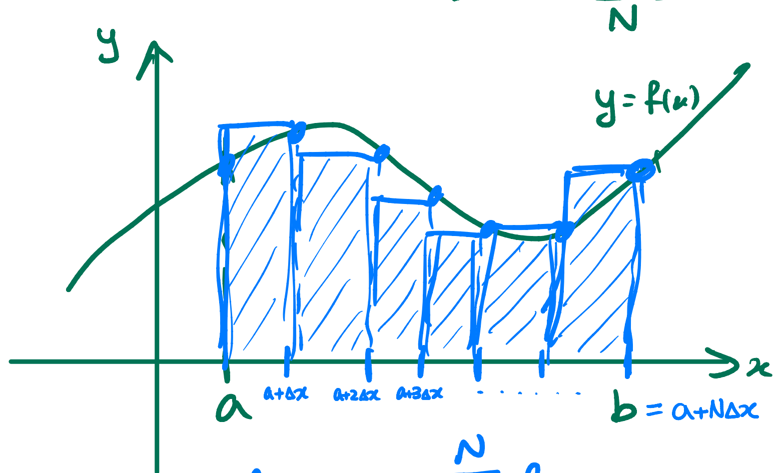
§15.1 Double Integrals, Iterated Integrals, Change of Order

Recall: Riemann sum and the definite integral from single-variable calculus.

$$\text{Area} = \int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\,\Delta x, \quad \text{where } x_k = a + k\Delta x, \quad \Delta x = \frac{b-a}{N}$$



$$\text{Area} \approx \sum_{k=1}^{N} f(x_k)\,\Delta x$$

Get EXACT area in the limit as $N \to 0$.

(as the width of each little rectangle goes to zero).

Also, same area w/ other choices of how to pick height so long as $x_k^* \in [a+k\Delta x, a+(k+1)\Delta x]$

such as: * Right end point
* left endpoint
* midpoint

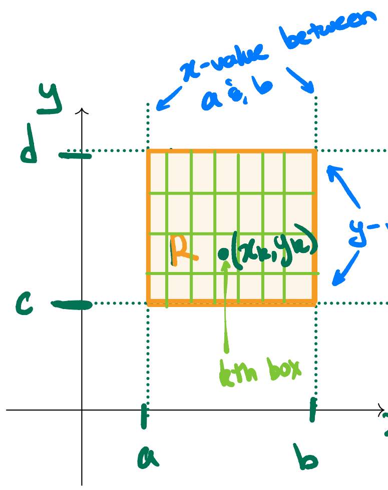
even other ways eg.
* average height (trapezoid rule)
* Simpson's rule (more complicated), etc. etc.

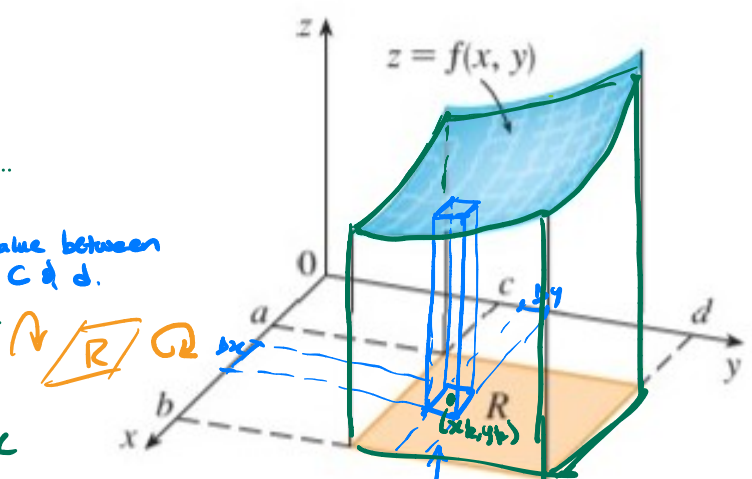
Double Integrals

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x, y)$ be a function defined on R such that $f(x, y) \geq 0$. Let S be the solid that lies above R and under the graph f.





This tall box has base $\Delta A_k = \Delta y \Delta x$ and height $f(x_k, y_k)$

So volume of tall box is $f(x_k, y_k) \Delta y \Delta x$

ADD THESE UP

Question: How can we estimate the volume of S?

$$\text{Volume} \approx \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} f(x_i, y_j) \Delta y \Delta x$$

in the limit

$$\text{Volume} = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

$$\overset{(alt)}{=} \iint_A f(x, y)\, dA \approx \sum_{k=1}^{N} f(x_k, y_k) \Delta A_k$$

8:15 (5 min) 8:20

Definition 79. The double integral of $f(x, y)$ over a rectangle R is

$$\iint_R \overbrace{f(x,y)}^{\text{integrand}}\,dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

region of integration

P is the "mesh size", largest size among all of the bases of the tall boxes.

if this limit exists.

Then we say f is integrable over the region R.

Eg.

- If f is continous over R, then f is integrable over R.
- Possible for f to still be integrable, even if not continuous
- Note: $\iint_R f(x,y)\,dA$ is SIGNED volume,

  So area under "the floor" $z=0$ counts as NEGATIVE volume.

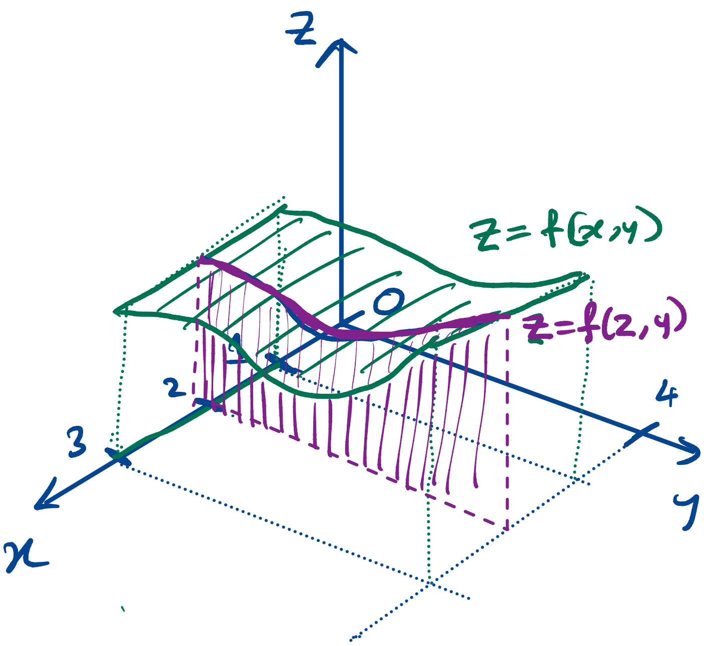
Question: How can we compute a double integral?

Answer: Iterate! Do one variable, then the other.

Let $f(x,y) = 2xy$ and lets integrate over the rectangle $R = [1,3] \times [0,4]$.

We want to compute $\int_1^3 \int_0^4 f(x,y)\ dy\ dx$, but lets consider the slice at $x = 2$.

What does $\int_0^4 f(2,y)\ dy$ represent here?



