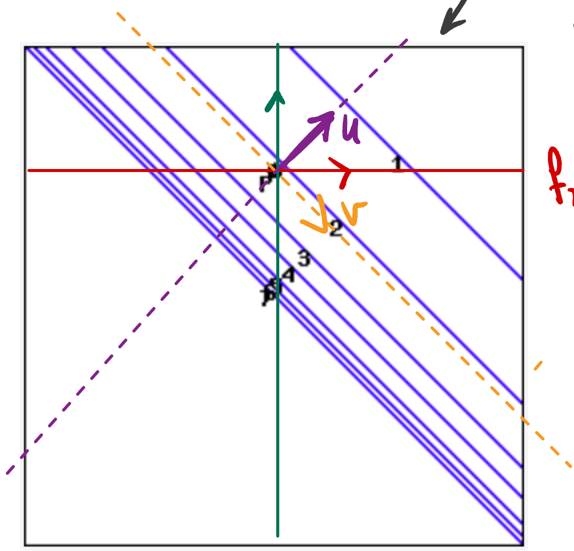
§14.5 Directional Derivatives & Gradient Vectors

Example 61. Recall that if $z = f(x,y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?

Level sets for $z=k$.
Shows all (x,y) pairs s.t. $f(x,y)=k$



f_x measures the change in z-value as x-value increases

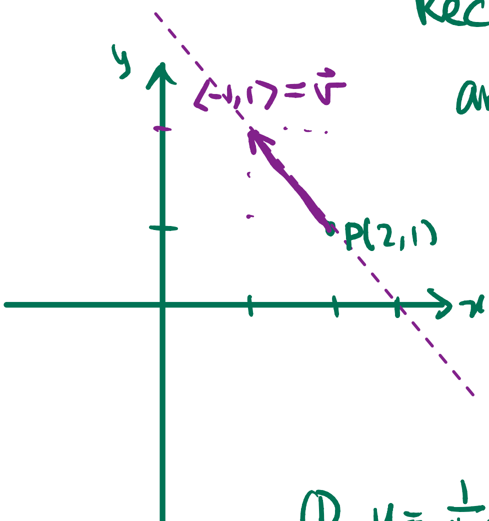
What about this direction parallel to the level sets? $z=k$ so $\boxed{D_v f(P)=0}$

f_y measures the change in z-value as y-value increases

What about THIS direction ORTHOGONAL to the level sets $z=k$?

Now $D_u f(P)$ is <u>largest</u> in this direction.

Let's go back to our hill example again, $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2,1)$ if we move in the direction $\langle -1, 1\rangle$?



Recall $Dh = \left[-\frac{1}{2}x \quad -\frac{1}{2}y\right]$

and so $h(2,1) = 4 - 1 - \frac{1}{4} = 11/4$

and $Dh\big|_{e(2,1)} = \left[-1 \quad -\frac{1}{2}\right]$

IDEA: normalize $\vec{v}$ to a unit vector & do difference quotient

① $u = \frac{1}{\|v\|}v = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$

② $\lim\limits_{t\to 0} \dfrac{h(\vec{p} + h\vec{u}) - h(\vec{p})}{t} = \lim\limits_{t\to 0} \dfrac{h(\langle 2,1\rangle + t\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle) - h(\langle 2,1\rangle)}{t}$

$= \lim\limits_{t\to 0} \dfrac{h(2 - t/\sqrt{2},\ 1 + t/\sqrt{2}) - 11/4}{t} = \lim\limits_{t\to 0} \frac{1}{2\sqrt{2}} - \frac{t}{4} = \boxed{\frac{1}{2\sqrt{2}}}$

Note:

$\frac{1}{t}\left(\cancel{4} - \frac{1}{4}\left(\cancel{4} - \frac{4t}{\sqrt{2}} + \frac{t^2}{2}\right) - \frac{1}{4}\left(\cancel{1} + \frac{2t}{\sqrt{2}} + \frac{t^2}{2}\right) - \frac{\cancel{11}}{4}\right) = \frac{1}{t}\left(\frac{t}{\sqrt{2}} - \frac{t^2}{8} - \frac{t}{2\sqrt{2}} - \frac{t^2}{8}\right)$

$= \frac{1}{\cancel{t}}\left(\frac{\cancel{t}}{2\sqrt{2}} - \frac{t^{\cancel{2}}}{4}\right)$

Definition 62. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{p}) = \lim_{h\to 0} \frac{f(\vec{p} + h\vec{u}) - f(p)}{h}$$

if this limit exists.

E.g. for our hill example above we have: $D_{\langle -1/\sqrt{2},\, 1/\sqrt{2}\rangle} h(2,1) = \frac{1}{2\sqrt{2}}$

Note that $D_{\mathbf{i}} f =$ f_x $\quad D_{\mathbf{j}} f =$ f_y $\quad D_{\mathbf{k}} f =$ f_z

(The regular or "Standard" directional derivatives)

Definition 63. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

$$\nabla f(\mathbf{p}) = Df(\vec{p})^T$$

or $= \langle f_{x_1}(\vec{p}), f_{x_2}(\vec{p}), \ldots, f_{x_n}(\vec{p}) \rangle$

Example $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ then $Dh = [-\frac{1}{2}x \quad -\frac{1}{2}y]$ and $\boxed{\nabla h = \begin{bmatrix} -1/2\, x \\ -1/2\, y \end{bmatrix}}$.

just TRANSPOSE!

