§13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

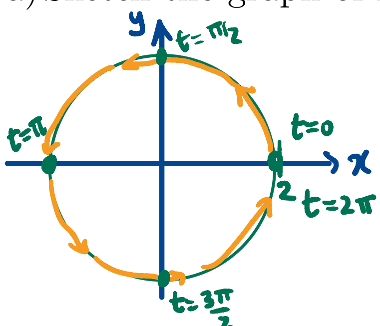
Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

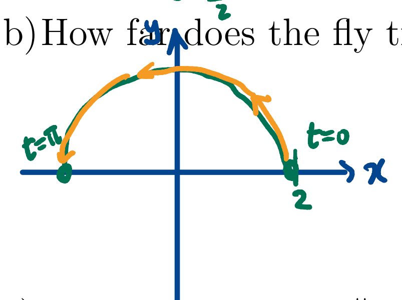
$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle,$$

where $0 \le t \le 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?



b) How far does the fly travel between $t = 0$ and $t = \pi$?



half circle so

$$L = \tfrac{1}{2} 2\pi r = \tfrac{1}{2} 2\pi(2) = \boxed{2\pi}$$

c) What is the speed $\|\mathbf{v}(t)\|$ of the fly at time t?

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2\sin t, -2\cos t\rangle$$

$$s(t) = \|\vec{v}(t)\| = \sqrt{(2\sin t)^2 + (-2\cos t)^2} = \sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= \underbrace{\sqrt{4}}_{2} * \underbrace{\sqrt{\sin^2 t + \cos^2 t}}_{1} = \boxed{2}$$

d) Compute the integral $\displaystyle\int_0^{\pi} \|\mathbf{v}(t)\|\, dt$. What do you notice?

$$\int_0^{\pi} \|\vec{v}(t)\|\, dt = \int_0^{\pi} 2\, dt = 2t \Big|_0^{\pi}$$

$$= 2\pi - 0 = \boxed{2\pi}$$

Same!

Definition 17. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $t=a$ to $t=b$ that is traced out exactly once is

$$L = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Example 18. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

$x(t)$ $y(t)$ $z(t)$

Step 0: need to find a & b the initial t-values

When $t=0$, $\vec{r}(0) = \langle 0,0,0\rangle$ doesn't work

Set $t=1$, $\vec{r}(1) = \langle 1, 1^2, 1^3\rangle = \langle 1,1,1\rangle$ ✓

Set $t=2$, $\vec{r}(2) = \langle 2, 2^2, 2^3\rangle = \langle 2,4,8\rangle$ ✓

So

$$L = \int_1^2 \sqrt{1^2 + (2t)^2 + (3t^2)^2}\, dt$$

$x'(t) = 1$
$y'(t) = 2t$
$z'(t) = 3t^2$

$$= \boxed{\int_1^2 \sqrt{1 + 4t^2 + 9t^4}\, dt}$$

Example 19. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = \langle 6\sin(2t), 6\cos(2t), 5t\rangle,\ 0 \le t \le 2\pi$.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle 12\cos 2t, -12\sin 2t, 5\rangle$$

$$\Rightarrow \quad |\vec{v}(t)|^2 = 144(\cos^2 2t + \sin^2 2t) + 25 = 169$$

$$\Rightarrow \quad |\vec{v}(t)| = 13$$

So,

$$L = \int_0^{2\pi} |\vec{v}(t)|\, dt = \int_0^{2\pi} 13\, dt = 13t\Big|_0^{2\pi} = \boxed{26\pi}$$

Check your intuition

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k}$, $0 \le t \le 8$.

$$L = \int_a^b \|\vec{v}(t)\| \; dt$$

$x(t) = t$ $\quad a = 0$
$y(t) = 0$ $\quad b = 8$
$z(t) = \frac{2}{3}t^{3/2}$

$$\vec{v}(t) = \vec{r}\,'(t) = 1 * \hat{\imath} + \overset{1}{\cancel{\tfrac{2}{3} * \tfrac{3}{2}}}\, t^{1/2}\, \hat{k}$$

So $\|\vec{v}(t)\| = \sqrt{1^2 + (t^{1/2})^2} = \sqrt{1+t}$

and

$$L = \int_0^8 \sqrt{1+t} \; dt$$

u-sub Box
$\boxed{u = 1+t \quad du = dt}$

$t = 0 \Rightarrow u = 1$
$t = 8 \Rightarrow u = 9$

$$= \int_1^9 \sqrt{u}\; du = \frac{2}{3} u^{3/2} \Big|_1^9$$

$$= \frac{2}{3} 9^{3/2} - \frac{2}{3} 1^{3/2}$$

$$= \frac{2}{3} 27 - \frac{2}{3} = \frac{2}{3} * 26 = \boxed{52/3}$$

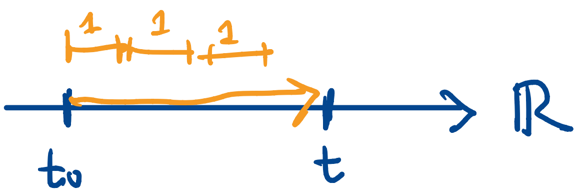
Check your intuition

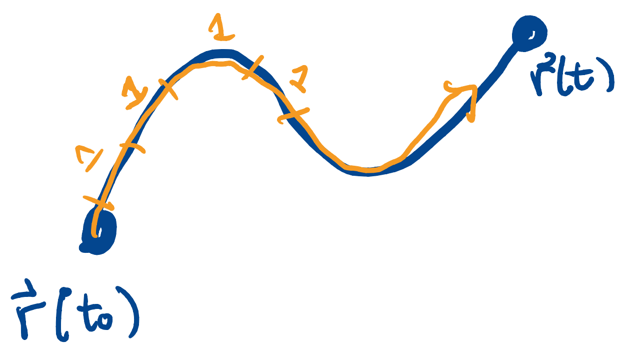
Arc length parametrization

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \int_{t_0}^{t} \|\vec{v}(\tau)\| \, d\tau$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.