Purple slice has area

$$\int_0^4 f(2,y)\,dy = \int_0^4 4y\,dy = 2y^2\Big|_0^4$$

$$= 32 - 0 = 32$$

If we move purple slice back and forth among ALL x-values between $x=1$ to $x=3$ and add up all the purple slices then we will get the whole volume!

add up all purple slices

$$\text{Vol} = \int_1^3 \underbrace{\int_0^4 f(x,y)\,dy}_{\text{purple slice area @ } x=k,\ k\in[1,3]}\,dx = \int_1^3 \underbrace{\int_0^4 2xy\,dy}_{\text{do inside integral first}}\,dx = \int_1^3 xy^2\Big|_0^4\,dx$$

$$= \int_1^3 16x\,dx$$

$$= 8x^2\Big|_1^3 = 8*9-8 = 72-8$$

$$= \boxed{64}$$ total volume

In general, if $f(x, y)$ is integrable over $R = [a, b] \times [c, d]$, then $\int_c^d f(2, y)\, dy$ represents:

The area of the cross section @ $x=2$

What about $\int_c^d f(x, y) dy$?

the area of the cross section @ $x =$ constant

Let $A(x) = \int_c^d f(x,y)\,dy$. Then,

$$\text{Volume} = \int_a^b A(x)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

This is called an Iterated integral (iterate: do one at a time)

Example 80. Evaluate $\int_1^2 \underbrace{\int_3^4 6x^2y\, dy}_{\text{do inside first}}\, dx. = \int_1^2 3x^2y^2\Big|_3^4\, dx$

$$= \int_1^2 3x^2(16-9)\,dx = \int_1^2 21x^2\,dx$$

$$= 7x^3\Big|_1^2 = 7*8-7 = 7(8-1) = \boxed{49}$$

Theorem 81 (Fubini's Theorem). *If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then*

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

"order doesn't matter"

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

8:37 (13 min) 8:50

Example 82. *You try it!* Integrate:

a) $\int_0^2 \int_{-1}^1 x - y \, dy \, dx$ **easy**

b) $\int_0^1 \int_0^1 \frac{y}{1+xy} \, dx \, dy$ **medium**

c) $\int_1^4 \int_1^e \frac{\ln x}{xy} \, dx \, dy$ HARD!

8:50

Example 82. *You try it!* Integrate:

a) $\int_0^2 \int_{-1}^1 x - y \, dy \, dx$ **easy**

$$= \int_0^2 xy - \tfrac{1}{2}y^2 \Big|_{-1}^{1} dx$$

$$= \int_0^2 x - (-x) - 0 \, dx$$

$$= \int_0^2 2x \, dx = x^2 \Big|_0^2 = \boxed{4}$$

b) $\int_0^1 \int_0^1 \frac{y}{1+xy} \, dx \, dy$ **medium**

$$\int_0^1 \int_0^1 \frac{y}{1+xy} \underbrace{dx}_{du} \, dy = \int_0^1 \ln(1+xy)\Big|_0^1 dy = \int_0^1 \ln(1+y) - \cancel{y\ln(1)}^{0} \, dy$$

u-sub

$u = 1+xy$
$du = y\,dx$

IBP $\int u\,dv = uv - \int v\,du$

$u = \ln(t)$ $dv = 1\,dt$
$du = \frac{1}{t}dt$ $v = t$

So $\int \ln t\,dt = t\ln t - \int dt$
$= t\ln t - t + C$

$$= (1+y)\ln(1+y) - (1+y)\Big|_0^1$$

$$= [2\ln 2 - 2] - [1\cancel{\ln(1)}^{0} - 1] = \boxed{2\ln 2 - 1}$$

c) $\int_1^4 \int_1^e \frac{\ln x}{xy} \, dx \, dy$ HARD!

$$= \int_1^4 \underbrace{\tfrac{1}{2}(\ln x)^2}_{*} \cdot \tfrac{1}{y} \Big|_1^e dy = \int_1^4 \left[\tfrac{1}{2}(\cancel{\ln e}^{1})^2 - \tfrac{1}{2}(\cancel{\ln 1}^{0})^2\right] \cdot \tfrac{1}{y} \, dy$$

IBP $\int u\,dv = uv - \int v\,du$

$u = \ln x$ $dv = \frac{1}{x}dx$
$du = \frac{1}{x}dx$ $v = \ln x$

$$= \int_1^4 \tfrac{1}{2} \cdot \tfrac{1}{y} \, dy = \tfrac{1}{2}\ln(y)\Big|_1^4 = \tfrac{1}{2}\ln 4 - \tfrac{1}{2}\ln \cancel{1} = \boxed{\ln 2}$$

$a\ln b = \ln(b^a)$
so
$\frac{1}{2}\ln 4 = \ln(\sqrt{4})$
$= \ln 2$

$$* = \int \frac{\ln x}{x} dx = (\ln x)^2 - \underbrace{\int \ln x \cdot \tfrac{1}{x} \, dx}_{*}$$

So $2* = (\ln x)^2 + C$
$* = \frac{1}{2}(\ln x)^2 + C$ ✓

8:50 (10 min) 9:00

Example 83. Compute $\displaystyle\iint_R xe^{e^{e^y}}\,dA$, where R is the rectangle $[-1,1]\times[0,4]$.

Hint: Fubini's Theorem.

IDEA? $$\int_{-1}^{1}\int_0^4 x e^{e^{e^y}}\,dy\,dx = ?$$

how to integrate $e^{e^{e^y}}$??

RUN AWAY!!

Better Idea? $$\int_0^4\int_{-1}^{1} x e^{e^{e^y}}\,dx\,dy = \int_0^4 e^{e^{e^y}}\tfrac{1}{2}x^2\Big|_{-1}^{1}\,dy$$

CONSTANT!!

0

$$= \int_0^4 e^{e^{e^y}}\left(\tfrac{1}{2}(1)^2-\tfrac{1}{2}(-1)^2\right)dy$$

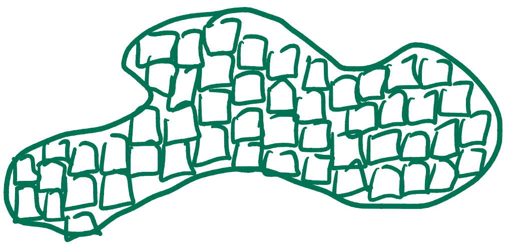
Never been so happy to get a zero from a calculation lol.

$$= \int_0^4 0\,dy$$

$$= \boxed{0}$$ hooray.

§15.2 Double Integrals on General Regions

Question: What if the region R we wish to integrate over is not a rectangle?

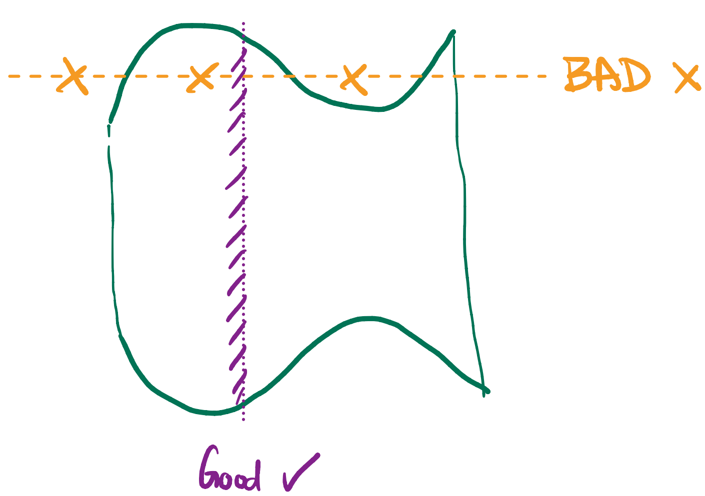


Same idea, just use little rectangles to approximate the region & height over each rectangle is $f(x_k, y_k)$.

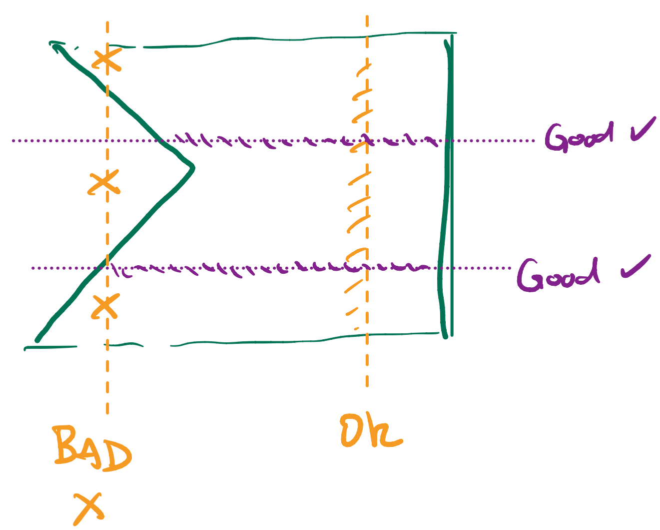
Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

IDEA: When choosing x first or y first it's more important now to choose wisely!!

