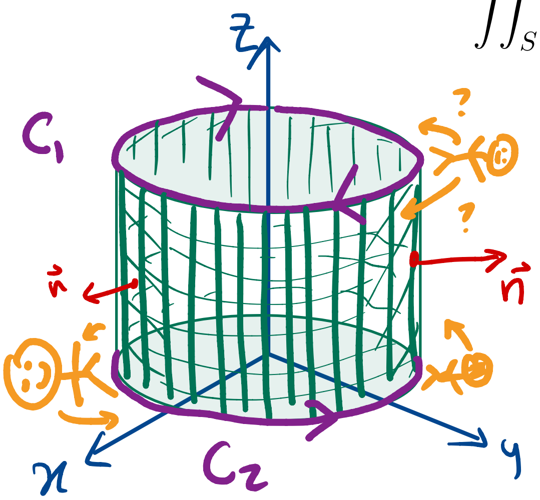
§16.7 Stokes' Theorem

Theorem 152 (Stokes' Theorem). *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$



Recall: the boundary is compatibly oriented if walking along the boundary with your head "up" in the direction of $\vec{n}$ the normal vector of S, then your LEFT HAND is pointing over S.

§16.6, 16.7 Page 160

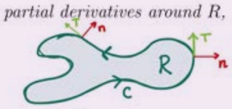
Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

$$\nabla \cdot (\nabla \times F) = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$
$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = \boxed{0} \text{ by Fubini's Thm!}$$

From last time:

If $F = \nabla \times G$ (recall $\nabla \times G$ is a vector field if G is a vector field) then $\operatorname{div} F = 0$. (easy)

Theorem 140 (Green's Theorem). *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal $\mathbf{n}$. If $\mathbf{F} = \langle P, Q \rangle$ has continuous partial derivatives around R, then*



a) Circulation form:

(a) $$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \underset{\text{notational equality}}{=} \int_C P\, dx + Q\, dy \underset{\text{G's T}}{=} \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \underset{\text{expand out}}{=} \iint_R Q_x - P_y\, dA$$

b) Flux form:

$(\nabla \times F) \cdot \hat{k}$ or $\nabla \cdot F$

(b) $$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\, dy - Q\, dx = \iint_R (\nabla \cdot \mathbf{F})\, dA = \iint_R P_x + Q_y\, dA$$

In fact, if the domain of F is simply connected (just like in Green's Theorem) then

$$\operatorname{div} F = 0 \implies F = \nabla \times G \text{ for some } G.$$

Summary:

If $\text{curl } F = \vec{0}$ then F is $\boxed{\underline{\text{conservative}}}$

and $F = \nabla f$. (and FToLI holds etc.)

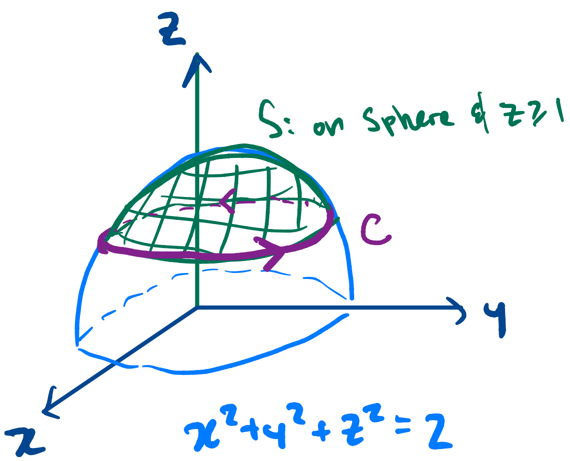
If $\text{div } F = 0$ and <u>domain of F</u> is simply connected, then F is the

$\boxed{\text{curl of some other vector field } G.}$

@ z=1 get 0

Example 153 (DD). Let $\mathbf{F} = \langle -y, x + (z-1)x^{x\sin(x)}, x^2+y^2\rangle$. Find $\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, d\sigma$ over the surface S which is the part of the sphere $x^2+y^2+z^2 = 2$ above $z = 1$, oriented away from the origin.