Idea: why should we be measuring out units in the parameter space $\mathbb{R}$?

More natural to measure units in the codomain of $\vec{r}(t)$, $\vec{r}: \mathbb{R} \to \mathbb{R}^n$.

Key point: $\frac{d}{ds}\vec{r}(s) = r'(s)$ has $\|r'(s)\| = 1$

Example 21. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle \underbrace{4\cos(t), 4\sin(t)}_{\text{not an arc-length parameterization b/c } \|r'(t)\| \neq 1.} \rangle, 0 \le t \le 2\pi$.

META

① Compute arc length function $s(t) = \int_{t_0}^{t} \|\vec{v}(t)\| \, dt$

② Solve $s = s(t)$ for $t = f(s)$

③ Substitute back into $\vec{r}(t)$ to obtain $\boxed{\vec{r}_1(s)}$. ← arc-length parameterization

$\vec{r}_2(s) = \vec{r}(f(s))$

① Find $s(t)$:

$\vec{v}(t) = \vec{r}\,'(t) = \langle -4\sin t, 4\cos t \rangle$,

so $\|\vec{v}(t)\|^2 = 16\sin^2 t + 16\cos^2 t = 16$

and $\|\vec{v}(t)\| = 4$.

Note: This integral is sometimes messy/hard in practice

$$s(t) = \int_{t_0}^{t} \|v(\tau)\| \, d\tau = \int_0^t 4 \, d\tau = 4\tau \Big|_0^t = 4t$$

② Solve for t:

$s = s(t) = 4t$ so $t = \frac{s}{4}$

← Note: Solving s in terms of t is sometimes messy/hard in practice

③ Sub into $\vec{r}(t)$:

$$\boxed{\vec{r}_2(s) = \vec{r}(t) = \vec{r}(s/4) = \langle 4\cos(s/4), 4\sin(s/4) \rangle, \quad 0 \le s \le 8\pi}$$

$t = 2\pi$

$\Rightarrow s = 4t = 8\pi$

Example 22. *You try it!* (a) Find an arc length parameterization of the portion of the helix $\mathcal{C}$ of radius 4 in $\mathbb{R}^3$ parametrized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle, 0 \le t \le \pi/2$.
(b) use $s(t)$ to find L the length of $\mathcal{C}$.

① Find $s(t)$:

$$\vec{v} = \frac{d\vec{r}}{dt} = \langle -4\sin t, 4\cos t, 3\rangle$$

$|\vec{v}|^2 = 16+9=25$
so $|\vec{v}|=5$

So,

$$s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t 5\, d\tau = 5\tau \Big|_0^t$$

and $s = 5t$ ///

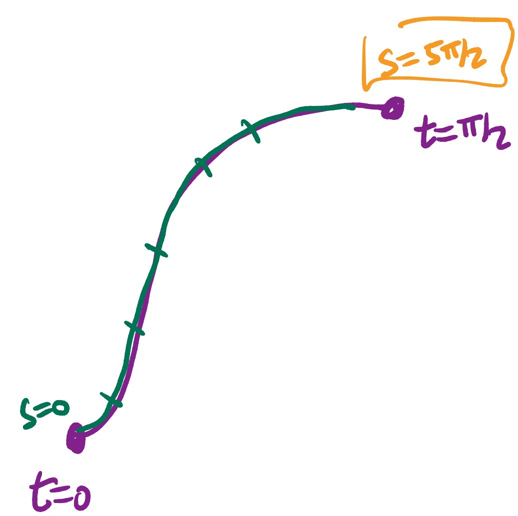
② Solve for t:

$$s = s(t) = 5t \quad \text{so} \quad t = f(s) = s/5$$

Sub into $\vec{r}$: $\vec{r}_2(s) = r(f(s))$

③
$$\boxed{\begin{gathered}\vec{r}_2(s) = \langle -4\sin(s/5), 4\cos(s/5), 3\rangle, \\ 0 \le s \le 5\pi/2\end{gathered}}$$

$\begin{cases} t=0 \\ t=\pi/2\end{cases} \Rightarrow \begin{matrix} s=0 \\ s=5\pi/2\end{matrix}$



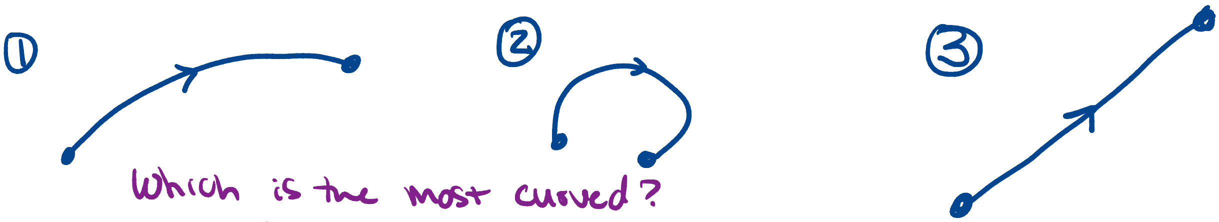
is the length of the curve!!

(b)
$$L = \int_0^{\pi/2} |\vec{v}|\, d\tau = s\Big|_{t=\pi/2} = \boxed{\frac{5}{2}\pi}$$

9:10 (5 min) 9:15

§13.3 & 13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.