Vertically simple R:



Horizontally simple R:

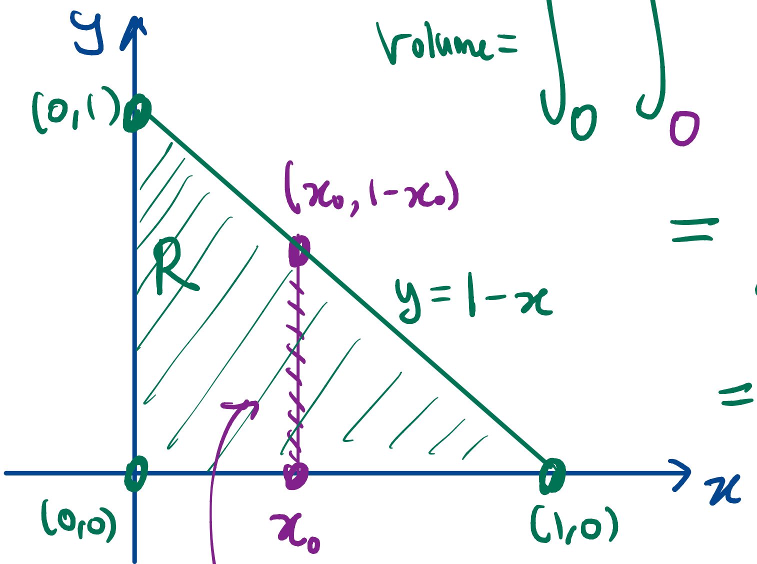


Not impossible to deal w/ but will need multiple integrals for each component yuck!

$\frac{1}{2}(1-x)^2 = \frac{1}{2}(1-2x+x^2) = \frac{1}{2} - x + \frac{1}{2}x^2$

Example 84. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple: ✓



For each $x_0 \in [0,1]$
y-value is between 0 and $1-x_0$

$$\text{volume} = \int_0^1 \int_0^{1-x} \underbrace{2-x-y}_{\text{function to integrate "height"}} \, dy \, dx$$

$$= \int_0^1 2y - xy - \frac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 2(1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \, dx$$

$$= \int_0^1 2 - 2x \cancel{-x} + x^2 - \frac{1}{2} \cancel{+x} - \frac{1}{2}x^2 \, dx$$

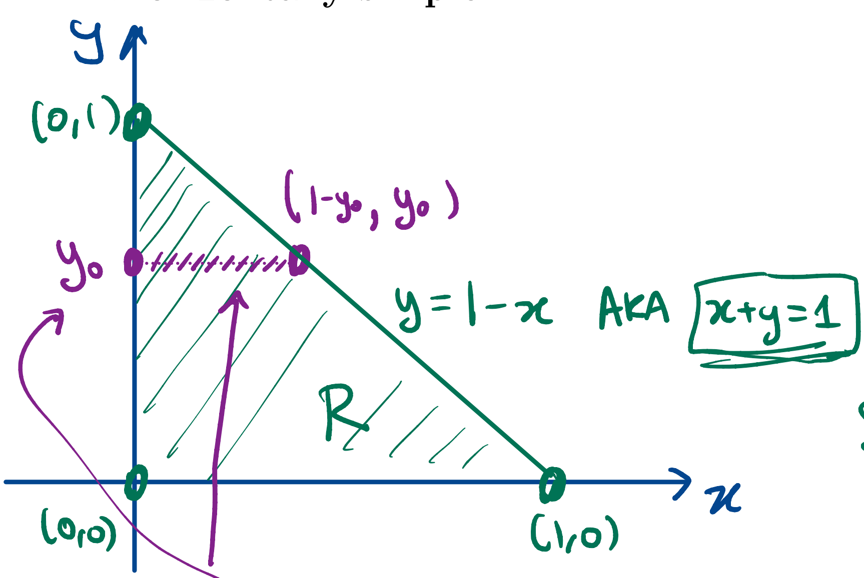
$$= \int_0^1 \frac{3}{2} - 2x + \frac{1}{2}x^2 \, dx$$

$$= \frac{3}{2}x - x^2 + \frac{1}{6}x^3 \Big|_0^1$$

$$= \left(\frac{3}{2} - 1 + \frac{1}{6}\right) - 0 = \frac{1}{2} + \frac{1}{6}$$

$$= \boxed{2/3}$$

Horizontally simple:



For each $y_0 \in [0,1]$
x-value is between 0 and $1-y_0$

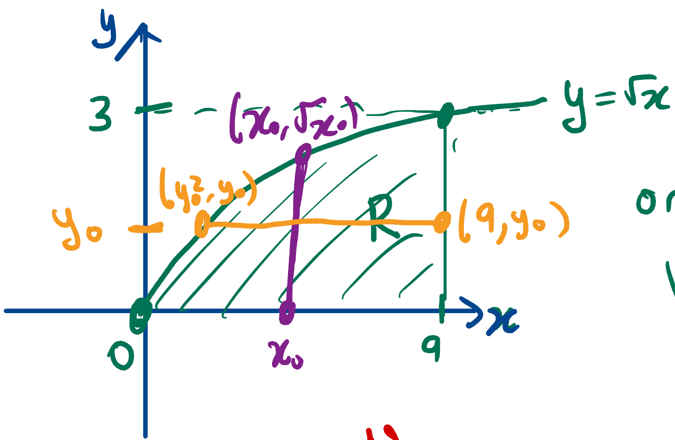
So volume is

$$\int_0^1 \int_0^{1-y_0} 2-x-y \, dx \, dy$$

$$= \boxed{2/3}$$ (same ans b/c same volume ☺)

9:25 (15 min) 9:40

Example 85. Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.



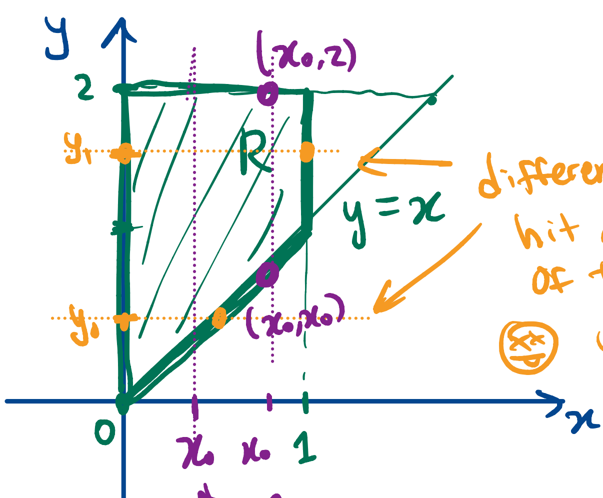
$$\text{Volume} = \int_0^9 \int_0^{\sqrt{x}} 1\ dy\ dx \quad \text{(vertical)}$$

or

$$\text{Volume} = \int_0^3 \int_{y^2}^9 1\ dx\ dy \quad \text{(horizontal)}$$

NOTE: outer integral must NOT have variable limits!!

Example 86. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\ dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.



different y_0's hit different sides of the region R

yuck!

all $x_0 \in [0,1)$ hit the same top & bot side of R.