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}} f(\mathbf{p}) = \nabla f \cdot \vec{u}$$

dot product of grad of f w/ direction vector u.

Example.

$$D_{\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} h(2,1) = \nabla h(2,1) \cdot \langle \tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle$$

$$= \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2}\right) = \boxed{\frac{1}{2\sqrt{2}}}$$

Example 64. *You try it!* Find the gradient vector and the directional derivative of each function at the given point **p** in the direction of the given vector **u**.

a) $f(x,y) = \ln(x^2+y^2)$, $\mathbf{p} = \langle -1, 1\rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 2x/(x^2+y^2) \\ 2y/(x^2+y^2) \end{bmatrix} \quad @ \ \vec{p} = \langle -1, 1\rangle \quad \nabla f(\vec{p}) = \begin{bmatrix} -2/2 \\ 2/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and $D_{\vec{u}} f(\vec{p}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = -\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \boxed{1/\sqrt{5}}$

b) $g(x,y,z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = \langle 1, 2, 1\rangle$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

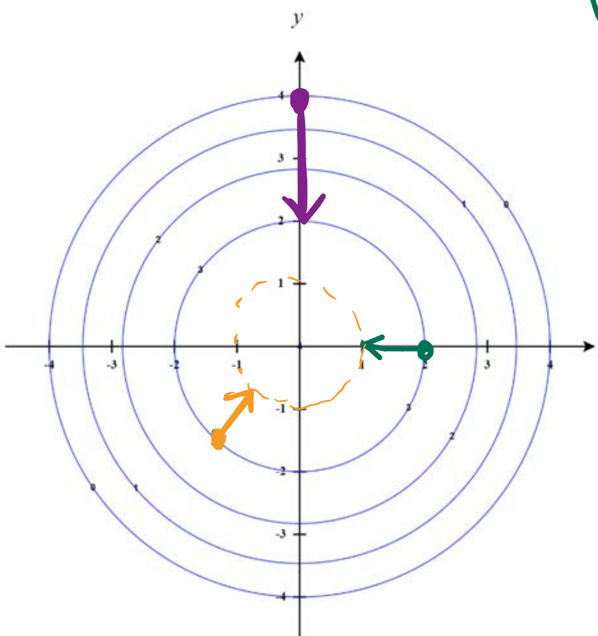
$$\nabla g = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 2x + 4y^2 \\ 8xy \\ 2z \end{bmatrix} \quad @ \ \vec{p} = \langle 1,2,1\rangle \quad \nabla g(\vec{p}) = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix}$$

$v = \langle 1, 2, -1\rangle \Rightarrow u = \frac{1}{\|v\|} v = \langle 1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\rangle$

So $D_{\vec{u}}\, g(\vec{p}) = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}}(18 + 32 - 2)$

$= \boxed{\frac{48}{\sqrt{6}}}$

Example 65. If $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2, 0), (0, 4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2, 0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.



$$\nabla h = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = \begin{bmatrix} -1/2\, x \\ -1/2\, y \end{bmatrix}$$

@ $(2,0)$ $\quad \nabla h(2,0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

@ $(0,4)$ $\quad \nabla h(0,4) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

@ $(-\sqrt{2},-\sqrt{2})$ $\quad \nabla h(-\sqrt{2},-\sqrt{2}) = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

What can we notice?

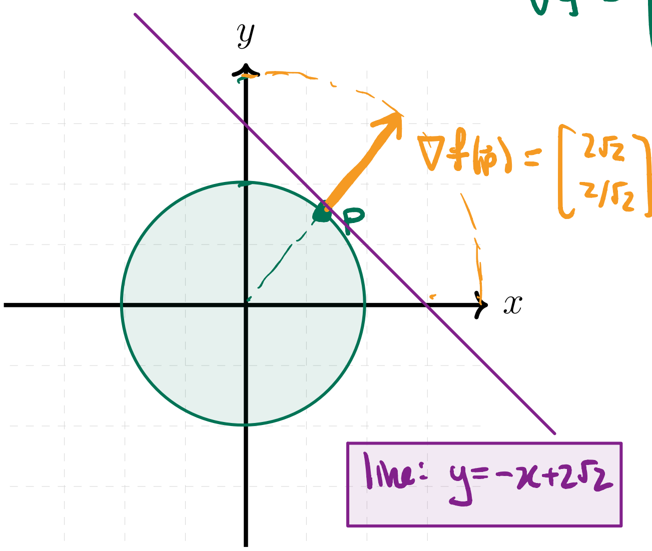
* all point towards center (highest point on hill)
* They all point perpendicular to level set lines
* Farther away from center means longer gradient (longer $\|\nabla h\|$)
* ∇h points in direction of greatest ascent
  Up the hill!

Note that the gradient vector ∇f is perpendicular to the level curves of the function $f(x,y)=z$.

Similarly, for $f(x, y, z)$, $\nabla f(a, b, c)$ is orthogonal to the LEVEL SURFACES

Example 66. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f\,|_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.

$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ @ $P(\sqrt{2}, \sqrt{2})$ $\quad \nabla f(\vec{P}) = \begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$



$8 + 8 = 16 = 4^2$

$\langle -1, 1 \rangle$ orth. to $\nabla f(\vec{p})$, so.