Which is the most curved?

2 > 1 > 3

Idea: scale 2nd derivative appropriately

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: $\vec{r}'(s)$
- In terms of any parameter t: $r'(t) / \|r'(t)\|$

NOTE: Since $\|\vec{r}'(s)\| = 1$ these are actually the same formula!

This lets us define the **curvature**, $\kappa(s) = \|T'(s)\|$

9:15 (15 min) 9:30

Example 23. In Example 21 we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

κ

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

$\vec{T}(s) = \vec{r}\,'(s)$ so $\vec{T}(s) = \left\langle 4 * \frac{-1}{4}\sin\left(\frac{s}{4}\right), 4 * \frac{1}{4}\cos\left(\frac{s}{4}\right)\right\rangle$

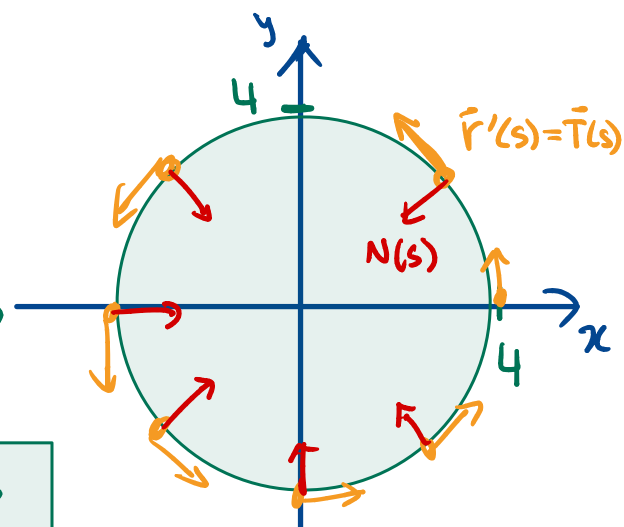
$$\boxed{\vec{T}(s) = \left\langle -\sin\left(\frac{s}{4}\right), \cos\left(\frac{s}{4}\right)\right\rangle}$$

So $\vec{T}\,'(s) = \left\langle \frac{-1}{4}\cos\left(\frac{s}{4}\right), -\frac{1}{4}\sin\left(\frac{s}{4}\right)\right\rangle$

$$\|\vec{T}\,'(s)\|^2 = \frac{1}{16}\cos^2(s/4) + \frac{1}{16}\sin^2\left(\frac{s}{4}\right) = \frac{1}{16}$$

and $\kappa(s) = \|\vec{T}\,'(s)\| = \sqrt{\frac{1}{16}} = \boxed{\frac{1}{4}}$

Sanity check!!



$$\vec{N}(s) = \frac{T'(s)}{\|T'(s)\|} = \left\langle \frac{-1/4\cos(s/4)}{1/4}, \frac{-1/4\sin(s/4)}{1/4} \right\rangle$$

$$\Rightarrow \boxed{\vec{N}(s) = \langle \cos(s/4), \sin(s/4) \rangle}$$

Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ $T'(s)/\|T'(s)\|$

unit length normal to tangent line

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) = \vec{r}'(t) / \|\vec{r}'(t)\|$
- $\mathbf{N}(t) = \vec{T}'(t) / \|\vec{T}'(t)\|$
- $\kappa(t) = \|T'(t)\| / \|\vec{r}'(t)\|$ $\boxed{\text{or}}$ $\dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$, $t \in \mathbb{R}$.

$\frac{d}{ds}\vec{r}(t) = \langle -2\sin t, 2\cos t, 1 \rangle$

$\|r'(t)\|^2 = \overbrace{4\sin^2 t + 4\cos^2 t}^{4} + 1 = 5$ and $\|r'(t)\| = \sqrt{5}$

so $\boxed{\vec{T}(t) = \langle -\frac{2}{\sqrt{5}}\sin t, \frac{2}{\sqrt{5}}\cos t, \frac{1}{\sqrt{5}} \rangle}$

$\vec{T}'(t) = \langle \frac{-2}{\sqrt{5}}\cos t, \frac{-2}{\sqrt{5}}\sin t, 0 \rangle$

and $\|\vec{T}'(t)\|^2 = \frac{4}{5}\cos^2 t + \frac{4}{5}\sin^2 t$
$= 4/5$ so $\|T(t)\| = \sqrt{4/5} = \frac{2}{\sqrt{5}}$

$\vec{N}(t) = \frac{1}{\|\vec{T}'(t)\|}\vec{T}'(t) = \frac{1}{2/\sqrt{5}} \langle \frac{-2}{\sqrt{5}}\cos t, \frac{-2}{\sqrt{5}}\sin t, 0 \rangle$

$= \frac{\sqrt{5}}{2} \langle \frac{-2}{\sqrt{5}}\cos t, \frac{-2}{\sqrt{5}}\sin t, 0 \rangle$

$\boxed{\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle}$

and $K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{2/\sqrt{5}}{\sqrt{5}} = \frac{1}{\sqrt{5}} * \frac{2}{\sqrt{5}} = \boxed{\frac{2}{5}}$

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}, \;\; t \in \mathbb{R}.$$

Step 1 Find $r'(t)$ and $\|r'(t)\|$.

$$v = \frac{dr}{dt} = (\cancel{-\sin t} + t\cos t + \cancel{\sin t})\hat{i} + (\cancel{\cos t} - (-t\sin t + \cancel{\cos t}))\hat{j} + 0\hat{k}$$

$$= t\cos t\,\hat{i} + t\sin t\,\hat{j} + 0\hat{k}$$

$|v|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2$ so $|v| = |t|$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$\boxed{T = \frac{v}{|v|} = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}}$$

$\frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}$ & $\left|\frac{dT}{dt}\right| = 1.$

so $\boxed{N = \frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}}$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$\boxed{\kappa = \frac{1}{|t|}}$$

$$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$$

IF time: (?)

