$

$$= \boxed{-7\hat{k}} \quad \text{or} \quad \boxed{\langle 0,0,-7\rangle.}$$

Sanity Check:

Note that $w = u \times v$ is really perp. to both u AND v, as desired.

A Geometric Interpretation of $\mathbf{u} \times \mathbf{v}$

The cross product $\mathbf{u} \times \mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n}$$

where $\mathbf{n}$ is a unit vector which is normal to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

Since $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Example 5. Find the area of the parallelogram determined by the points P, Q, and R.

$$P(1,1,1),\ Q(2,1,3),\ R(3,-1,1)$$

$$\vec{PQ} = \langle 2-1, 1-1, 3-1 \rangle = \langle 1, 0, 2 \rangle$$

$$\vec{PR} = \langle 3-1, -1-1, 1-1 \rangle = \langle 2, -2, 0 \rangle$$

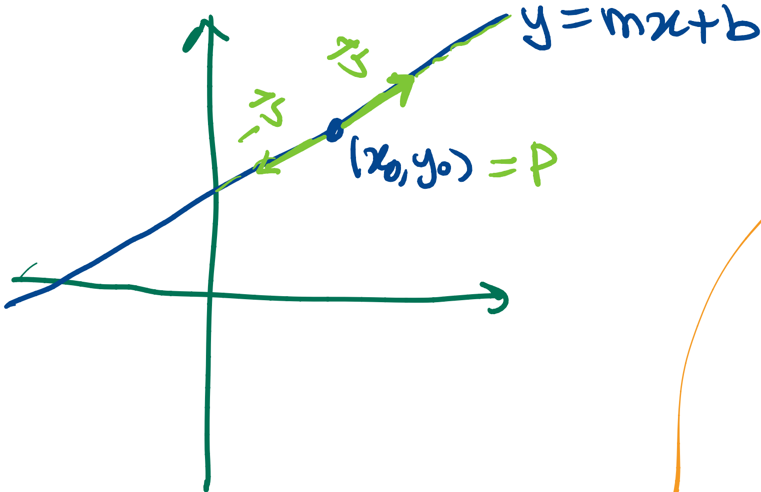
So $$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}$$

$$= 4\hat{i} + 4\hat{j} - 2k \text{ or } \langle 4, 4, -2 \rangle.$$

So $$|\vec{PQ} \times \vec{PR}| = \sqrt{4^2 + 4^2 + (-2)^2} = \sqrt{36} = \boxed{6}$$

§12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:



The line is all points

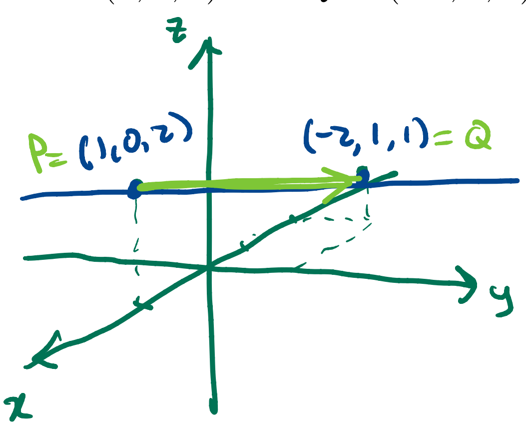
$$\boxed{\ell(t) = \overrightarrow{OP} + t * \vec{v}}, \quad t \in \mathbb{R}$$

P is any point on the line

$\vec{v}$ is a vector in the same direction as the line.

vector equation

Example 6. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.



$$\overrightarrow{PQ} = \langle -3, 1, -1 \rangle$$

So

$$\boxed{\ell(t) = \langle 1, 0, 2 \rangle + t \langle -3, 1, -1 \rangle \text{ for } t \in \mathbb{R}}$$

or

$$\boxed{\ell(t) = \langle 1-3t, t, 2-t \rangle, \quad t \in \mathbb{R}}$$

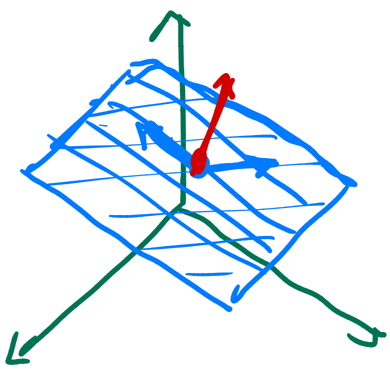
vector eqns.

parametric eqns

$$\begin{cases} x = 1 - 3t \\ y = t \\ z = 2 - t \end{cases}, \quad t \in \mathbb{R}$$

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction **n**, called the *normal vector*.