$\ell(t) = \langle \sqrt{2}, \sqrt{2} \rangle + t\langle 1, -1 \rangle$

(so $m = -1$)

$y = -x + b$ and passes thru $(\sqrt{2}, \sqrt{2})$

so $\sqrt{2} = -\sqrt{2} + b$

$\Rightarrow 2\sqrt{2} = b$

ChatGPT Share

what's the latex symbol \nabla

// DO PA??

$\boxed{\text{20 min}}$

The LaTeX symbol `\nabla` renders as:

$$\nabla$$

It is called **"nabla"** or the **del operator**, and it is used primarily in vector calculus. It represents the vector differential operator:

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right]$$

Common Uses:

- **Gradient** of a scalar function f:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\right]$$

- **Divergence** of a vector field $\vec{F}$:

$$\nabla \cdot \vec{F}$$

- **Curl** of a vector field $\vec{F}$:

$$\nabla \times \vec{F}$$

- **Laplacian** of a scalar field f:

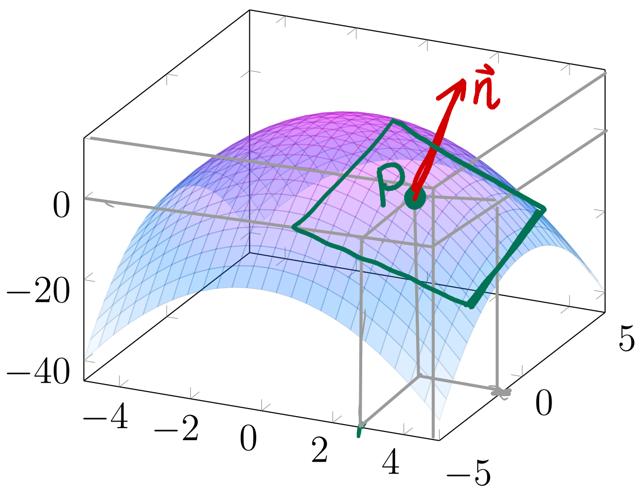
$$\nabla^2 f = \nabla \cdot \nabla f$$

So in summary, `\nabla` is a compact and powerful symbol in multivariable calculus, especially when working with fields and differential operators.

Tools

§14.6 Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

* need $\vec{n}$ normal vector & use plane equation

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

where $\vec{n}=\langle a,b,c\rangle$.

But $\vec{n}=\nabla F(\vec{p})$ is normal to the surface $F(\vec{p})=k$. (!!!)

So ① Identify $F(x,y,z)$ such that $F(x,y,z)=k$ defines the surface
② Compute the gradient ∇F & evaluate @ $\vec{P}$
③ write plane equation

Soln. ① $F(x,y,z) = x^2+y^2+z$

② $\nabla F = \langle 2x, 2y, 1\rangle$ @ $\vec{P}=\langle -1,3,0\rangle$ get $\nabla F(\vec{p}) = \langle -2, 6, 1\rangle$

③ $\boxed{-2(x+1)+6(y-3)+z=0}$

Example 67. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = \overbrace{4 - x^2 - y}^{f(x,y)}$$

① Identify F: $F(x,y,z) = 4 - x^2 - y - z$

② Find $\vec{n}$: $\nabla F = \langle -2x, -1, -1 \rangle$ @ $\vec{p} = \langle -2, 1, -1 \rangle$ then

$$\nabla F(\vec{p}) = \langle 4, -1, -1 \rangle$$

③ Plane eqn: $\boxed{4(x+2) - (y-1) - (z+1) = 0}$

or expanding the z part and then rearranging we have

$$z = \underbrace{-1}_{f(a,b)} + \underbrace{4}_{f_x(a,b)}\underbrace{(x+2)}_{(x-a)} \underbrace{-(y-1)}_{+f_y(a,b)(y-b)}$$

$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

this is just the linearization of $z = f(x,y)$!

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This should look familiar: it's the linearization

Example 68. *You try it!* Consider the surface in $\mathbb{R}^3$ containing the point P and defined by

$$x^2 + 2xy - y^2 + z^2 = 7,\ P(1, -1, 3).$$

Identity the function $F(x, y, z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

surface is level set

① $F(x,y,z) = x^2 + 2xy - y^2 + z^2 = k$, $k = 7$

② $\nabla F = \begin{bmatrix} 2x + 2y \\ 2x - 2y \\ 2z \end{bmatrix}$ @ $(1,-1,3)$ $\nabla F(1,-1,3) = \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} = \vec{n}$

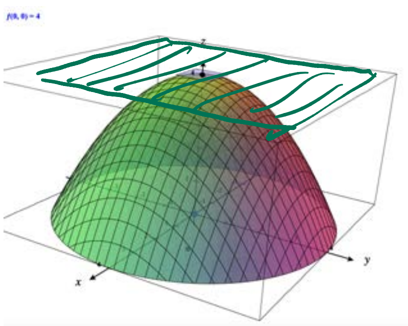
③ plane eqn becomes $\cancel{0(x-1)} + \boxed{4(y+1) + 6(z-3) = 0}$

§14.7 Optimization: Local & Global

Gradient: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$.

What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?



$\nabla h = \left[-\frac{1}{2}x \quad -\frac{1}{2}y\right]$ So @ $(0,0)$

$\nabla h(0,0) = [0 \quad 0]$

means "the path of steepest ascent is NOT TO MOVE at all!"