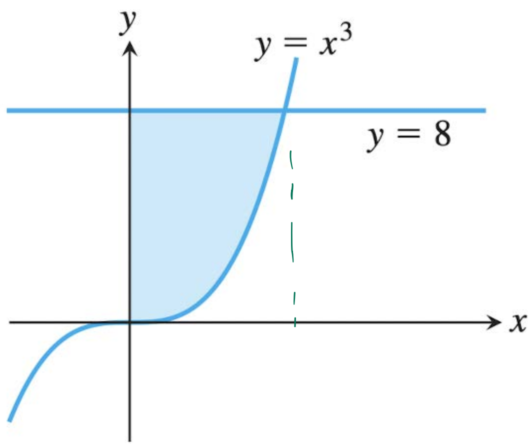
yay tré cool!

$$\text{Volume} = \int_0^1 \int_x^2 6x^2y\ dy\ dx$$

pick an x-value in the interval $[0,1)$

Then integrate for y-values in the range $[x, 2]$

Example 87. *You try it!* Write the two iterated integrals for $\iint_R 1 \; dA$ for the region R which is bounded by $x = 0, y = 8$, and $y = x^3$.



(Vertical)

$$\text{volume} = \int_0^2 \int_{x^3}^8 1 \; dy \, dx$$

(horizontal)

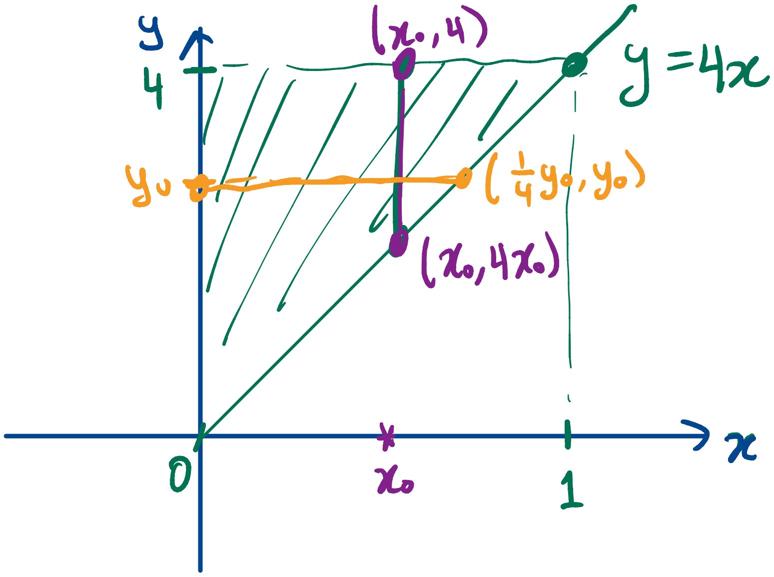
$$\text{volume} = \int_0^8 \int_0^{\sqrt[3]{y}} 1 \; dx \, dy.$$

9:45 (5 min) 9:50

Example 88. Sketch the region of integration for the integral

$$V = \int_0^1 \int_{4x}^4 f(x, y)\, dy\, dx. \quad \text{(vertical)}$$

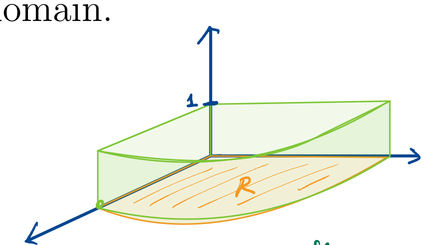
Then write an equivalent iterated integral in the order $dx\, dy$.



$$V = \int_0^4 \int_0^{1/4\, y_0} f(x,y)\, dx\, dy. \quad \text{(horizontal)}$$

§15.3 Area & Average Value

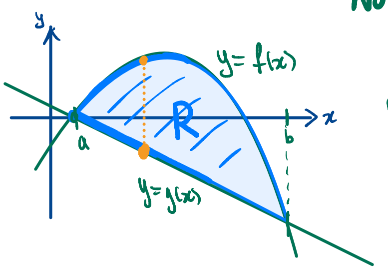
Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.



Area R * 1 = $\iint_R 1\,dA$
base height "volume"

Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \iint_R 1\,dA$$



NOTE:

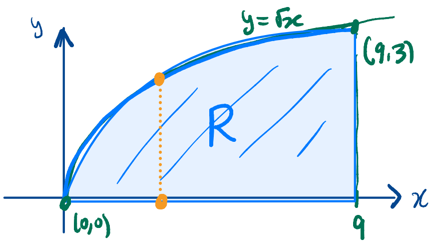
$$\int_a^b \int_{g(x)}^{f(x)} 1\,dy\,dx = \int_a^b y \Big|_{g(x)}^{f(x)} dx$$

$$= \int_a^b f(x) - g(x)\,dx$$

TOP − BOT

This is just the old formula from Calc II !!

Example 89. Find the area of the region R bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.



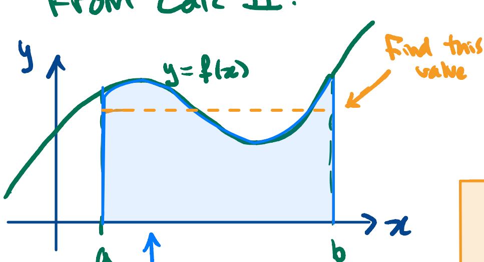
$$\text{Area} = \int_0^9 \int_0^{\sqrt{x}} 1\,dy\,dx = \int_0^9 y \Big|_0^{\sqrt{x}} dx = \int_0^9 \sqrt{x}\,dx$$

$$= \frac{2}{3} x^{3/2} \Big|_0^9 = \frac{2}{3} 9^{3/2} - 0 = \frac{2}{3} * 27 = 2 * 9$$

$$= \boxed{18}$$

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

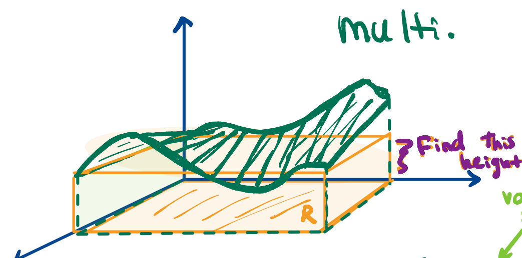
From Calc II.



blue area should match orange area

$$f_{avg} = \frac{1}{\text{Area}R} \iint_R f(x,y)\,dA$$

Same idea for multi.



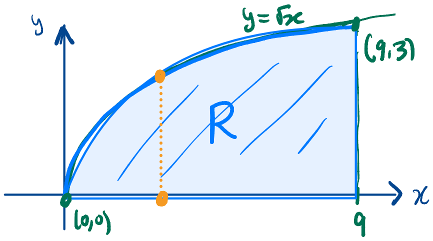
volume of solid over R w/ top height z = f(x,y)

So $(\text{Avg}_{[a,b]} f(x)) *$ length of $[a,b]$
$= \int_a^b f(x)\,dx$

Want Area R * height $= \iint_R f(x,y)\,dA$

The "AVG VALUE" of f over R

Example 90. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x,y) = 4xy^2$.



Area R = 18 From previous page, so

$$\text{Avg}_R f = \frac{1}{\text{Area}R} * \text{Vol}_R f$$

$$\text{Vol}_R f = \int_0^9 \int_0^{\sqrt{x}} 4xy^2\, dy\, dx = \int_0^9 \frac{4}{3} xy^3 \Big|_0^{\sqrt{x}} dx$$

$$= \int_0^9 \frac{4}{3} x\left((\sqrt{x})^3 - 0\right) dx = \int_0^9 \frac{4}{3} x \cdot x^{3/2}\, dx$$

$$= \int_0^9 \frac{4}{3} x^{5/2}\, dx = \frac{4}{3} * \frac{2}{7} x^{7/2} \Big|_0^9 = \frac{8}{21}\left(9^{7/2} - 0\right)$$

$$= \frac{8}{21} 3^7 = \frac{8}{7} 3^6$$

$$\text{So } \text{Avg}_R f = \frac{1}{18} * \text{Vol} = \frac{1}{\cancel{2}\cancel{18}} * \frac{\cancel{8}^{4}}{7} 3^{\cancel{6}^{4}} = \frac{4}{7} * 3^4 = \frac{4*81}{7}$$

$$= \boxed{\frac{324}{7}}$$

Example 91. *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

$$\text{Vol} = \int_0^2\int_0^2 x^2+y^2\,dy\,dx = \int_0^2 x^2y + \tfrac{1}{3}y^3\Big|_0^2$$

$$= \int_0^2 2x^2+\tfrac{8}{3}\,dx = \tfrac{2}{3}x^3+\tfrac{8}{3}x\Big|_0^2 = \tfrac{16}{3}+\tfrac{16}{3} = 32/3$$

and Area R = 4

So $\text{Avg}_R(f) = \frac{1}{\text{Area }R} * \text{Vol} = \frac{1}{4} * \frac{32}{3} = \boxed{8/3}$

Example 92. Find the average value of the function $f(x,y) = \sin(x+y)$ on (a) the region $R_1 = [0,\pi] \times [0,\pi]$, and (b) the region $R_2 = [0,\pi] \times [0,\pi/2]$.
Hint: choose your order of integration carefully!