Need:

* point & two l.i. vectors in plane
* 3 points
* point and one normal vector

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$ax + by + cz = d \quad \text{(two free vars)}$$

Solns are (x,y,z) that satisfy

$$\langle x,y,z\rangle \cdot \langle a,b,c\rangle = d \quad (1)$$

So given a fixed point (x_0,y_0,z_0) in the plane then

$$\langle x_0,y_0,z_0\rangle \cdot \langle a,b,c\rangle = d \quad (2)$$

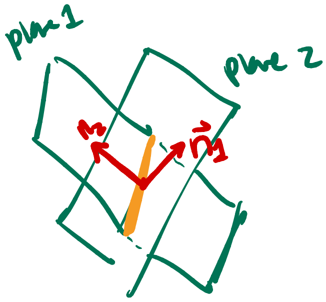
So combining (1) & (2) we get

$$(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle)\cdot\langle a,b,c\rangle = 0$$

or $\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$

plane passing through (x_0,y_0,z_0) w/ normal vec. $\vec{n}=\langle a,b,c\rangle$.

Example 7. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect ① and ② find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.



plane 1 $\quad 0x + y - z = -2 \quad \vec{n}_1 = \langle 0, 1, -1 \rangle$

plane 2 $\quad x - y + 0z = 0 \quad \vec{n}_2 = \langle 1, -1, 0 \rangle$

So $\vec{v}$ vector in line is parallel to $\vec{n}_1 \times \vec{n}_2$ (key idea)

① Planes meet since $\vec{n}_1 \neq c\vec{n}_2$ (planes not parallel)

② $\vec{n}_1 \times \vec{n}_2 = \langle 0, 1, -1 \rangle \times \langle 1, -1, 0 \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= -\hat{i} - \hat{j} - k = \langle -1, -1, -1 \rangle$$

Sanity Check ✓

So line passing through $(-8, 0, 2)$ and parallel to $\vec{v}$ is

$$\boxed{\ell(t) = \langle -8, 0, 2 \rangle + t\langle -1, -1, -1 \rangle}$$

§12.6 Quadric Surfaces

Definition 8. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

① $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ Sphere of radius r centered at (a,b,c)

② $x^2+y^2=1$ cylinder of radius 1 extending along z-axis (from lecture on §12.1)

The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 9. Use cross-sections to sketch and identify the quadric surface $x = z^2+y^2$.

$\boxed{\text{Horizontal cross sections}}$

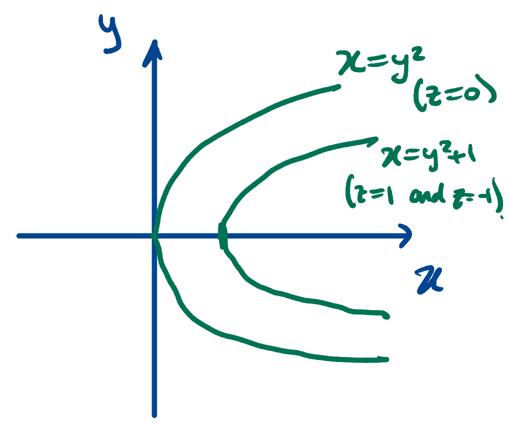
Idea: "cut" the surface by choosing a fixed value $x=\text{const}$, $y=\text{const}$ or $z=\text{const}$.

e.g. set $z=\text{const}$

$z=0 \Rightarrow x=0+y^2$
$\Rightarrow x=y^2$ parabola

$z=1 \Rightarrow x=1+y^2$ parabola

$z=-1 \Rightarrow x=(-1)^2+y^2$
$\Rightarrow x=1+y^2$ same parabola



In general when $z=k \Rightarrow x=k^2+y^2$ parabola opening right w/ x-intercept k^2.

What about $y=\text{const}$ or $z=\text{const}$?

Example 9. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

(cont.)