Tangent plane equation for $\underline{z = f(x,y)}$

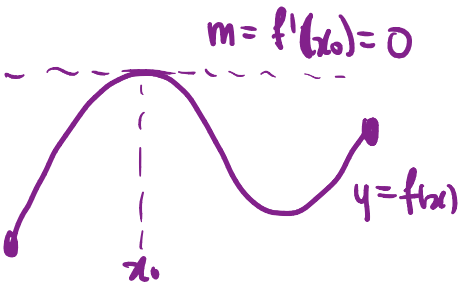
$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

In this case

$$z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$z = 4 + 0x + 0y$$

Sim. idea



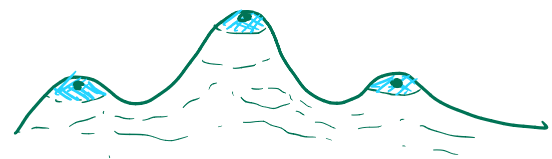
So just $\boxed{z = 4}$

parallel to the xy-plane (the floor)

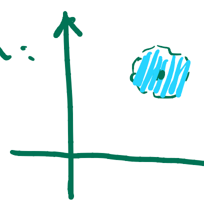
Definition 68. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

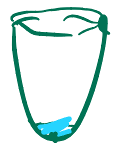
- $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a local Minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)

Graph:  ← all the graph points have smaller (or equal) z-value (height) near the local maximum.

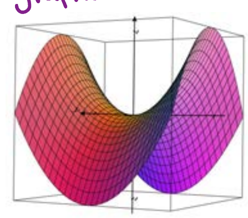


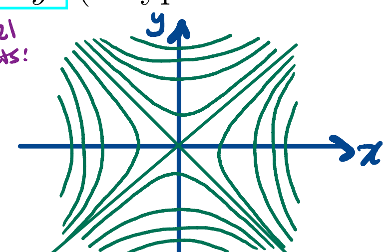
"local" max b/c can be several peaks

Domain:  ← For all (x,y) in the DISK the function value $f(x,y) \leq f(a,b)$

 local min vs. local max 

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $\boxed{z = x^2 - y^2}$ (a hyperbolic paraboloid!) near $(0, 0)$.

graph: 

level sets: 

This is called a Saddle point

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

MAX/MIN occurs @ (a,b) if

either $Df(a,b) = [0 \quad 0]$

$\Leftrightarrow \nabla f(a,b) = \langle 0, 0 \rangle = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

OR $Df(a,b)$ does not exist (DNE)

e.g.  $f(x,y) = \sqrt{x^2+y^2}$

$Df = \left[\frac{x}{\sqrt{x^2+y^2}} \quad \frac{y}{\sqrt{x^2+y^2}} \right]$

Df is $\boxed{\text{DNE}}$ @ $(0,0)$

Definition 69. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ $[0 \quad 0]$ or where $Df(a, b)$ is DNE is called a critical point of f.

Example 70. Find the critical points of the function

$$f(x,y) = x^3 + y^3 - 3xy.$$

$$Df = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix}$$

Need $Df(a,b) = [0 \quad 0]$

So need to find all pairs (a,b) s.t. $\begin{cases} 3a^2 - 3b = 0 \\ 3b^2 - 3a = 0 \end{cases}$

So $\Rightarrow \begin{cases} 3a^2 = 3b \\ 3b^2 = 3a \end{cases}$

$\Rightarrow \begin{cases} a^2 = b & (1) \\ b^2 = a & (2) \end{cases}$

Sub (1) into (2) to get $(a^2)^2 = a^4 = a$

$\Rightarrow a^4 - a = 0$

$\Rightarrow a(a^3 - 1) = 0$

$\Rightarrow a = 0$ or $a^3 = 1$

$\Rightarrow \boxed{a = 0 \text{ or } a = 1}$ and $b^2 = a$ so $(a,b) \in \{(0,0), (1,1)\}$

Set of critical points.

Q: Should we worry about crit points from Df being DNE?

A: nothing to worry about here. The function $f(x,y) = x^3 + y^3 - 3xy$ is continuous everywhere in its domain $\mathbb{R}^2$ (as are ALL POLYNOMIAL functions!)

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0,0)$.

a) $f(x,y) = 3x + y^3 + 2y^2$

$Df = [3 \quad 3y^2+4y]$

NO CRIT POINTS since $Df \neq [0 \quad 0]$ for any $(x,y) \in \mathbb{R}^2$, so $\boxed{\text{NO}}$

b) $g(x,y) = \cos(x) + \sin(x)$

$Dg = [-\sin x \quad \cos x] = [0 \quad 0]$ if $x = \frac{\pi}{4} + k\frac{\pi}{2}$, $k \in \mathbb{Z}$.

but $Df(0,0) \neq [0 \quad 0]$ so $\boxed{\text{NO}}$

c) $h(x,y) = \dfrac{4}{x^2+y^2}$

$Dh = \left[\frac{-8x}{(x^2+y^2)^2} \quad \frac{-8y}{(x^2+y^2)^2}\right]$ is DNE @ $(0,0)$