$\boxed{\text{T/F}}$

$\vec{r}(s) = \langle 2s^2+1, s\rangle$, $s \in \mathbb{R}$ is an arc-length parametrization of the parabola $x = 2y^2 + 1$.

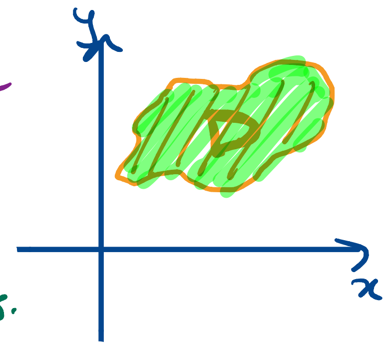
§14.1 Functions of Multiple Variables

Definition 26. A Function of two variables is a rule that assigns to each pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

$$\underset{\text{Function name}}{f} : \underset{\substack{\text{The domain of } f \\ \text{some subset of } \mathbb{R}^2}}{D} \to \underset{\text{The codomain of } f}{\mathbb{R}}, \text{ where } D \subseteq \mathbb{R}^2$$

Note:
D is the set of pairs (x,y) where $f(x,y)$ is actually defined

Range of f is the collection of actual outputs.

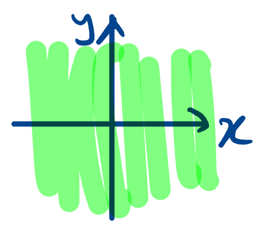


Example 27. Three examples are

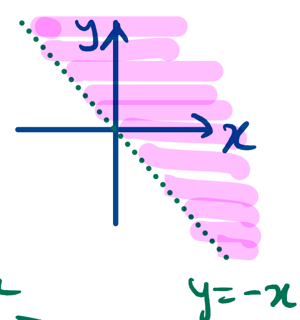
$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \frac{1}{\sqrt{x + y}}.$$

Example 28. Find the largest possible domains of f, g, and h.

$D_f = \{(x,y) \in \mathbb{R}^2\} = \mathbb{R}^2$, all real numbers



$D_g = \{(x,y) \in \mathbb{R}^2 \mid x+y>0\} = \{(x,y) \in \mathbb{R}^2 \mid y>-x\}$



$D_h = \{(x,y) \in \mathbb{R} \mid \underbrace{\sqrt{x+y} \neq 0}_{\text{from } \div 0}, \underbrace{x+y \geq 0}_{\text{from } \sqrt{\cdot}}\} = $ " same as D_g.

WARNING: domain is NEVER an interval!

Definition 29. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

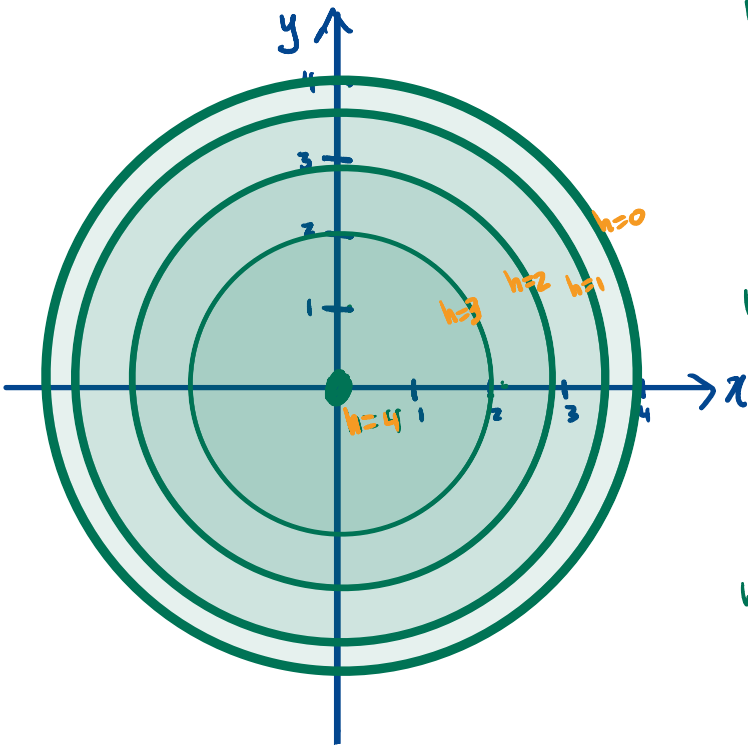
8:15 (10 min) 8:25

require $h \geq 0$ so $\frac{1}{4}x^2+\frac{1}{4}y^2 \leq 4$

Example 30. Suppose a small hill has height $h(x,y) = 4 - \dfrac{1}{4}x^2 - \dfrac{1}{4}y^2$ m at each point (x,y). How could we draw a picture that represents the hill in 2D?

$$D_h = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 16\}$$

Idea: draw level sets for $h=k$ with some fixed height



when $h=0 \Rightarrow 4-\frac{1}{4}x^2-\frac{1}{4}y^2=0$

$\Rightarrow 4=\frac{1}{4}x^2+\frac{1}{4}y^2$

$\Rightarrow 16=x^2+y^2$

Circle centered @ $(0,0)$ w/ $r=4$

When $h=1 \Rightarrow 4-\frac{1}{4}x^2-\frac{1}{4}y^2=1$

$\Rightarrow 4-1=\frac{1}{4}x^2+\frac{1}{4}y^2$

$\Rightarrow 12=x^2+y^2$

" "