Cross sections parallel to back wall / yz-plane

Set $x = \text{const}$

$x=0 \Rightarrow 0 = z^2 + y^2$ single soln $(z,y) = (0,0)$

$x=1 \Rightarrow 1 = z^2+y^2$ circle

$x=-1 \Rightarrow -1 = z^2+y^2$ no soln

$x=2 \Rightarrow 2 = z^2+y^2$ circle

$x=4 \Rightarrow 4 = z^2+y^2$ circle

Prev. page

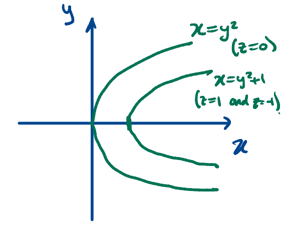
e.g. set $z = \text{const}$

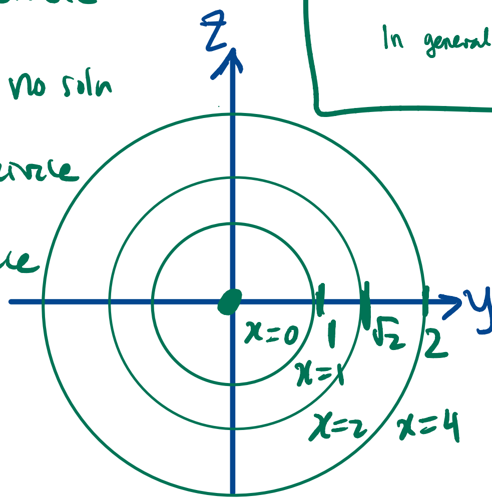
$z=0 \Rightarrow x = 0+y^2$
$\Rightarrow x = y^2$ parabola

$z=1 \Rightarrow x = 1+y^2$ parabola

$z=-1 \Rightarrow x = (-1)^2+y^2$
$\Rightarrow x = 1+y^2$ same parabola

In general when $z=k \Rightarrow x = k^2+y^2$ parabola opening right w/ x-intercept k^2.





So cross sections of surface in cuts parallel to yz-plane are circles.

If $x=k$ then $k = z^2+y^2$ circle of radius $\sqrt{k}$ centred at $(0,0)$.