BUT $(0,0)$ not in the DOMAIN OF h!! So $\boxed{\text{NO}}$

d) $k(x,y) = x^2 + y^2$

$Dk = [2x \quad 2y]$ and

$Dk(0,0) = [0 \quad 0]$ so $\boxed{\text{yes}}$

8:45 3 min 8:48

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x,y)$ at (a,b) is

$$Hf(a,b) = D^2 f(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

Theorem 72 (2nd Derivative Test). *Suppose (a,b) is a critical point of $f(x,y)$ and f has continuous second partial derivatives. Then we have:*

- *If* $\det(Hf(a,b)) > 0$ *and* $f_{xx}(a,b) > 0$, *$f(a,b)$ is a local minimum*
- *If* $\det(Hf(a,b)) > 0$ *and* $f_{xx}(a,b) < 0$, *$f(a,b)$ is a local maximum*
- *If* $\det(Hf(a,b)) < 0$, *f has a saddle point at (a,b)*
- *If* $\det(Hf(a,b)) = 0$, *the test is inconclusive.*

NOTE $Hf = D^2 f$ is symmetric so @(a,b)
① all eigenvalues are real
② matrix is orthogonally diagonalizable, ... etc.

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.

  when $n=2$ $\det A = \lambda_1\lambda_2 > 0$ iff λ_1, λ_2 nonzero w/ same sign

- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 73. Classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$ from Example 70.

$$Df = [\underbrace{3x^2 - 3y}_{f_x} \quad \underbrace{3y^2 - 3x}_{f_y}]$$

$$(a,b) \in \{(0,0), (1,1)\}$$

Set of critical points.

$f_{xx} = 6x \qquad f_{xy} = -3$

$f_{yx} = -3 \qquad f_{yy} = 6y$

So $D^2 f = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$

So, @ $(0,0)$ $D^2f(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ and $\det D^2f(0,0) = -9 > 0$

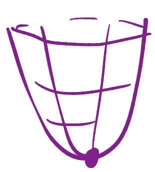
So $(0,0)$ is location of a $\boxed{\text{saddle point}}$

and @ $(1,1)$ $D^2f(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$ and $\det D^2f(1,1) = 36 - 9 = 27 > 0$

and $f_{xx}(1,1) = 6 > 0$

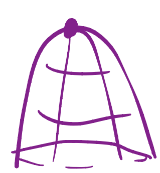
So $(1,1)$ is location of a $\boxed{\text{local minimum}}$

Recall:



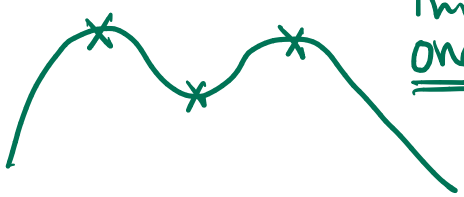
concave UP $\Rightarrow$ local MIN.

and

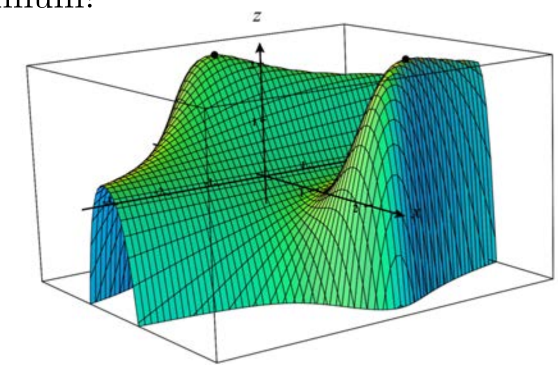


concave DOWN $\Rightarrow$ local MAX.

Two Local Maxima, No Local Minimum: The function $\boxed{g(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2}$ has two critical points, at $(-1,0)$ and $(1,2)$. Both are local maxima, and the function never has a local minimum!



this can't happen in one variable case



A global maximum of $f(x,y)$ is like a local maximum, except we must have $f(a,b) \geq f(x,y)$ for **all** (x,y) in the domain of f. A global minimum is defined similarly.

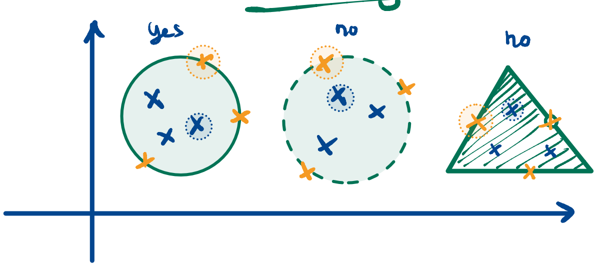
Theorem 74. *On a closed & bounded domain, any continuous function $f(x,y)$ attains a global minimum & maximum.*