$r=\sqrt{12}=2\sqrt{3}$ ≈ 3.46

when $h=2 \Rightarrow 4-\frac{1}{4}x^2-\frac{1}{4}y^2=2$

$\Rightarrow 2=\frac{1}{4}x^2+\frac{1}{4}y^2$

$\Rightarrow 8=x^2+y^2$

" "

$r=\sqrt{8}=2\sqrt{2}$ ≈ 2.83

when $h=3 \Rightarrow 4-\frac{1}{4}x^2-\frac{1}{4}y^2=1$

$\Rightarrow 4-3=\frac{1}{4}x^2+\frac{1}{4}y^2$

$\Rightarrow 4(1)=x^2+y^2$

" "

$r=2$

When $h=4 \Rightarrow 4-\frac{1}{4}x^2-\frac{1}{4}y^2=0$

$\Rightarrow 4-4=\frac{1}{4}x^2+\frac{1}{4}y^2$

$\Rightarrow 0=\frac{1}{4}x^2+\frac{1}{4}y^2$ point

In 3D, it looks like this.

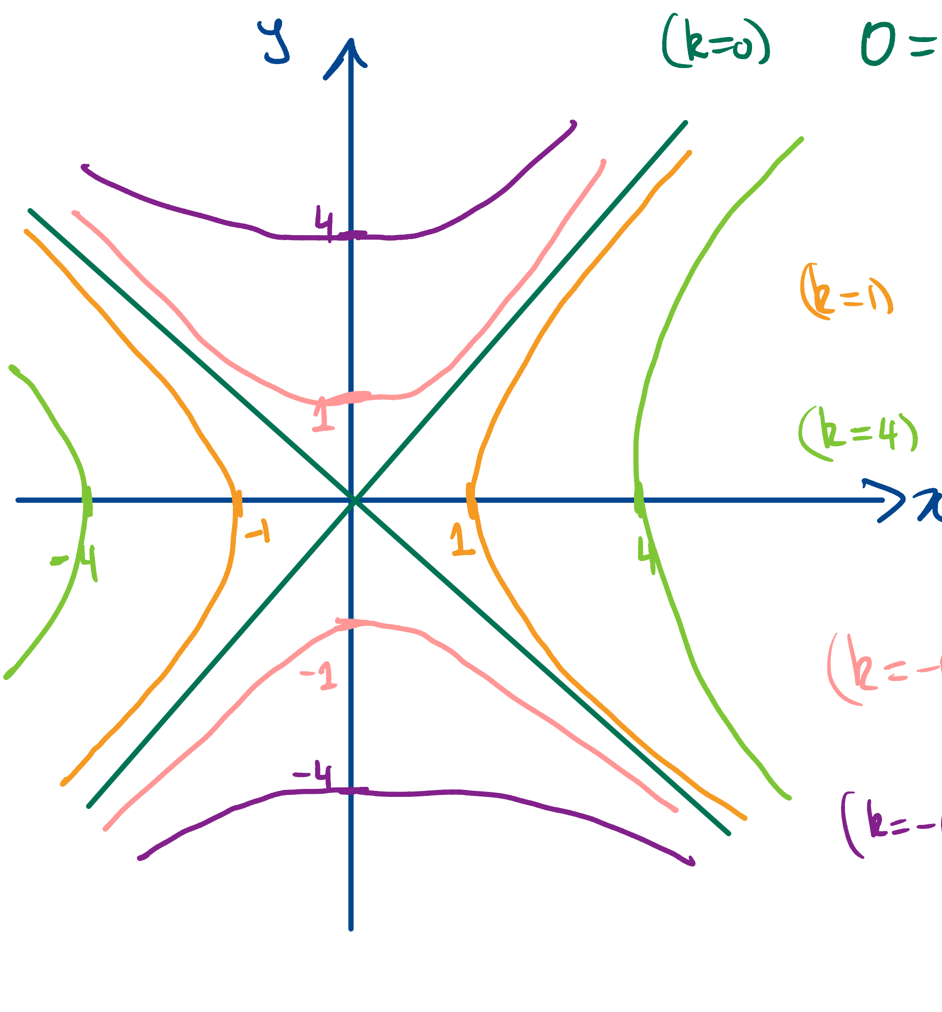
8:25 (10 min) 8:35

Definition 31. The contours (also called level sets) of a function f of two variables are the curves with equations $k = f(x,y)$, where k is a constant (in the range of f). A plot of contours for various values of z is a countour map (or level curve plot).

Some common examples of these are:

- topography maps e.g. for hiking
- electric field plots
- ?

Example 32. Create a contour diagram of $f(x, y) = x^2 - y^2$ Try $k=0,1,4$.