(Option 1)



Flux of curl (across S) $= \iint_S (\nabla \times F)\cdot n\, d\sigma$

(yuck!)
- * parametrize S w/ $\vec{r}(u,v)$
- * compute $\vec{r}_u \times \vec{r}_v \sim \vec{n}$
- * compute $\nabla \times F$
- * then integrate a surface integral after substituting.

} complicated technical long & often HARD

Flow of F (across C) $= \oint_C F\cdot T\, ds$

(this one!)
- * parametrize C w/ $\vec{r}(t)$
- * compute $r'(t)$
- * integrate a line integral after sub.

$x^2+y^2+1=2$
$\Rightarrow r = 1.$

$C: \vec{r}(t) = \langle \cos t, \sin t, 1\rangle, \quad t \in [0, 2\pi]$

$\vec{r}'(t) = \langle -\sin t, \cos t, 0\rangle$

and $\text{Flow} = \int_0^{2\pi} \langle -\sin t, \cos t + 0, \overbrace{\cos^2 t + \sin^2 t}^{1}\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$

$= \int_0^{2\pi} \sin^2 t + \cos^2 t + 0\, dt = \int_0^{2\pi} 1\, dt = t\Big|_0^{2\pi}$

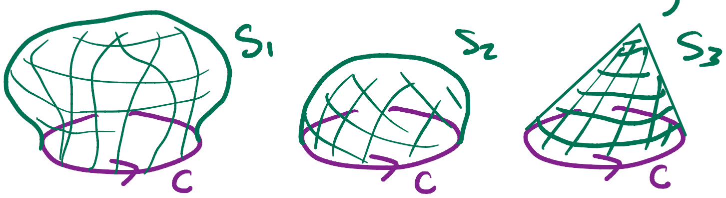
$= 2\pi - 0 = \boxed{2\pi}$

Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C?

For any F, $\oint_C F\cdot T\,ds = \iint_{S_1} (\nabla\times F)\cdot n\,d\sigma$

$= \iint_{S_2} (\nabla\times F)\cdot n\,d\sigma$

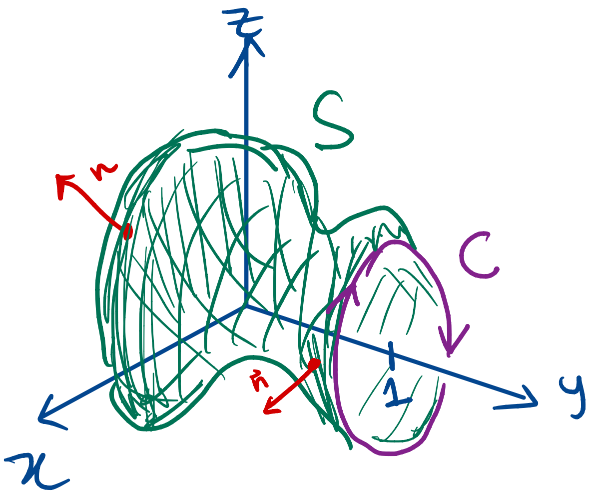
etc. !? alert!!



1 @ y=1 0

Example 154. Suppose curl $\mathbf{F} = \langle y^{y^y}\sin(z^2), (y-1)e^{x^{x^x}}+2, -ze^{x^{x^x}}\rangle$. Compute the net flux of the curl of $\mathbf{F}$ over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y-axis in the plane $y = 1$.

lit. does not matter what this is!