14.1 Functions of Several Variables **795**

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

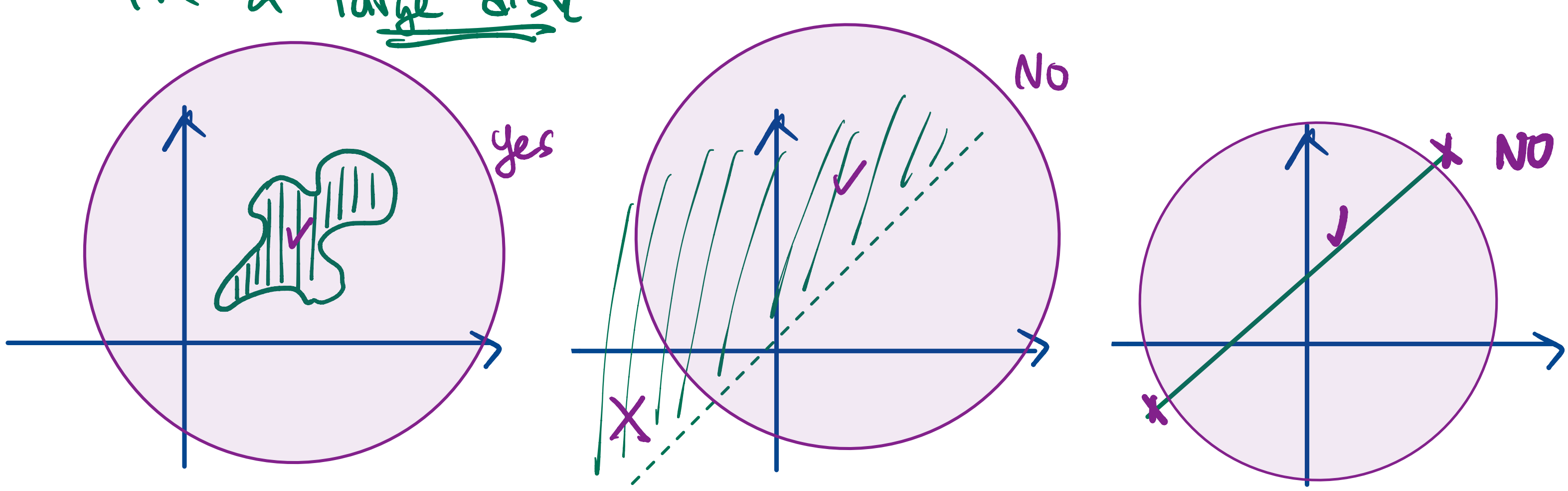
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

Closed: Contains all boundary points



DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Bounded: Contained in a large disk



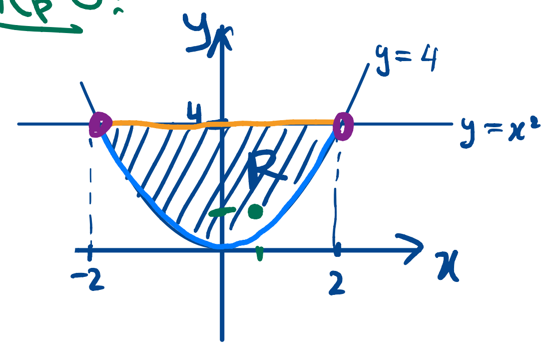
Strategy for finding global min/max of $f(x,y)$ on a closed & bounded domain R

Need $\nabla f(a,b) = \vec{0}$

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary. ?
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 75. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Step 0:



Step 1:

$$Df = [8x - 4y \quad -4x + 2]$$

Crit points are (a,b) s.t.

$$\begin{cases} 8a - 4b = 0 \\ -4a + 2 = 0 \end{cases} \Rightarrow \begin{cases} 8a = 4b \\ 2 = 4a \end{cases} \Rightarrow \begin{cases} 2a = b \\ a = 1/2 \end{cases} \Rightarrow \begin{cases} b = 1 \\ a = 1/2 \end{cases}$$

So $(x,y) = (1/2, 1)$ is the only crit point. and it is in R ✓

is $(1/2, 1)$ in R??

is $y \geq x^2$ @ $(1/2, 1)$?

$1 \geq \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ ✓ yes.

Step 2: Find crit points of f on boundary.

@ $y = x^2$ $\quad f(x,y) = f(x, x^2) = 4x^2 - 4x\,x^2 + 2x^2 = 6x^2 - 4x^3$

@ $y = 4$ $\quad f(x,y) = f(x,4) = 4x^2 - 4x*4 + 2*4 = 4x^2 - 16x + 8$

Find crit points

Example 76. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

(Cont.)

Step 2: Find crit points of f on boundary.

@ $y=x^2$ $f(x,y) = f(x, x^2) = 4x^2 - 4x\,x^2 + 2x^2 = 6x^2 - 4x^3$

$g(x) = 6x^2 - 4x^3$

$\Rightarrow g'(x) = 12x - 12x^2 = 12x(1-x) = 0$ if $x=0$ or $x=1$.

So @ $y=x^2$ crit points are $\boxed{(0,0) \text{ \& } (1,1)}$

@ $y=4$ $f(x,y) = f(x,4) = 4x^2 - 4x*4 + 2*4 = 4x^2 - 16x + 8$

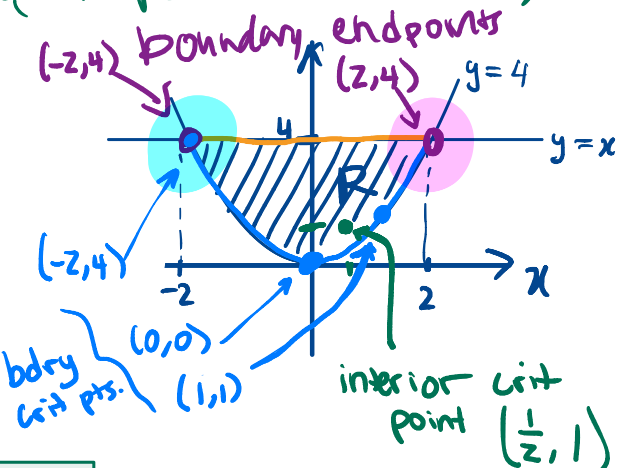
$g(x) = 4x^2 - 16x + 8$

$\Rightarrow g'(x) = 8x - 16 = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2.$