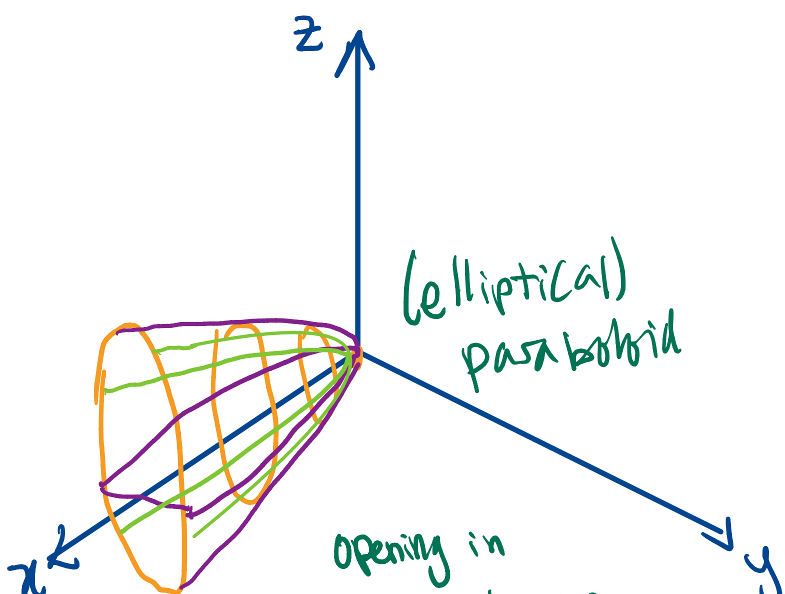
Last is $y = \text{const}$

Cross sections parallel to xz-plane

$y=0 \Rightarrow x = z^2+0$ parabola

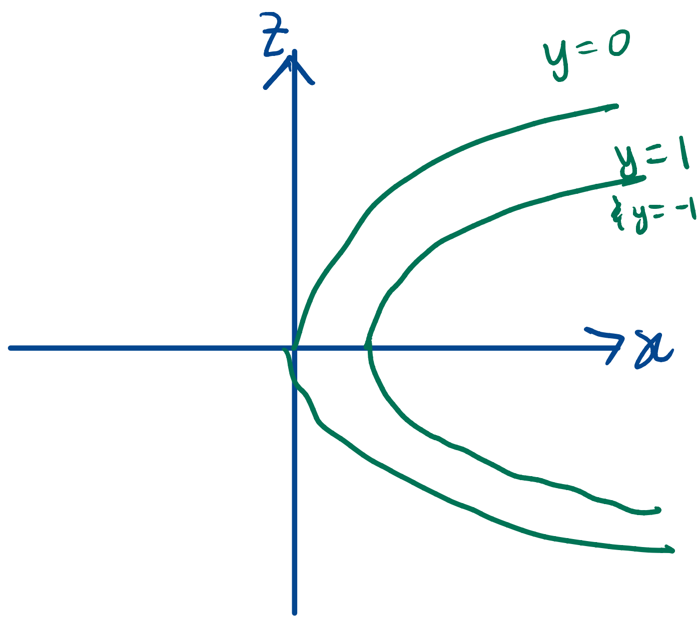
$y=1 \Rightarrow x = z^2+1$ parabola

$y=-1 \Rightarrow x = z^2+(-1)^2$ same parabola



(elliptical) paraboloid

Opening in x-direction



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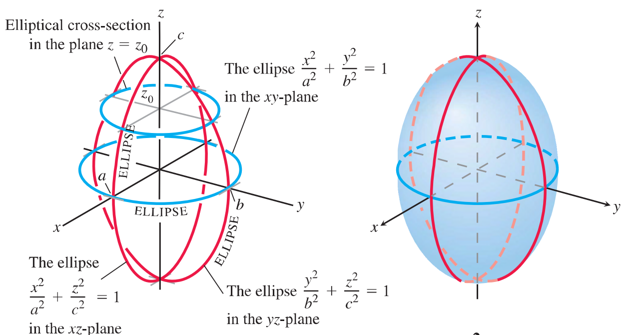
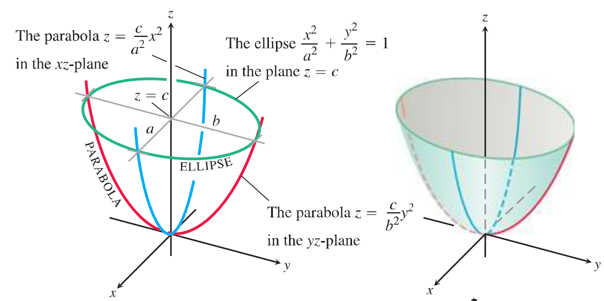
★ Things to note

TABLE 12.1 Graphs of Quadric Surfaces



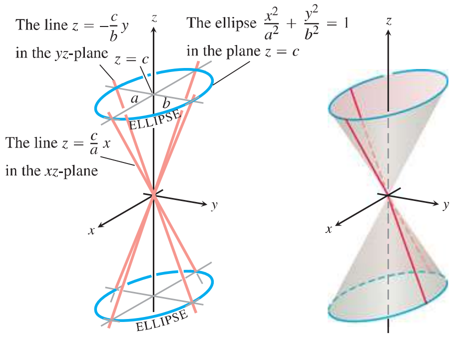
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Coeff of x^2, y^2, z^2 all have the same sign



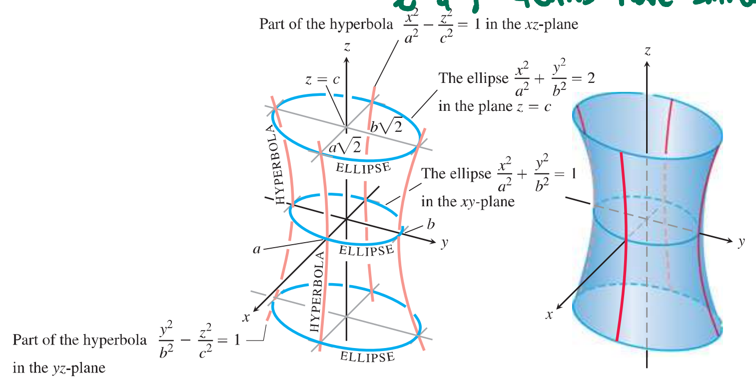
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