(k=0) $0 = x^2 - y^2 \Rightarrow x^2 = y^2$

$\Rightarrow |x| = |y|$

So $x = y$ or $x = -y$

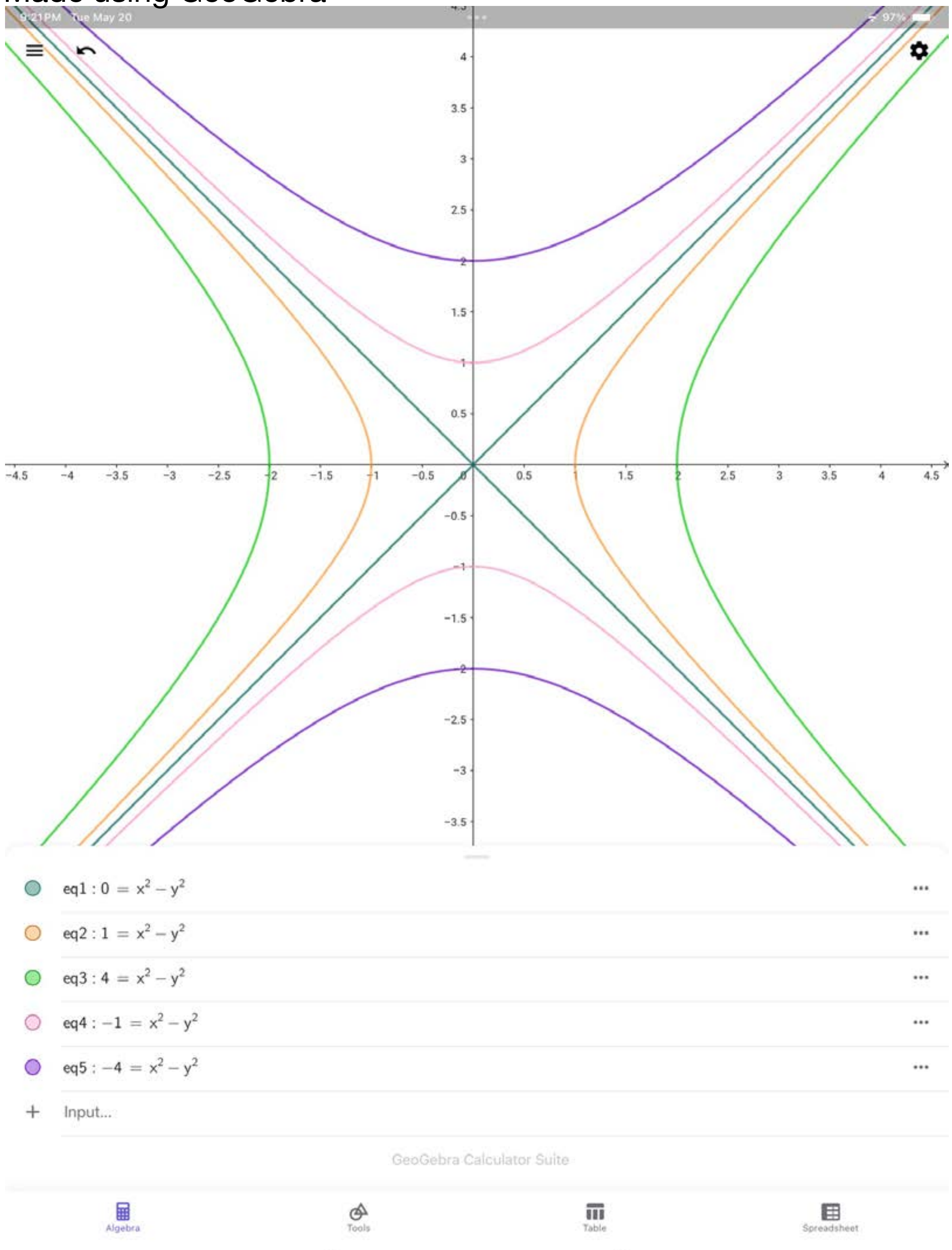
(k=1) $1 = x^2 - y^2$ hyperbola opening sideways

(k=4) $4 = x^2 - y^2$ hyperbola opening sideways

(k=−1) $-1 = x^2 - y^2$ hyperbola opening up/down

(k=−4) $-4 = x^2 - y^2$ hyperbola opening up/down

Made using GeoGebra



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Definition 32. The traces of a surface are the curves of intersection of the surface with planes parallel to the xz-plane or yz-plane. (setting $y=k$ or $x=k$ respectively)

Example 33. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant. $x \geq 0, y \geq 0, z \geq 0$.