So @ $y=4$ crit point is $\boxed{(2,4)}$

Step 3: Evaluate f @ crit points & endpoints of boundary

(x,y)	$f(x,y)$
$(\frac{1}{2}, 1)$	1
$(0,0)$	0
$(1,1)$	2
$(-2,4)$	
$(2,4)$	



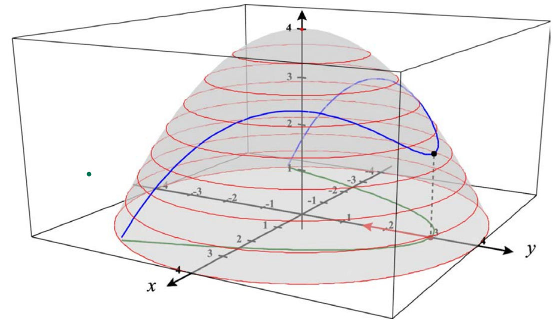
$\boxed{\begin{array}{l} 56: \text{ MAX of } f \text{ on } R \text{ @ } (-2,4) \\ -8 \quad \text{MIN of } f \text{ on } R \text{ @ } (2,4) \end{array}}$

§14.8 Constrained Optimization, Lagrange Multipliers

Goal: Maximize or minimize $f(x, y)$ or $f(x, y, z)$ subject to a *constraint*, $g(x, y) = c$.

Example 77. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

largest h-value



Objective function:

$$h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

(Thing to maximize)

Constraint equation:

$$\underbrace{y + 0.5x^2}_{g(x,y)} = 3$$

(condition that must be satisfied)

IDEA: Solve $\boxed{\nabla h = \lambda \nabla g}$ & $\boxed{g(x,y) = 3}$

$\nabla h = [-\frac{1}{2}x \;\; -\frac{1}{2}y]$ and $\nabla g = [x \;\; 1]$

So $\begin{cases} \nabla h = \lambda \nabla g \\ g(x,y) = 3 \end{cases} \Rightarrow \begin{cases} ① -\frac{1}{2}x = \lambda x \\ ② -\frac{1}{2}y = \lambda \\ ③ y + \frac{1}{2}x^2 = 3 \end{cases} \Rightarrow \begin{cases} ① x(\frac{1}{2} + \lambda) = 0 \\ ② y = -2\lambda \\ ③ y = 3 - \frac{1}{2}x^2 \end{cases}$

FROM ① either $x = 0$ or $\lambda = -\frac{1}{2}$.

If $x = 0$

$\begin{cases} ② & y = -2\lambda \\ ③ & y = 3 - 0 \end{cases} \Rightarrow \begin{cases} \lambda = -\frac{3}{2} \\ y = 3 \end{cases}$ & $x = 0$, so $\boxed{(x,y) = (0,3)}$ & $\lambda = -3/2$

If $\lambda = -1/2$

$\begin{cases} ② & y = -2(-\frac{1}{2}) \\ ③ & y = 3 - \frac{1}{2}x^2 \end{cases} \Rightarrow \begin{cases} ② & y = 1 \\ ③ & 1 = 3 - \frac{1}{2}x^2 \end{cases} \Rightarrow \frac{1}{2}x^2 = 2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$