(a) $R_1 = [0,\pi] \times [0,\pi]$, $\text{Area}(R_1) = \pi^2$

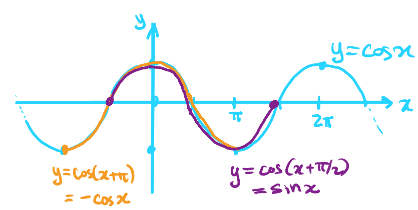
So $\text{Avg}_{R_1}(f) = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx$

$= \frac{1}{\pi^2}\int_0^\pi -\cos(x+y)\Big|_0^\pi dx = \frac{1}{\pi^2}\int_0^\pi -\cos(x+\pi) + \cos(x)\,dx$

$= \frac{1}{\pi^2}\int_0^\pi 2\cos x\,dx = 2\sin x\Big|_0^\pi = 2(\overset{0}{\cancel{\sin\pi}} - \overset{0}{\cancel{\sin 0}})$

$= \boxed{0}$

(b) $R_2 = [0,\pi] \times [0,\pi/2]$ $\text{Area}(R_2) = \pi^2/2$



So $\text{Vol.} = \int_0^\pi \int_0^{\pi/2} \sin(x+y)\,dy\,dx = \int_0^\pi -\cos(x+y)\Big|_0^{\pi/2} dx$

$= \int_0^\pi -\cos(x+\pi/2) + \cos(x)\,dx$

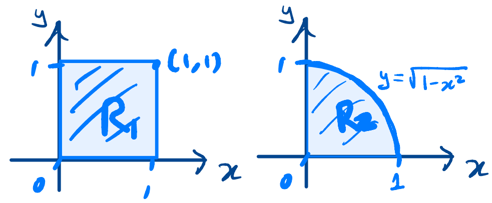
$= \int_0^\pi \sin x + \cos x\,dx = -\cos x + \sin x\Big|_0^\pi$

$= -\cos\pi + \overset{0}{\cancel{\sin\pi}} - (-\cos 0 + \overset{0}{\cancel{\sin 0}}) = -(-1) + 1 = 2$

Must've been neg. on other half of R_1 !!

So $\text{Avg}_{R_2}(f) = \frac{1}{\pi^2/2} * 2 = \boxed{4/\pi^2}$

Example 93. *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of f over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.



$\text{Avg}_{R_1}(f)$ or $\boxed{\text{Avg}_{R_2}(f)}$?

guess b/c ÷ smaller area, so larger total ?

$$V_{R_1} = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^1 = \int_0^1 \tfrac{1}{2}x(1-0)\,dx = \int_0^1 \tfrac{1}{2}x \, dx$$

$$= \tfrac{1}{4}x^2\Big|_0^1 = 1/4 \qquad \text{So } \text{Avg}_{R_1}(f) = \frac{1}{\text{Area } R_1} V_{R_1} = \frac{1}{1} * \frac{1}{4} = \boxed{\frac{1}{4}}$$

$$V_{R_2} = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \tfrac{1}{2}x(1-x^2)\,dx$$

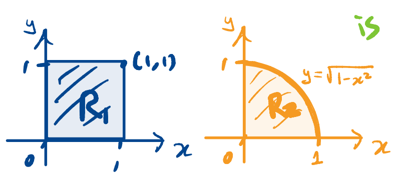
$$= \int_0^1 \tfrac{1}{2}(x - x^3)\,dx = \tfrac{1}{2}\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1 = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2} * \tfrac{1}{4} = 1/8$$

$$\text{So } \text{Avg}_{R_2}(f) = \frac{1}{\text{Area } R_2} V_{R_2} = \frac{1}{\pi/4} * \frac{1}{8} = \boxed{\frac{1}{2\pi}} \approx \boxed{\frac{1}{6}}$$

Well I'll be...

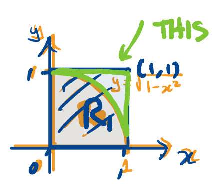
how's that work...

In retrospect, what seems to be happening



is that in the outer region of R_1, (the part not contained in R_2)

THIS BIT HERE



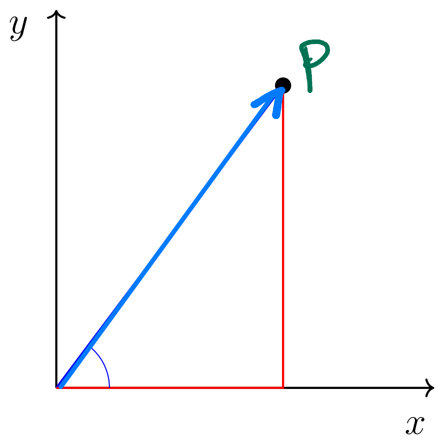
in the part of R_1 not contained in R_2 the value of $f(x,y) = xy$ is "particularly large" since

who woulda think it.

xy is small when x or y is small. And apparently this more than compensates for ÷ by larger area.

§15.4 Double Integrals in Polar Coordinates

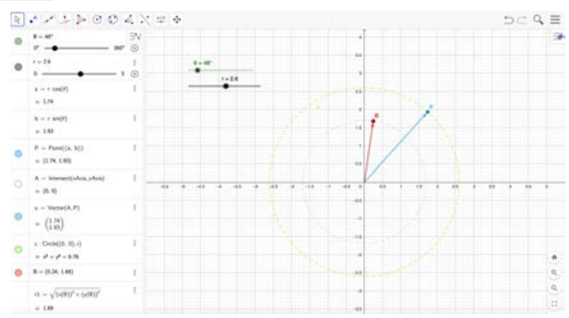
Review of Polar Coordinates



Cartesian coordinates: Give the distances in x and y directions from $(0,0)$

Polar coordinates:

- r = distance from $(0,0)$ to (x,y)
- θ = angle between the ray $\overrightarrow{OP}$ and the positive x-axis



We can use trigonometry to go back and forth.