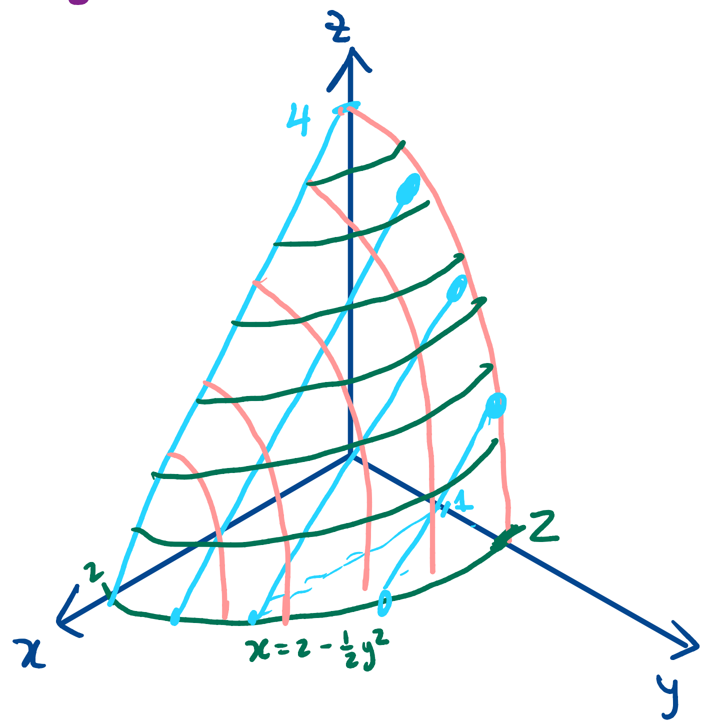
Contours

If $z=0$, then $0 = 4-2x-y^2$

so $x = 2 - \frac{1}{2}y^2$

If $z = k > 0$, then $k = 4-2x-y^2$

So $x = \frac{4-k}{2} - y^2$



traces w/ $y=k$

If $y=0$, then $z = 4-2x$

If $y=1$, then $z = 4-2x-1$

So $z = 3-2x$

If $y=k>0$, then $z = 4-2x-k^2$

So $z = (4-k^2)-2x$

traces w/ $x=k$

If $x=0$, then $z = 4-y^2$

If $x=k>0$, then $z = 4-k^2-y^2$

So $z = (4-k^2)-y^2$

Let's check our work: https://tinyurl.com/math2551-2var-graph

Definition 34. A Function of three variables is a rule that assigns to each triple of real numbers (x, y, z) in a set D a unique output denoted by $f(x, y, z)$.

Note: The graph of $w = f(x,y,z)$ lives... in $\mathbb{R}^4$!

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 35. Describe the largest possible domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

The only issue is $\div$ by 0. So need to avoid

$$4 - x^2 - y^2 - z^2 = 0$$

$$\Rightarrow \quad 4 = x^2 + y^2 + z^2$$

So D_f is all of $\mathbb{R}^3$ except the sphere of radius 2 centered at $(0,0,0)$.

Example 36. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

If $g(x,y,z) = k$ then

$k = 0 \Rightarrow 2x^2 + y^2 + z^2 = 0$ is the point $(0,0,0)$

$k = 1 \Rightarrow 2x^2 + y^2 + z^2 = 1$ ellipsoid

$k > 1 \Rightarrow 2x^2 + y^2 + z^2 = k$ larger ellipsoid

$k < 0 \Rightarrow 2x^2 + y^2 + z^2 = -1$ no soln.

9:00 (8 min) 9:08

§14.2 Limits & Continuity

Definition 37. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

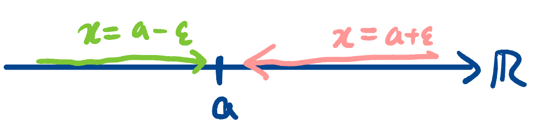
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

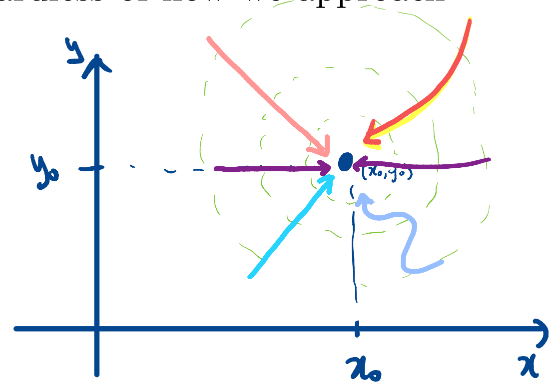
$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if and only if $f(x, y)$ approaches the value L regardless of how we approach the point (x_0, y_0).

From Calc I. $\lim_{x\to a} f(x) = L$ meant $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$



Now,



Definition 38. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists (approach (x_0,y_0) from any direction)
2. $f(x_0,y_0)$ exists (evaluating at (x_0,y_0) is defined)
3. Those two values are equal

Idea: However you approach (x_0,y_0) should get $f(x,y)$ closer to L.

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

FACT $f_1(x,y) = x$ & $f_2(x,y) = y$ are both continuous

So e.g. * $f(x,y) = xy$
* $f(x,y) = xy + y$
* $f(x,y) = (xy + y)\,xy$

all cont.

also $f(x,y) = e^{xy+y}$ etc.

composition of $(x,y) \mapsto xy+y$ with $t \mapsto e^t$

9:08 (12 min) 9:20

Example 39. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \overbrace{\frac{\sqrt{2x-y}-2}{2x-y-4}}^{F(x,y)}$, if it exists.

Try to evaluate?

$$f(2,0) = \frac{\sqrt{2\cdot 2-0}-2}{2\cdot 2-0-4} = \frac{\sqrt{4}-2}{4-4} = \frac{2-2}{4-4} = \frac{0}{0}$$

undefined but of indeterminate form.

Try L'Hopitals Rule? (Similar problem for warm up?)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{(\sqrt{x}-2)'}{(x-4)'}$$

$$= \lim_{x\to 4}\frac{1/2\sqrt{x}}{1}$$

$$= \lim_{x\to 4}\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

but how to do this w/ 2 vars?

Maybe try algebra? (manipulate the expression before taking the limit)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} * \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$= \lim_{x\to 4}\frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$$

$$= \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

try this on our 2-variable function?

$$\lim_{(x,y)\to(2,0)}\frac{\sqrt{2x-y}-2}{2x-y-4} = \lim_{\text{''}}\frac{\overbrace{\sqrt{2x-y}-2}^{(a\,-\,b)}}{2x-y-4} * \frac{\overbrace{\sqrt{2x-y}+2}^{(a\,+\,b)}}{\sqrt{2x-y}+2}$$

$$= \lim_{\text{''}}\frac{\overbrace{\cancel{(2x-y)-4}}^{a^2\,-\,b^2}}{\cancel{(2x-y-4)}*(\sqrt{2x-y}+2)}$$

$(a-b)(a+b) = a^2-b^2$ w/ $a=\sqrt{2x-y}$, $b=2$

$$= \lim_{(x,y)\to(2,0)}\frac{1}{\sqrt{2x-y}+2}$$

$$= \frac{1}{\sqrt{2\cdot 2-0}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

just a coincidence!!