So $\boxed{(x,y) = (2,1) \text{ or } (-2,1)}$ & $\lambda = -1/2$

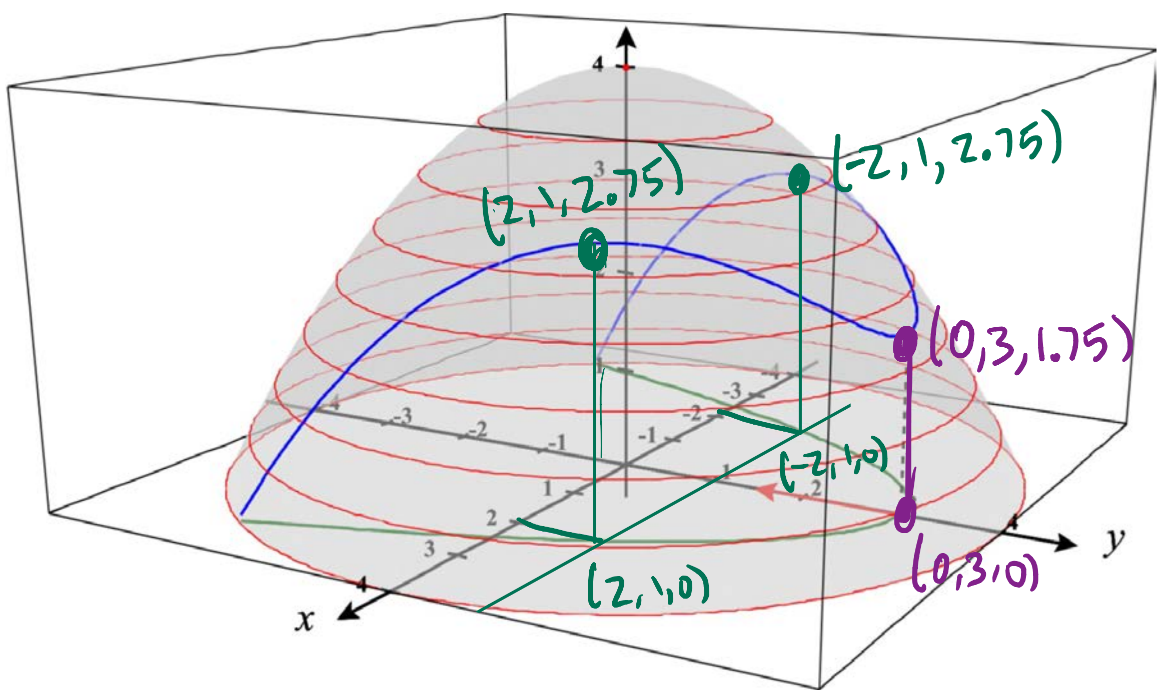
9:25 5 min 9:30

Example 77. A new hiking trail has been constructed on the hill with height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

(Cont.) $\boxed{(x,y) = (0,3)}$ & $\boxed{(x,y) = (2,1) \text{ or } (-2,1)}$

Solve $\nabla h = \lambda \nabla g$ & $g(x,y) = 3$.

(x,y)	$h(x,y)$
$(0,3)$	1.75
$(2,1)$	2.75
$(-2,1)$	2.75



So MAX h-value on the new hiking trail is $h = 2.75$ which occurs @ $(2,1)$ & $(-2,1)$.

9:30 10 min 9:40

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Objective Function: $z^2 = xy + 4$? NO

$d(x,y,z) = \sqrt{x^2+y^2+z^2}$ yes! "distance from origin" is thing to minimize.

Constraint: $g(x,y,z) = xy + 4 - z^2 = 0$

IDEA FROM CALC 1: Minimize instead

$d^2 = f(x,y,z) = x^2+y^2+z^2$

(no square root, easier w/ same answer)

Solve $\nabla f = \lambda \nabla g$ & $g(x,y,z)=0$

$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \nabla g = \begin{bmatrix} y \\ x \\ -2z \end{bmatrix}$

So Solve
$$\begin{cases} ① \ 2x = \lambda y \\ ② \ 2y = \lambda x \\ \boxed{③ \ 2z = -2\lambda z} \\ ④ \ xy + 4 - z^2 = 0 \end{cases}$$

Start here: $z + \lambda z = 0$

$\Rightarrow z(\lambda+1) = 0$

So $\boxed{z=0 \text{ or } \lambda = -1}$

Case 1: if $z=0$ then

$$\begin{cases} (1a) \ 2x = \lambda y \\ (2a) \ 2y = \lambda x \\ (3a) \ xy = -4 \end{cases}$$

plug (1a) into (2a)

$2y = \lambda\left(\frac{\lambda y}{2}\right) \Rightarrow 4y = \lambda^2 y$

$\Rightarrow y(4-\lambda^2) = 0$

$\Rightarrow \boxed{y=0, \lambda=2 \text{ or } \lambda=-2}$

If $y=0$ then (1a) $x=0$, (3a) $0=-4$ ~~X~~ impossible!

If $\lambda=-2$ then

$$\begin{cases} (1a) \ 2x = -2y \\ (2a) \ 2y = -2x \\ (3a) \ xy = -4 \end{cases} \Rightarrow \begin{matrix} x = -y \\ xy = -4 \end{matrix}$$

$\Rightarrow \boxed{(x,y,z) = (2,-2,0) \text{ or } (-2,2,0)}$

If $\lambda=2$ then

$$\begin{cases} (1a) \ 2x = 2y \\ (2a) \ 2y = 2x \\ (3a) \ xy = -4 \end{cases} \Rightarrow \begin{matrix} x = y \\ xy = -4 \end{matrix}$$

~~X~~ impossible!

9:40 10 min 9:50

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

(Cont.)

So Solve
$$\begin{cases} ①\ 2x = \lambda y \\ ②\ 2y = \lambda x \\ ③\ 2z = \lambda(-2z) \\ ④\ xy + 4 - z^2 = 0 \end{cases}$$

Case 2 : $\lambda = -1$

1b $2x = -y$
2b $2y = -x$
3b $xy + 4 - z^2 = 0$

plug in 1b into 2b
$2(-2x) = -x$
$\Rightarrow 4x = x$
$\Rightarrow x = 0$
and so $y = 0$

Then 3b says $0 + 4 - z^2 = 0$
$\Rightarrow z^2 = 4$
$\Rightarrow z = \pm 2$.

So $\boxed{(x,y,z) = (0,0,\pm 2)}$

time to evaluate!

(x,y,z)	$f(x,y,z) = x^2+y^2+z^2$
$(2,-2,0)$	$4+4+0 = 8$
$(-2,2,0)$	$4+4+0 = 8$
$(0,0,2)$	$0+0+4 = 4$
$(0,0,-2)$	$0+0+4 = 4$

MAX distance is $\sqrt{8} = 2\sqrt{2}$

MIN distance is $\sqrt{4} = 2$

So the points on the surface $g(x,y)=0$ w/ MIN distance to origin are

$\boxed{(0,0,2)\ \&\ (0,0,-2)\ \text{w/ distance } 2}$