z but no z^2 term
x^2 & y^2 terms have same sign



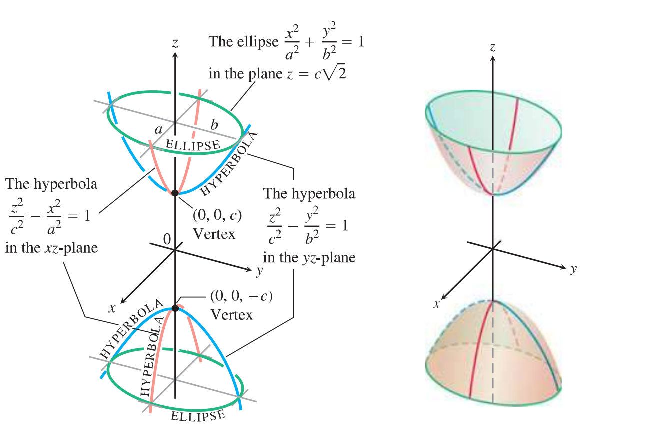
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

no const term.
x^2 & y^2 different sign than z^2 term



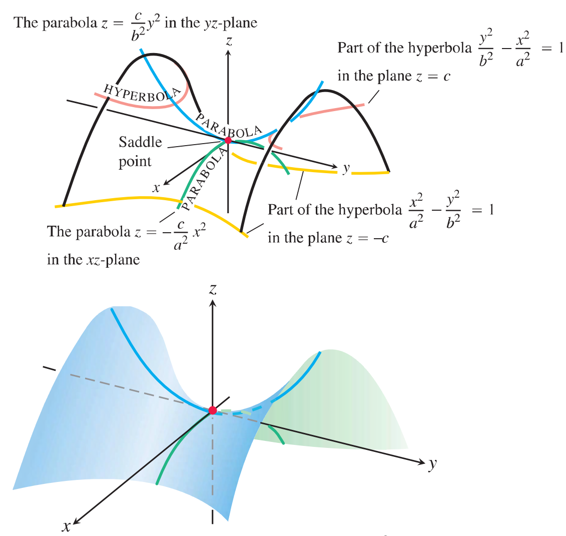
HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Const term & x^2 & y^2 pos coeff, z^2 neg.



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Const term, x^2 & y^2 neg. coeff
z^2 pos coeff.



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

z but no z^2, x^2 & y^2 opp. sign

§13.1 Curves in Space & Their Tangents

The description we gave of a line last week generalizes to produce other one-dimensional graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well. We said that a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ produces a straight line when graphed.

Fixed $\vec{v}$ & $\vec{r}_0$, t is a real variable

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

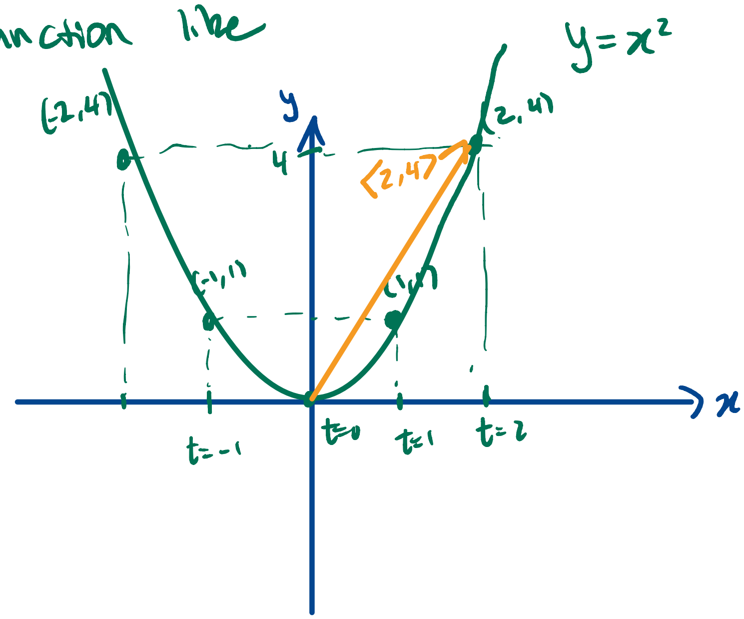
You have seen several examples already:

* lines $\vec{r}(t) = \vec{OP} + t\vec{v}$, $t \in \mathbb{R}$

* Circles $\vec{r}(t) = \langle \cos\theta, \sin\theta \rangle$ $0 \le \theta \le 2\pi$

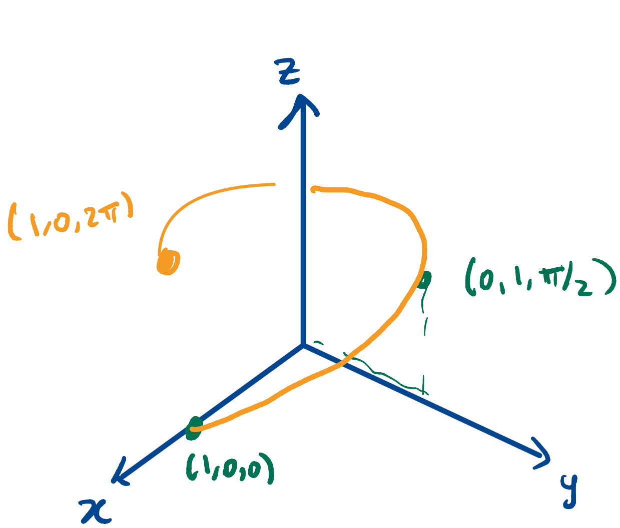
but can be literally any function like

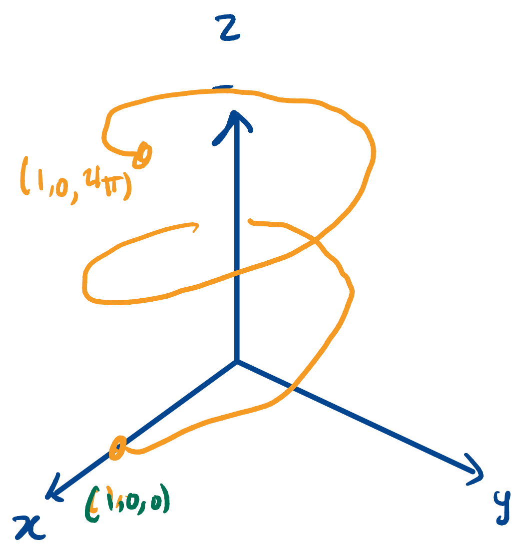
$\vec{r}(t) = \langle t, t^2 \rangle$



Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called <u>parametrizing</u> the curve C, and $\mathbf{r}$ is called a <u>parametrization</u> of C.

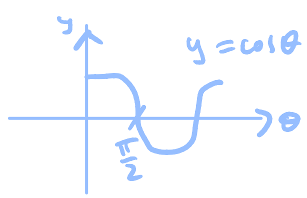
Example 10. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

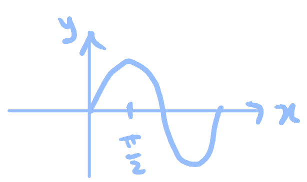




$t=0$ $\quad \vec{r}_1(0) = \langle \cos(0), \sin(0), 0 \rangle = \langle 1, 0, 0 \rangle$

$\vec{r}_1(\frac{\pi}{2}) = \langle \cos(\pi/2), \sin(\pi/2), \pi/2 \rangle = \langle 0, 1, \pi/2 \rangle$





punchline: different graph/function even when same rule but DIFFERENT DOMAIN

TECHNOLOGY TIME

Check your intuition

§13.2: Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 11. Compute $\lim_{t \to e} \langle t^2, 2, \ln(t) \rangle = L$

$$L = \left\langle \lim_{t \to e} t^2 , \lim_{t \to e} 2 , \lim_{t \to e} \ln(t) \right\rangle$$

$$= \langle e^2, 2, \ln e \rangle = \boxed{\langle e^2, 2, 1 \rangle}$$

And with continuity:

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Example 12. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \frac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

Since $f(t) = t$, $g(t) = \frac{-1}{t^2-4}$, and $h(t) = \sin t$ all are continuous on their domains, the function $\vec{r}(t)$ is continuous on

$$D_f \cap D_g \cap D_h = \mathbb{R} \cap \Big[\underbrace{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}_{\text{So this.}} \Big] \cap \mathbb{R}$$

Idea: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

And with derivatives:

Example 13. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$, find $\mathbf{r}'(t)$.

$$\boxed{\vec{r}\,'(t) = \langle 2 - t, 1 \rangle}$$

Q: what is $\vec{r}\,'(0)$? $r'(2)$?

$$\boxed{r'(0) = \langle 2, 1 \rangle \qquad r'(1) = \langle 0, 1 \rangle}$$

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives Velocity vector at time t
- $|\mathbf{r}'(t)|$ gives Speed (scalar) at time t
- $\mathbf{r}''(t)$ gives accelleration vector at time t

Let's see this graphically

Example 14. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$ at time $t = 2$.

