✓✓???

§15.5-15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

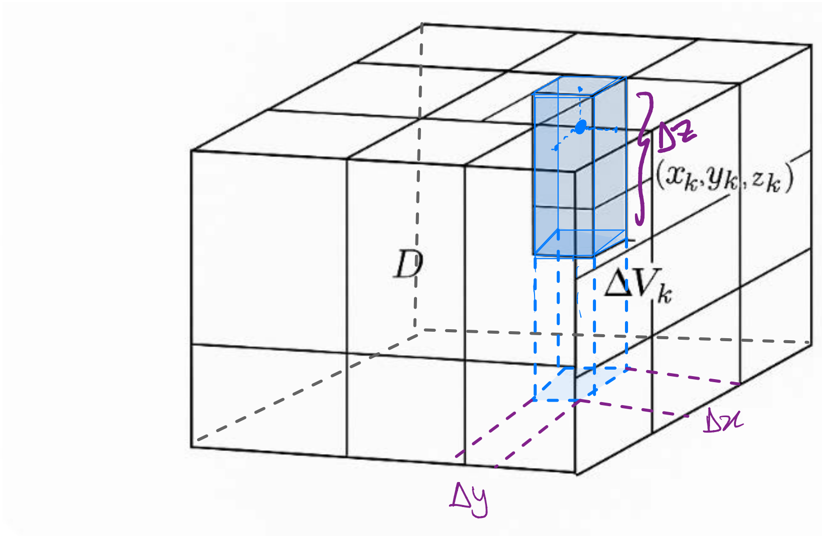
$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

https://strawpoll.com/61gD939KLZw

$f : \mathbb{R}^3 \to \mathbb{R}$

$D \subseteq \mathbb{R}^3$ domain of f



$$D = \underbrace{[a,b]\times[c,d]\times[e,g]}$$

plane rectangle

and

rectangular prism

and

box

$\Delta V_k = \Delta x \Delta y \Delta z$

Taking the limit gives a

triple integral : $\iiint_D f(x,y,z)\,dV$

Important special case:

$$\iiint_D 1\,dV = \text{Volume}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

$$\int_a^b\int_c^d\int_e^g f(x,y,z)\,dz\,dy\,dx = \int_c^d\int_e^g\int_a^b f(x,y,z)\,dx\,dz\,dy.$$

Other important spatial applications:

doesn't seem to appear in any sample exams?