ShT Flux of curl over S = Flow around C

$$\iint_S \text{Curl} F\cdot n\,d\sigma = \oint_C F\cdot T\,ds$$

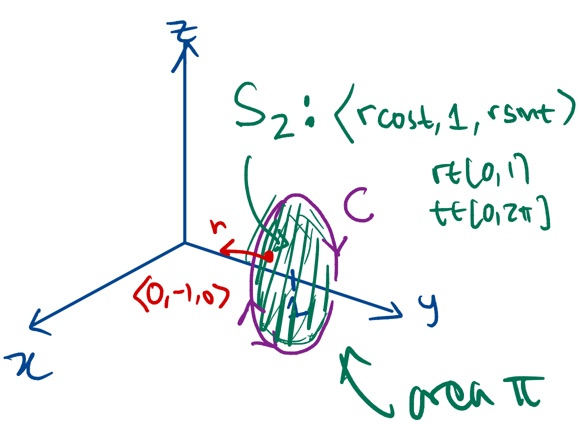
$$= \int_0^{2\pi}\langle \sin(\cos^2 t), 2, -\cos t\, e^{\sin t^{\sin t^{\sin t}}}\rangle\cdot\langle\dots\rangle dt$$

?? abort!!

C: $r(t) = \langle \cos t, 1, \sin t\rangle$, $t\in[0,2\pi]$

$r'(t) = \langle -\sin t, 0, \cos t\rangle$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$



$$\iint_{S_2} \text{Curl} F\cdot n\,d\sigma = \iint_S \text{Curl} F\cdot\langle 0,-1,0\rangle\,d\sigma$$

@y=1

$$= \iint_S 0 + (0+2)(-1) + 0\,d\sigma$$

@r(t)

$$= -2\iint_{S_2} 1\,d\sigma$$

$= -2 *$ surface area of S_2

$= \boxed{-2\pi}$

§16.8 Divergence Theorem

Theorem 155 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Example 156. Let $\mathbf{F} = \langle y^{1234} e^{\sin(yz)}, y - x^{z^x}, z^2 - z \rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.

Example 156. Let $\mathbf{F} = \langle y^{1234} e^{\sin(yz)}, y - x^{z^x}, z^2 - z \rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.

Closed surface means: $\boxed{\text{encloses a closed and bounded 3D region}}$

§16.8 Divergence Theorem

WARNING! S must be closed!!

Theorem 155 (Divergence Theorem)**.** *Let S be $\boxed{\text{a closed surface}}$ oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma \overset{DT}{=} \iiint_D \nabla\cdot\mathbf{F}\, dV. = \iiint_D \operatorname{div}\mathbf{F}\, dV$$

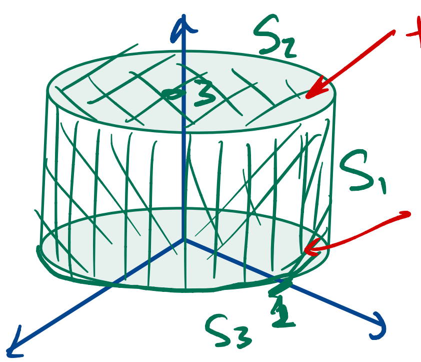
Example 156. Let $\mathbf{F} = \langle \underbrace{y^{1234}e^{\sin(yz)}}_{P}, \underbrace{y - x^{z^x}}_{Q}, \underbrace{z^2 - z}_{R}\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.

$S = S_1 \cup S_2 \cup S_3$

*3 parametrizations
*3 cross products finding
*3 surface integrals

$\iint_{S_i} \mathbf{F}\cdot\mathbf{n}\, d\sigma$ no way!

STOP THE MADNESS!!



top & bottom must be included!!

by DT

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint_D \operatorname{div}\mathbf{F}\, dV = \iiint_D 0 + \cancel{1} + (2z - \cancel{1})\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^3 2z * r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 rz^2\Big|_0^3 dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 9r\, dr\, d\theta = \int_0^{2\pi} \frac{9}{2}r^2\Big|_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{9}{2}\, d\theta = \frac{9}{2}\theta\Big|_0^{2\pi} = \boxed{9\pi}$$

Cylindrical coords

D: $\theta \in [0, 2\pi]$
$r \in [0, 1]$
$z \in [0, 3]$

Closed, closed, or closed?