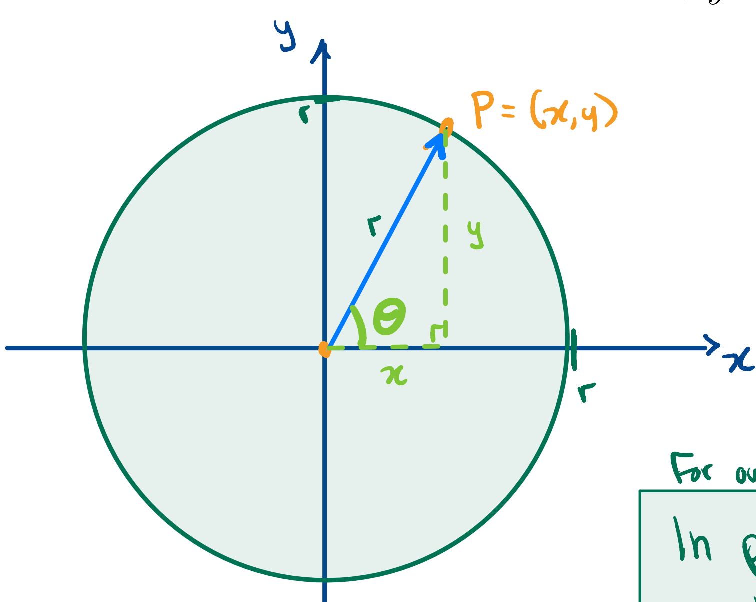
https://www.geogebra.org/classic/thaxxzzp

Polar to Cartesian:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$



Warning: many possible (r,θ) for one particular (x,y).

e.g. $(r,\theta)_P = (-1, \frac{\pi}{4})_P = (1, \frac{5\pi}{4})_P = (-\sqrt{2}/2, -\sqrt{2}/2)_C$

polar coordinates

Cartesian/rectangular coordinates

For our class:

$\boxed{\text{In problems/your solns, please limit } r \geq 0 \text{ \& } \theta \in [0, 2\pi] \text{ or } \theta \in [-\pi, \pi].}$

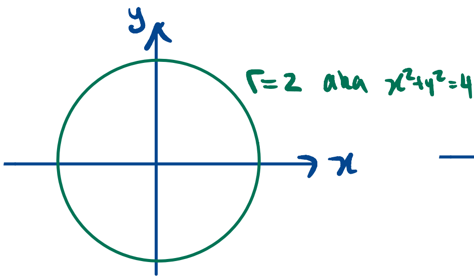
Example 94. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

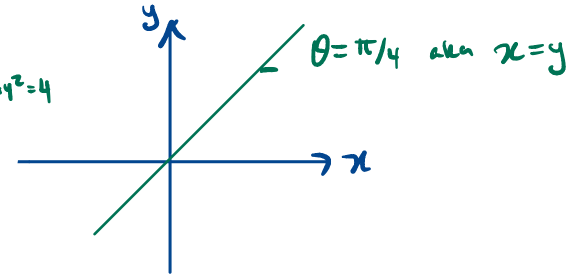
$r = \sqrt{1^2+1^2} = \sqrt{2}$

and $\theta = \tan^{-1}(1/1) = \tan^{-1}(1) = \pi/4$

So $(1,1)_{\mathcal{C}} = \boxed{(\sqrt{2}, \pi/4)_{\mathcal{P}} = (r,\theta)}$

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the xy-plane.**





c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

@ $x = r\cos\theta$, $y = r\sin\theta$

Idea: $f(r,\theta) = r$

Show work: $f(x,y) = \sqrt{x^2+y^2} = \sqrt{(r\cos\theta)^2+(r\sin\theta)^2}$

$= \sqrt{r^2(\cos^2\theta+\sin^2\theta)} = \sqrt{r^2} = r$

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

① $r = \sqrt{x^2+y^2}$

② $r\sin\theta = y \Rightarrow \sin\theta = \frac{y}{r}$

So $r = 2\sin\theta$ becomes

$\sqrt{x^2+y^2} = 2\cdot\frac{y}{\sqrt{x^2+y^2}}$

$\Rightarrow \boxed{x^2+y^2 = 2y}$

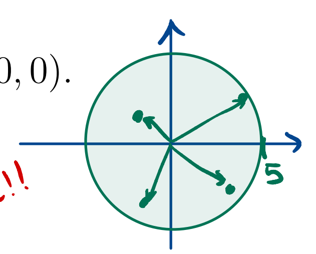
Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

Example 95. Compute the area of the disk of radius 5 centered at (0, 0).

maybe this?

$\text{Area} = \iint_R 1\, dA \overset{??}{\neq} \int_0^5 \int_0^{2\pi} 1\, d\theta\, dr = \int_0^5 \theta\Big|_0^{2\pi} dr$

$= \int_0^5 2\pi\, dr = 2\pi r\Big|_0^5 = \boxed{10\pi}$ didn't work!!



r values in $[0,5]$

θ values in $[0,2\pi)$

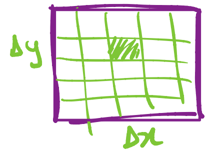
Remember: In polar coordinates, the area form $dA =$???

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r,\theta)$ on R, compute $\iint_R f(r,\theta)\ dA$.

Example 96. $\boxed{\text{Compute the area of the disk of radius 5 centered at } (0,0).}$

Cont.

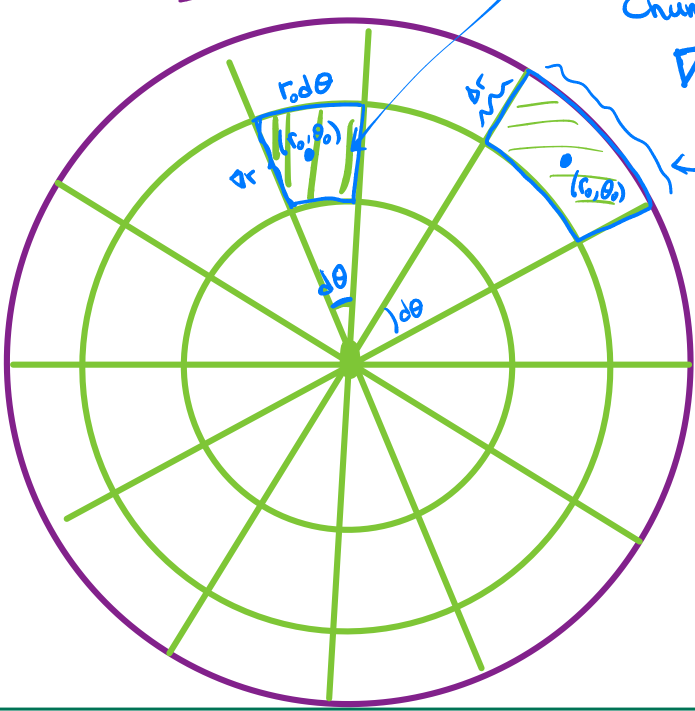
recall.



$dA_k = \Delta x \Delta y$

So $\iint_R 1\, dA = \int_a^b \int_c^d 1\, dx\, dy$

but NOW.



So this little chunk has area

$\Delta A_k = r_1 \Delta\theta\, dr$

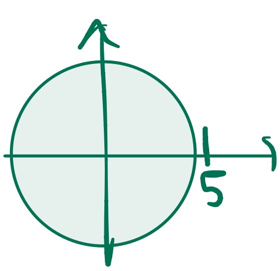
← WAY WIDER NOW!!
the size of the little chunk DEPENDS ON r!!!

need the extra r!!

So $\iint_R 1\, dA = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} 1\ r\, dr\, d\theta$

Let's try it out!

Disk area $\iint_R 1\, dA = \int_0^5 \int_0^{2\pi} 1 * r\, d\theta\, dr$



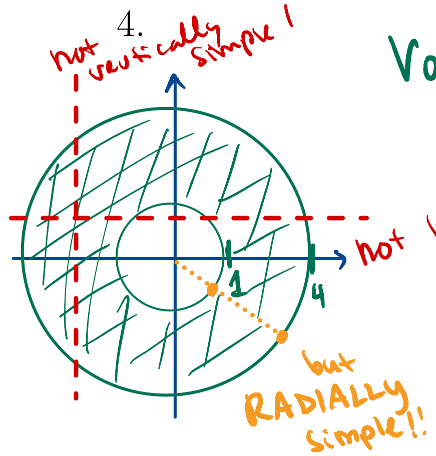
$= \int_0^5 \theta r \Big|_0^{2\pi} dr = \int_0^5 2\pi r\, dr$

$= \pi r^2 \Big|_0^5 = \boxed{25\pi}$ ✓ hooray!