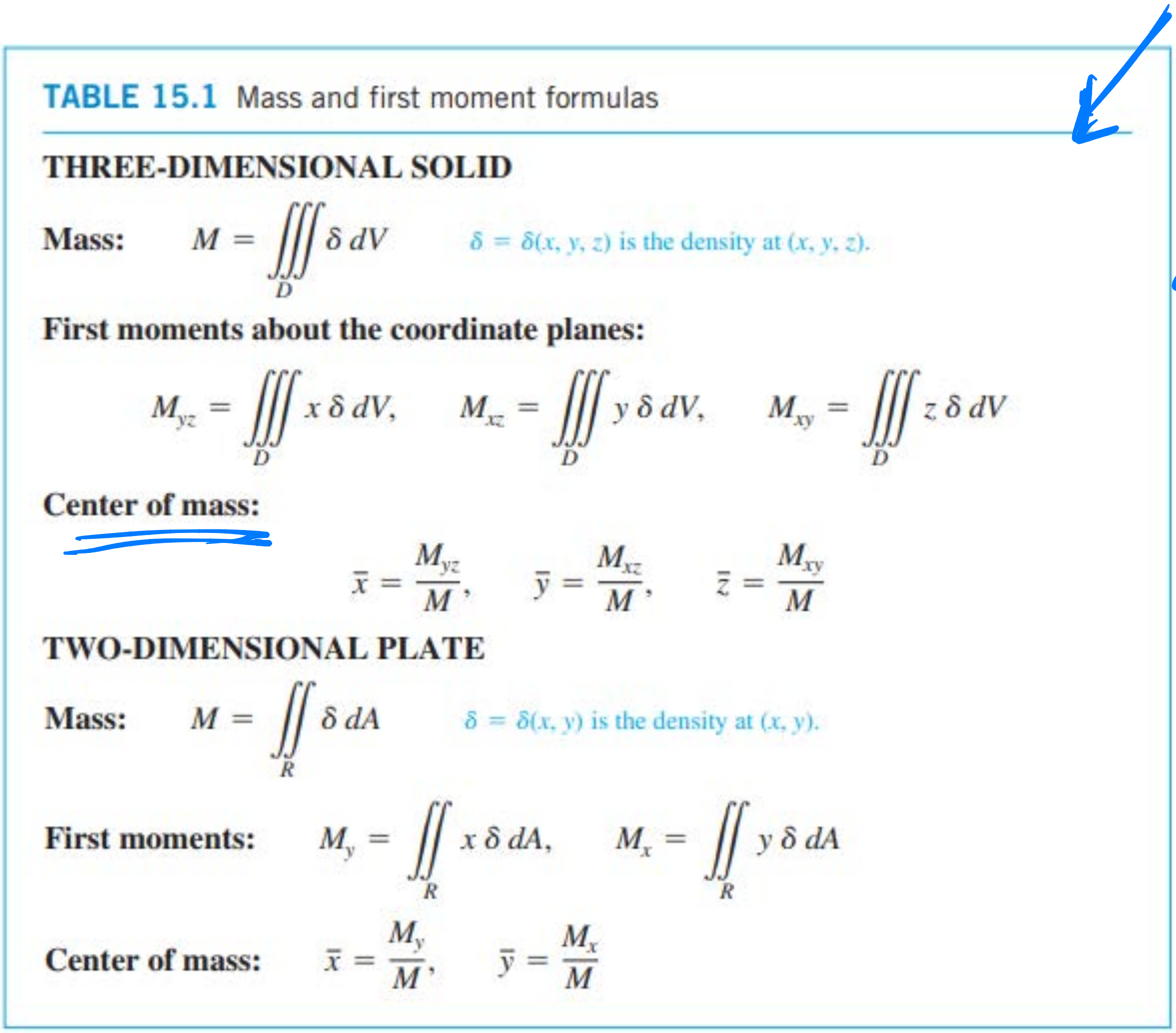


TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta\,dV$ $\quad \delta = \delta(x,y,z)$ is the density at (x,y,z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\,dV, \qquad M_{xz} = \iiint_D y\,\delta\,dV, \qquad M_{xy} = \iiint_D z\,\delta\,dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta\,dA$ $\quad \delta = \delta(x,y)$ is the density at (x,y).

First moments: $M_y = \iint_R x\,\delta\,dA, \qquad M_x = \iint_R y\,\delta\,dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$

Example 102. 1. **How to do the computation:**

Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} 1 \; dz\, dy\, dx. = \int_0^1 \int_0^{2-x} z \Big|_0^{2-x-y} dy\, dx$

$$= \int_0^1 \int_0^{2-x} 2-x-y \; dy\, dx = \int_0^1 2y - xy - \frac{1}{2}y^2 \Big|_0^{2-x} dx$$

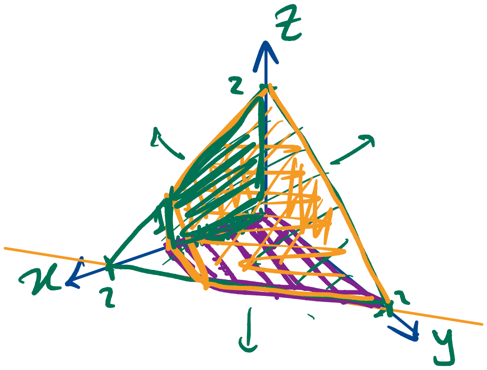
$$= \int_0^1 2(2-x) - x(2-x) - \frac{1}{2}(2-x)^2 \; dx = \int_0^1 (2-x)\left[(2-x) - \frac{1}{2}(2-x)\right] dx$$

$$= \int_0^1 (2-x)^2 \left[1 - \frac{1}{2}\right] dx = \frac{1}{2}\int_0^1 (2-x)^2 \, dx = \frac{1}{2}\int_0^1 4-4x+x^2 \, dx$$

$$= \frac{1}{2}\left(4x - 2x^2 + \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2}\left(\left(4-2+\frac{1}{3}\right) - (0-0+0)\right)$$

$$= \frac{1}{2}\cdot\frac{7}{3} = \boxed{7/6}$$

2. **What does it mean:** What shape is this the volume of?