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

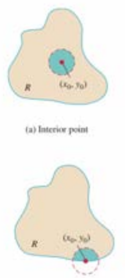
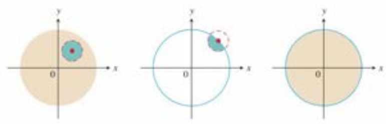


FIGURE 14.2 Interior points and boundary points of a plane region R. An interior point is necessarily a point of R. A boundary point of R need not belong to R.



$\{(x, y) \mid x^2 + y^2 < 1\}$ Open unit disk. Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$ Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \le 1\}$ Closed unit disk. Contains all boundary points.

FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

As with a half-open interval of real numbers $[a, b)$, some regions in the plane are neither open nor closed. If you start with the open disk in Figure 14.3 and add to it some, but not all, of its boundary points, the resulting set is neither open nor closed. The boundary points that *are* there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

① a region in $\mathbb{R}^2$ is closed if it contains all its boundary points

e.g. absolute MAX/MIN on closed & bounded regions in $\mathbb{R}^2$

EXAMPLE 7 Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$ (Figure 16.19).

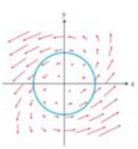


FIGURE 16.19 The vector field $\mathbf{F}$ and curve $\mathbf{r}(t)$ in Example 7.

Solution On the circle, $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$$

gives

$$\begin{aligned} \text{Circulation} &= \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} (1 - \sin t \cos t)\, dt \\ &= \left[t - \frac{\sin^2 t}{2}\right]_0^{2\pi} = 2\pi. \end{aligned}$$

As Figure 16.19 suggests, a fluid with this velocity field is circulating *counterclockwise* around the circle, so the circulation is positive. ■

Flux Across a Simple Closed Plane Curve

A curve in the xy-plane is **simple** if it does not cross itself (Figure 16.20). When a curve starts and ends at the same point, it is a **closed curve** or **loop**. To find the rate at which a fluid is entering or leaving a region enclosed by a smooth simple closed curve C in the xy-plane, we calculate the line integral over C of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. We use only the normal component of $\mathbf{F}$, while ignoring the tangential component, because the normal component leads to the flow across C. The value of this integral is the *flux* of $\mathbf{F}$ across C. *Flux* is Latin for *flow*, but many flux calculations involve no motion at all. If $\mathbf{F}$ were an electric field or a magnetic field, for instance, the integral of $\mathbf{F} \cdot \mathbf{n}$ is still called the flux of the field across C.

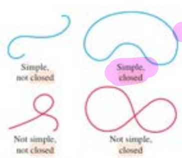


FIGURE 16.20 Distinguishing curves that are simple or closed. Closed curves are also called loops.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. \tag{6}$$

② a loop is closed if it has same starting & ending point.

e.g. line integrals for Flux/Flow around closed loops like circles.

and

$$\begin{aligned} \iint_S G(x, y, z)\, d\sigma &= \iint_R \left(\sqrt{x}\sqrt{1 + y^2}\right)\sqrt{1 + y^2}\, dx\, dy \\ &= \int_0^1 \int_0^{1-x} \sqrt{x}(1 + y^2)\, dy\, dx \\ &= \int_0^1 \sqrt{x}\left[(1 - x) + \frac{1}{3}(1 - x)^3\right] dx && \text{Integrate and evaluate.} \\ &= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2}\right) dx && \text{Routine algebra} \\ &= \left[\frac{8}{9}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{27}x^{9/2}\right]_0^1 \\ &= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945} \approx 0.30. \end{aligned}$$ ■