Example 40. *You try it!* Evaluate $\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$, if it exists.

$$\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y+1}{y-\sin x} = \frac{\cos(0)+1}{0-\sin(\pi/2)} = \frac{1+1}{0-1} = \frac{2}{-1}$$

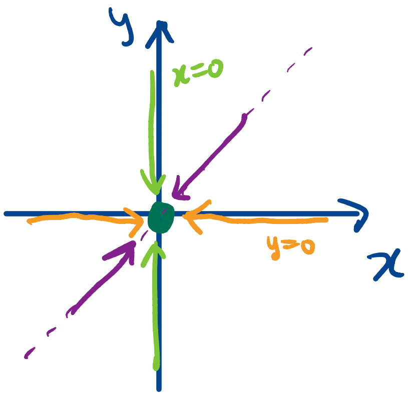
$$= \boxed{-2}$$

9:25 (15 min) 9:40

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 41. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.



Try evaluating? @ $(0,0)$ $\frac{0^2}{0^2+0^2} = \frac{0}{0}$ indeterminant.

Try setting $y=0$ and evaluating along the ~~x-axis~~

@ $y=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1.$ maybe?

Ok now how about along the y-axis

@ $x=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$? crumbs!

remember we're not evaluating at $y=0$!!

So $\Rightarrow$ Since the limit approaches different values along different paths, then the limit does not exist (DNE).

(If time) Can approach along line $x=y$. !!

@ $x=y$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$

This idea is called the **two-path test:**

If we can find <u>two paths approaching</u> to (x_0, y_0) along which <u>the limit of $f(x,y)$</u> takes on two different values, then <u>does not exist (is DNE)</u>.

Example 42. Show that the limit

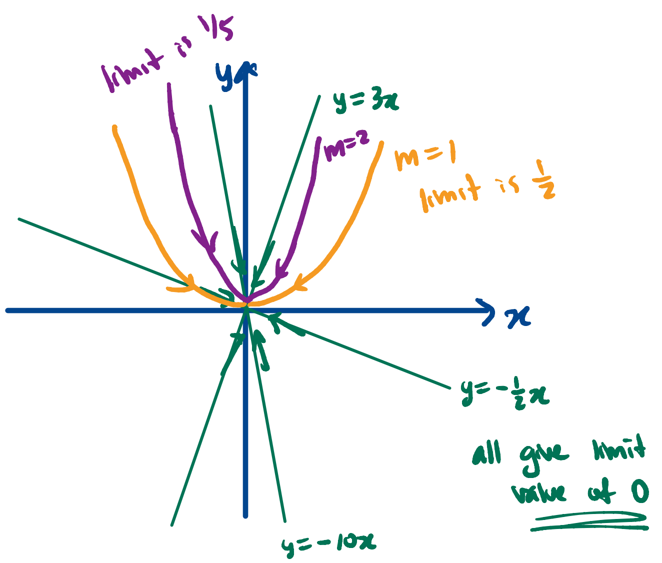
$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

Along any line $y=mx$ we have

Since $\frac{x^3}{x^4}\to 0$

@ (x,mx) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,mx)} \frac{x^2mx}{x^4+(mx)^2} = \lim_{x\to 0}\frac{mx^3}{x^4+m^2x^2} = 0$



What about something like $y=mx^2$ though?

@ (x,mx^2)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,mx^2)\to(0,0)} \frac{x^2mx^2}{x^4+(mx^2)^2}$$

$$= \lim_{x\to 0} \frac{x^4}{x^4+m^2x^4} = \lim_{x\to 0}\frac{\cancel{x^4}}{\cancel{x^4}(1+m^2)}$$

$$= \lim_{x\to 0}\frac{1}{1+m^2} = \frac{1}{1+m^2}$$

Since limit is different along two different paths

(e.g. $y=x$ limit is 0
and $y=x^2$ limit is ½)

$\boxed{\text{by the TWO\_PATH TEST the limit is DNE.}}$

Example 43. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test. Hint: try two parabolas.

$$f(x,y) = \frac{x^4}{x^4+y^2} \qquad \text{Let } y = mx^2$$

Then $$f(x, mx^2) = \frac{x^4}{x^4+m^2x^4} = \frac{1}{1+m^2}.$$

So $$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx^2}} f(x,y) = \frac{1}{1+m^2}.$$

So, by Two-path test the limit is DNE. ///

Example 44. [Challenge:] Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4+y^2}$$

does exist using the Squeeze Theorem.

Theorem 45 (Squeeze Theorem). *If* $f(x,y) = g(x,y)h(x,y)$, *where* ① $\boxed{\lim_{(x,y)\to(a,b)} g(x,y) = 0}$ *and* $\boxed{|h(x,y)| \leq C}$ *for some constant* C *near* (a,b), *then* $\lim_{(x,y)\to(a,b)} f(x,y) = 0$.

bound this

This goes to 0

Idea: $\dfrac{x^4 y}{x^4+y^2} = \dfrac{x^4}{x^4+y^2} * y$

① $g(x,y) = y$ tends to 0 as $(x,y)\to(0,0)$ ✓

② Notice $y^2 \geq 0$ so $x^4+y^2 \geq x^4$

$\Rightarrow 1 \geq \dfrac{x^4}{x^4+y^2} \geq 0$ (also non-neg)

$h(x,y) = \dfrac{x^4}{x^4+y^2}$ satisfies $|h(x,y)| \leq 1$. (for all (x,y)).

$\underline{\underline{C=1}}$.

So $\boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 0 \text{ by Squeeze Theorem}}$