line $\begin{cases} z=0 \\ y=2-x \end{cases}$

$$\int_0^1 \int_0^{2-x} \int_0^{\boxed{2-x-y}} 1 \; dz\, dy\, dx.$$

This is a $\boxed{}$ $\rightarrow z = 2-x-y$

$\Rightarrow x+y+z=2$

$x \in [0,1]$

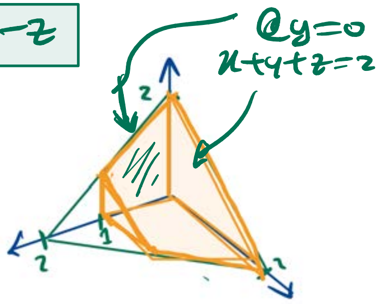
$y \in [0, 2-x]$

$z \in [0, 2-x-y]$

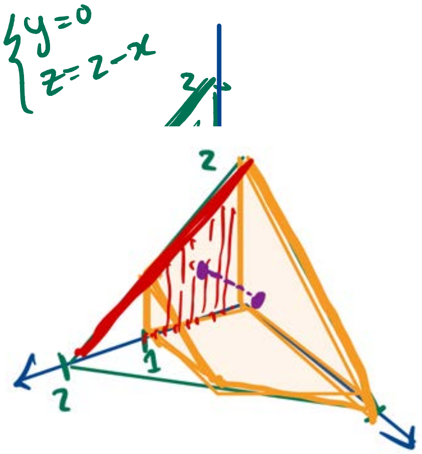
3. **How to reorder the differentials:** Write an equivalent iterated integral in the order $dy\; dz\; dx$.

$$\int_0^1 \int_0^{2-x} \int_0^{\boxed{\cancel{2-x-y}} \rightarrow ???\ \boxed{2-x-z}} 1 \; dy\, dz\, dx$$

@y=0 $x+y+z=2$



line $\begin{cases} y=0 \\ z=2-x \end{cases}$



rewrite

$x+y+z=2$

Solve for $y = 2-x-z$

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\; dy\; dz$

(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\; dx\; dy\; dz$

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz$

$$= \int_1^e \int_1^{e^2} \frac{1}{yz} \ln(x) \Big|_1^{e^3} dy\, dz = \int_1^e \int_1^{e^2} \frac{1}{yz} * 3 \; dy\, dz$$

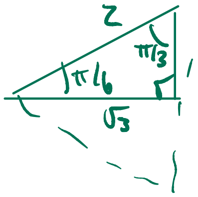
$$= \int_1^e \frac{3}{z} \ln y \Big|_1^{e^2} dz = \int_1^e \frac{3}{z}(2-0)\, dz = 6 \ln z \Big|_1^e = 6-0$$

$$= \boxed{6}$$

(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\, dx\, dy\, dz$

$$= \int_0^{\pi/3} \int_0^1 y \sin z \cdot x \Big|_{-2}^{3} dy\, dz = \int_0^{\pi/3} \int_0^1 5y \sin z\, dy\, dz$$

$$= \int_0^{\pi/3} \frac{5}{2} \sin z \cdot y^2 \Big|_0^1 dz = \int_0^{\pi/3} \frac{5}{2} \sin z\, dz$$

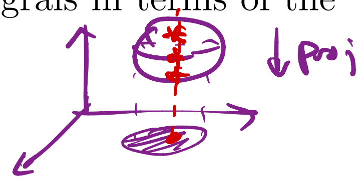


$$= -\frac{5}{2} \cos z \Big|_0^{\pi/3} = -\frac{5}{2}\left[\cos(\pi/3) - \cos(0)\right]$$

$$= -\frac{5}{2}\left(\frac{1}{2} - 1\right)$$

$$= \boxed{5/4}$$

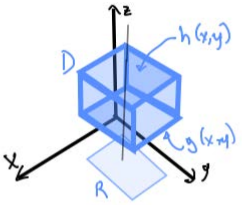
We will think about converting triple integrals to iterated integrals in terms of the projection of D on one of the coordinate planes.