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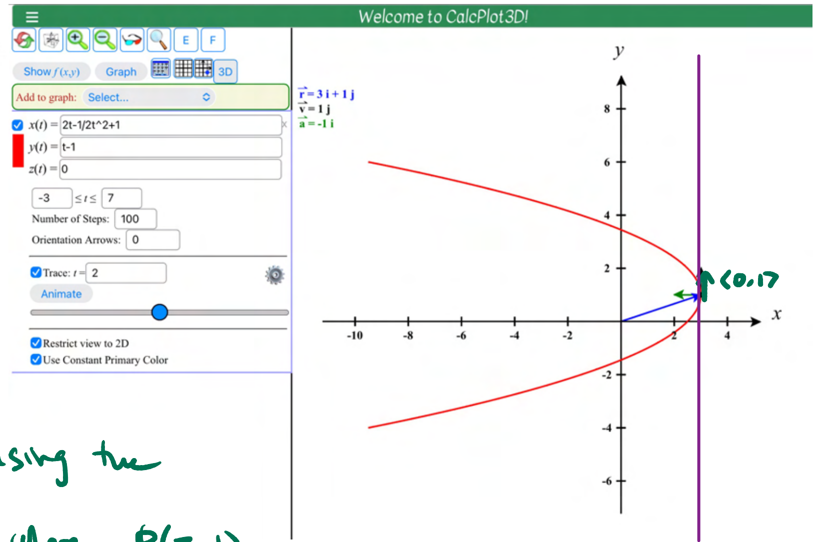

$\vec{r}'(t) = \langle 2 - t, 1\rangle$

$\vec{r}'(2) = \langle 0, 1\rangle$ this vector is parallel to the tangent line.

note that $\vec{r}(2) = \langle 4 - \frac{1}{2}2^2 + 1, 2 - 1\rangle = \langle 3, 1\rangle$

So $P(3,1)$ is on the curve, so using the formula $\ell(t) = \vec{OP} + t\vec{v}$ where $P(3,1)$ and $\vec{v} = \vec{r}'(2)$

We get $\boxed{\ell(t) = \langle 3,1\rangle + t\langle 0,1\rangle,\ t \in \mathbb{R}}$

is a parametrization of the line tangent to the curve at $t = 2$.

Same idea: $\int_a^b \langle f(t), g(t), h(t)\rangle\, dt$

$= \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$

And with integrals:

Example 15. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle\ dt$.

So $\int_0^1 \langle t, e^{2t}, \sec^2 t\rangle\, dt = \langle \frac{1}{2}t^2, \frac{1}{2}e^{2t}, \tan t\rangle \Big|_0^1$

$= \langle \frac{1}{2}, \frac{1}{2}e, \tan(1)\rangle - \langle 0, \frac{1}{2}, 0\rangle$

$= \boxed{\langle \frac{1}{2}, \frac{1}{2}(e-1), \tan(1)\rangle}$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 16. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle\ m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, use calculus to reconstruct his flight path.

$\int \vec{v}(t)\, dt = \vec{s}(t)$ (integral of velocity is position)

So $\vec{s}(t) = \int \langle -200\sin 2t, 200\cos t, 400 - \frac{400}{1+t}\rangle\, dt$

$= \langle \frac{200}{2}\cos 2t + C_1, 200\sin t + C_2, 400t - 400\ln(1+t) + C_3\rangle$

@ $t=0$

$\vec{s}(0) = \langle 100 + C_1, C_2, C_3\rangle = \langle 0,0,0\rangle$ so $C_1 = -99$, $C_2 = 1$, $C_3 = 1$

and $\boxed{\vec{s}(t) = \langle 100\cos 2t - 100, 200\sin t, 400t - 400\ln(1+t)\rangle}$