Remember: In polar coordinates, the area form $dA = \boxed{r\, dr\, d\theta}$

Example 96. Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.

convert to polar expression using $x = r\cos\theta$, $y = r\sin\theta$



$$\text{Vol} = \iint_D e^{-(x^2+y^2)}\,dA = \int_1^2\int_0^{2\pi} e^{-r^2}\,\underbrace{r\,d\theta\,dr}_{dA}$$

dA don't forget $\boxed{\text{the } r}$!!!

$$= \int_1^2 r e^{-r^2}\theta\Big|_0^{2\pi}\,dr = \int_1^2 2\pi r e^{-r^2}\,dr$$

u-sub

$\boxed{\begin{array}{l} u = -r^2 \\ du = -2r\,dr \end{array}}$

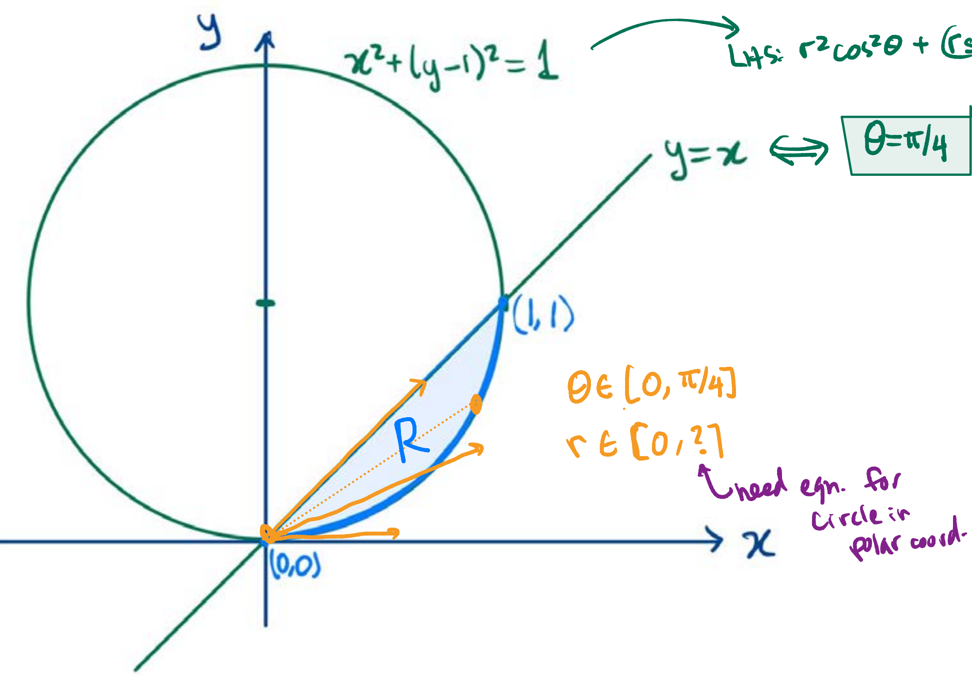
so $-\frac{1}{2}du = r\,dr$

$$= \int_*^* 2\pi e^u * -\tfrac{1}{2}\,du = -\pi e^u\Big|_*^*$$

$$= -\pi e^{-r^2}\Big|_1^2 = -\pi\left(e^{-4} - e^{-1}\right)$$

$$= \boxed{\pi\left(1/e - 1/e^4\right)}$$

Example 97. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.



LHS: $r^2\cos^2\theta + (r\sin\theta - 1)^2 = \overbrace{r^2\cos^2\theta + r^2\sin^2\theta}^{1} - 2r\sin\theta + 1$

$= r^2 - 2\sin\theta + 1$

LHS $= 1$ so

$r^2 - 2\sin\theta + 1 = 1$

$\Rightarrow r^2 - 2\sin\theta = 0$

$\Rightarrow r(r - 2\sin\theta) = 0$

So $r = 0$ or $\boxed{r = 2\sin\theta}$

So $x^2 + (y-1)^2 = 1$ in polar coord. is just $r = 2\sin\theta$.

$$\text{So Area} = \iint_R f(x,y)\,dA = \int_0^{\pi/4}\int_0^{2\sin\theta} 1 * r\,dr\,d\theta = \int_0^{\pi/4} \tfrac{1}{2}r^2\Big|_0^{2\sin\theta}\,d\theta$$

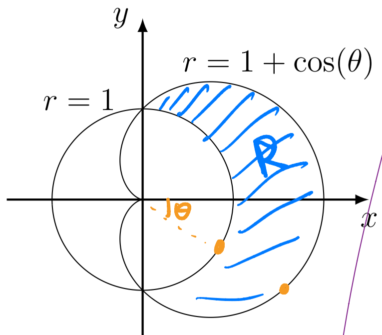
power reducing formula

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$= \int_0^{\pi/4} \tfrac{1}{2}\cdot 4\sin^2\theta\,d\theta = \int_0^{\pi/4} 1 - \cos 2\theta\,d\theta = \theta - \tfrac{1}{2}\sin 2\theta\Big|_0^{\pi/4}$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\overset{1}{\cancel{\sin\frac{\pi}{2}}}\right] - \left[0 - \frac{1}{2}\overset{0}{\cancel{\sin 0}}\right) = \boxed{\frac{\pi}{4} - \frac{1}{2}}$$

Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



$\theta \in [-\pi/2, \pi/2]$
$r \in [1, 1+\cos\theta]$

So $f(x,y) = x$
$f(r,\theta) = r\cos\theta$

don't forget r!!

So $\iint_R f(x,y)\, dA = \boxed{\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\cos\theta * r\, dr\, d\theta}$

Stop here ☺

$= \int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \cos\theta \Big|_1^{1+\cos\theta} d\theta$

oh lol. oops.

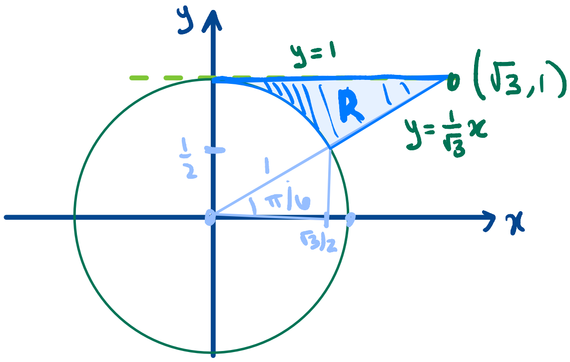
$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\left[(1+\cos\theta)^3\cos\theta - \cos\theta\right] d\theta$

$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta\left[(1+\cos\theta)^3 - 1\right] d\theta$

hmmm...

$= \ldots = \boxed{5\pi/8}$

Example 100. Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. Then, evaluate the original integral to find the value of $\iint_R f(x,y)\, dA$.



$$\int_{\pi/6}^{\pi/2}\int_{1}^{\csc\theta} r^2\cos\theta\; dr\; d\theta$$

$y = r\sin\theta \Leftrightarrow r = y\csc\theta$

So $r=\csc\theta \Leftrightarrow y=1.$

$f(r,\theta) = r\cos\theta$ Since $dA = r * dr\,d\theta$

$= \sqrt{x^2+y^2}$

$$\text{So Vol} = \boxed{\int_0^{\sqrt{3}/2}\int_{\sqrt{1-x^2}}^{1} x \; dy\, dx + \int_{\sqrt{3}/2}^{\sqrt{3}}\int_{\frac{1}{\sqrt{3}}x}^{1} x \; dy\, dx}$$

$$\text{Vol} = \int_{\pi/6}^{\pi/2}\int_1^{\csc\theta} r^2\cos\theta\, dr\, d\theta = \int_{\pi/6}^{\pi/2} \frac{1}{3}r^3\cos\theta\Big|_1^{\csc\theta} d\theta = \int_{\pi/6}^{\pi/2}\frac{1}{3}\csc^3\theta\cos\theta\, d\theta$$