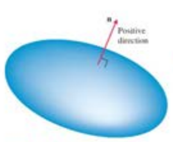


FIGURE 16.49 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

Orientation of a Surface

The curve C in a line integral inherits a natural orientation from its parametrization $\mathbf{r}(t)$ because the parameter belongs to an interval $a \le t \le b$ directed by the real line. The unit tangent vector $\mathbf{T}$ along C points in this forward direction. For a surface S, the parametrization $\mathbf{r}(u, v)$ gives a vector $\mathbf{r}_u \times \mathbf{r}_v$ that is normal to the surface, but if S has two "sides," then at each point the negative $-(\mathbf{r}_u \times \mathbf{r}_v)$ is also normal to the surface, so we need to choose which direction to use. For example, if you look at the sphere in Figure 16.38, at any point on the sphere there is a normal vector pointing inward toward the center of the sphere and another opposite normal pointing outward. When we specify which of these normals we are going to use consistently across the entire surface, the surface is given an *orientation*. A smooth surface S is **orientable** (or **two-sided**) if it is possible to define a field of unit normal vectors $\mathbf{n}$ on S which varies continuously with position. Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we usually choose $\mathbf{n}$ on a closed surface to point outward.

Once $\mathbf{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector $\mathbf{n}$ at any point is called the **positive direction** at that point (Figure 16.49).

The Möbius band in Figure 16.50 is not orientable. No matter where you start to construct a continuous unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out. The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exists.

FIGURE 16.50 To make a Möbius band, take a rectangular strip of paper $abcd$, give the end bc a single twist, and paste the ends of the strip together to match a with c and b with d. The Möbius band is a nonorientable or one-sided surface.

Surface Integrals of Vector Fields

In Section 16.2 we defined the line integral of a vector field along a path C as $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent vector to the path pointing in the forward oriented direction. By analogy we now have the following corresponding definition for surface integrals.

③ closed surface in space $\mathbb{R}^3$ is a smooth surface that encloses a closed and bounded 3D region.

lol

④ closed region in $\mathbb{R}^3$ is same defn as ① (but in $\mathbb{R}^3$)

Math 2551 Worksheet: Review for Exam 3

1. Set up an iterated integral in spherical coordinates for $\iiint_E z^2\, dV$ where E is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=25$ and inside $z=-\sqrt{\frac{1}{3}(x^2+y^2)}$.

2. Set up an integral that computes the volume of the solid which is bounded above by the cylinder $z=4-x^2$, on the sides by the cylinder $x^2+y^2=4$, and below by the xy-plane using

   (a) Cartesian coordinates

   (b) cylindrical coordinates

   Which integral would you rather evaluate and why?

3. Find an integral that computes the mass of the wire which lies along the curve $y^2=x^3$ from $(0,0)$ to $(1,-1)$ and has density function $\rho(x,y)=2xy^2$.

4. Show that the field $\mathbf{F}=2x\mathbf{i}-y^2\mathbf{j}-\frac{4}{1+z^2}\mathbf{k}$ is conservative, find a potential function, and use it to compute the integral

$$\int_C 2x\ dx - y^2\ dy - \frac{4}{1+z^2}\ dz$$

   where C is any path from $(0,0,0)$ to $(3,3,1)$.

5. Compute $\int_C (6y+x)\ dx + (y+2x)\ dy$ using any method, where C is the circle $(x-2)^2+(y-3)^2=4$.

6. Find the flux of the field $\mathbf{F}=y\mathbf{i}-x\mathbf{j}+\mathbf{k}$ through the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant in the direction away from the origin.

7. Use Stokes' theorem to show that the circulation of the field $\mathbf{F}=\langle 2x,2y,2z\rangle$ around the boundary curve C of **any** smooth orientable surface S in $\mathbb{R}^3$ is 0.

8. Find the outward flux of $\mathbf{F}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/\sqrt{x^2+y^2+z^2}$ through the boundary S of the "thick sphere" D given by the points satisfying $1\le x^2+y^2+z^2\le 4$.