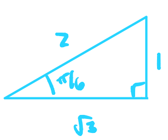
$$= \frac{1}{3}\int_{\pi/6}^{\pi/2}\frac{\cos\theta}{\sin^3\theta}\, d\theta = \frac{1}{3}\int_{1/2}^{1}\frac{1}{u^3}\, du = \frac{1}{3} * \frac{-1}{2}u^{-2}\Big|_{1/2}^{1} = -\frac{1}{6}\left(1-\frac{1}{(1/2)^2}\right)$$

u-sub box

$\boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array}}$

$\theta = \pi/6 \Rightarrow \sin\theta = 1/2$

$\theta = \pi/2 \Rightarrow \sin\theta = 1$



$$= -\frac{1}{6}(1-4) = 3/6$$

$$= \boxed{1/2}$$

Tips and tricks

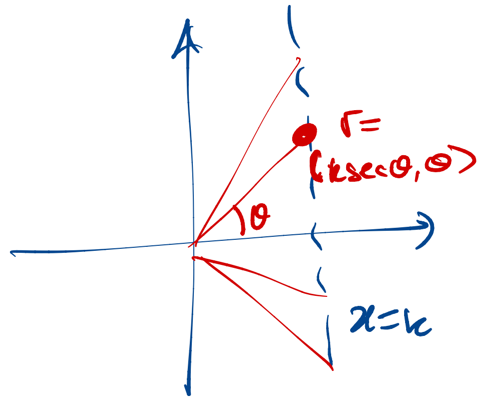
① $x = r\cos\theta$
② $y = r\sin\theta$

For horizontal lines such as $x = 2$:

use ① get $2 = r\cos\theta \Rightarrow r = 2\sec\theta$

in general

if $x = k \Rightarrow r = k\sec\theta$



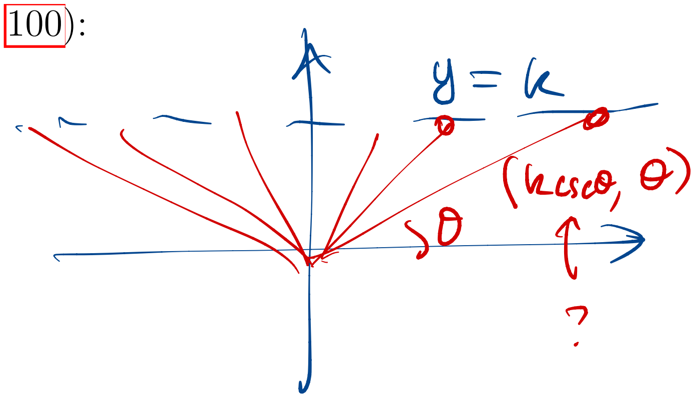
For vertical lines such as $y = 1$ (e.g., Example 100):

use ② get

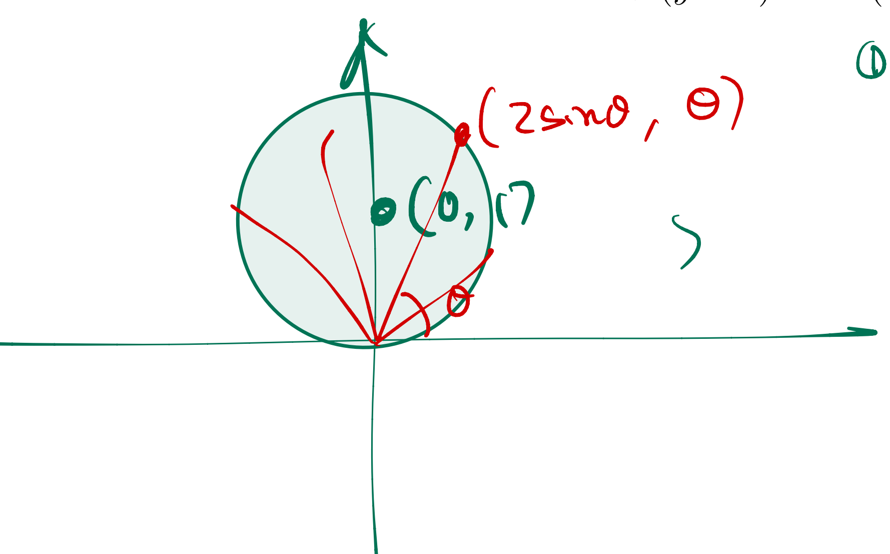
$1 = r\sin\theta$

so $r = \csc\theta$

So in general $y = k$
$\Rightarrow r = k\csc\theta$



For off-set circles such as $x^2 + (y-1)^2 = 1$ (e.g., Example 98):



① & ② CTS.

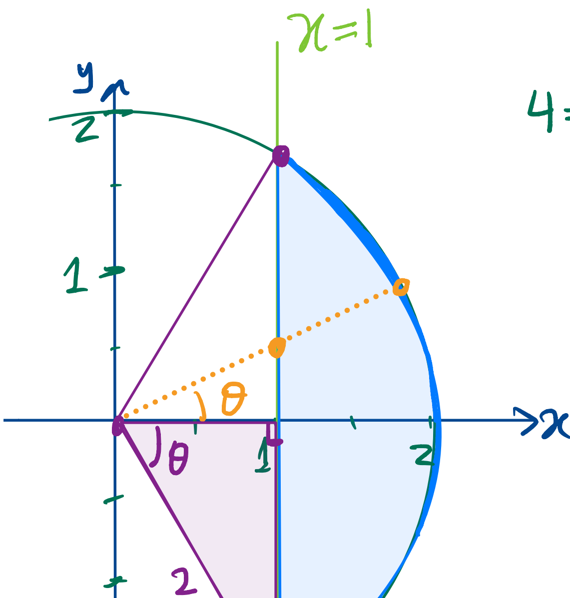
$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$

$\Rightarrow r^2 - 2r\sin\theta = 0$

$\Rightarrow \boxed{r = 2\sin\theta}$

Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

The θ-value is between $(1,-\sqrt{3})$ & $(1,\sqrt{3})$ so $\theta \in [-\pi/3, \pi/3]$.



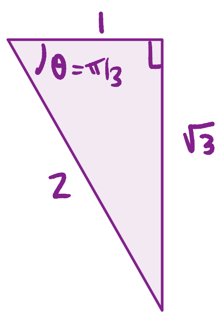
$$4 = x^2 + y^2 = r^2 \Rightarrow r = \pm 2, \ r \geq 0$$

so $r = 2$ on circle.

$$x = r\cos\theta \Rightarrow r = \frac{x}{\cos\theta} = x\sec\theta$$

@ $x=1$ $\quad r = \sec\theta$ on line $x = 2$.

So $\theta \in [-\pi/3, \pi/3]$
and $r \in [\sec\theta, 2]$



So $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$

$$\text{Area} = \int_{-\pi/3}^{\pi/3} \int_{\sec\theta}^{2} 1 * r \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2} r^2 \Big|_{\sec\theta}^{2} d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(\sec^2\theta - 4\right) d\theta$$

$$= \frac{1}{2}\tan\theta - 2\theta \Big|_{-\pi/3}^{\pi/3} = \tan(\pi/3) - 4 * \pi/3 = \boxed{\sqrt{3} - 4\pi/3}$$

Function is odd

Math 2551 Worksheet: Exam 2 Review

1. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0)? Give reasons for your answers.
   (a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.
   (b) The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.
   (c) The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f.
   (d) At (x_0, y_0), the vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.
2. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi), x = e^t$, and $y = \ln(t + 1)$.
3. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.
4. Test the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points and find the function's value at these points.
5. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.
6. Evaluate the integral $\int_0^1 \int_{2y}^2 4\cos(x^2)\, dx\, dy$. Describe why you made any choices you did in the course of evaluating this integral.
7. If $f(x, y) \geq 2$ for all (x, y), is it possible that the average value of $f(x, y)$ on a unit disk centered at the origin is $\frac{2}{\pi}$?
8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.