$(x,y) \in \text{Proj}_z(D)$ iff $(x,y,z) \in D$ for some $z \in \mathbb{R}$

Case 1: z-**simple**) region. If R is the projection of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x,y)$ and $z = g(x,y)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz \right) dy\, dx$$



Case 2: y-**simple**) region. If R is the projection of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x,z)$ and $y = g(x,z)$, then

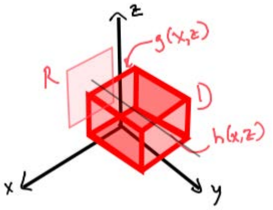
dx dz also ok.

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\, dy \right) dz\, dx$$

no y's here either

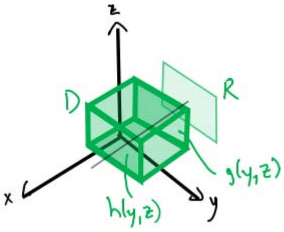
no y's in the limits

y-variable last



Case 3: x-**simple**) region. If R is the projection of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y,z)$ and $x = g(y,z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\, dx \right) dz\, dy$$

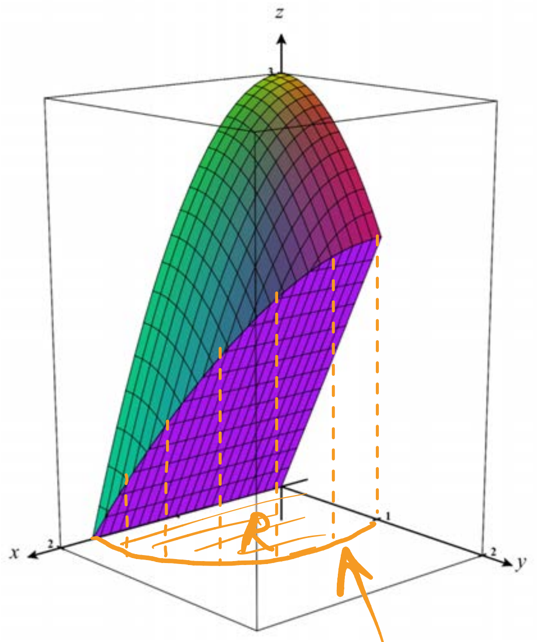


Example 104. Write an integral for the mass of the solid D in the first octant with $2y \le z \le 3 - x^2 - y^2$ with density $\delta(x, y, z) = x^2y + 0.1$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

Case 1:

z-simple.

$$\iint_R \int_{g(x,y)}^{h(x,y)} \delta(x,y,z)\, dz\, dy\, dx = M$$



How to find R projection of D solid?

⊛ $(x,y) \in R$ if $(x,y,z) \in D$ for some $z \in \mathbb{R}$

Top surface $z = 3 - x^2 - y^2$

Bot plate/plane $z = 2y$

Intersect top & bot

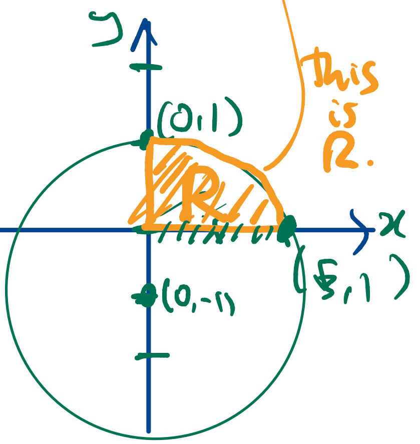
$3 - x^2 - y^2 = 2y$

$\Rightarrow 3 = x^2 + y^2 + 2y$

$\Rightarrow 4 = x^2 + (y+1)^2$

↑

$x = \sqrt{4 - (y+1)^2}$



$$\boxed{M = \int_0^1 \int_0^{\sqrt{4-(y+1)^2}} \int_{2y}^{3-x^2-y^2} x^2y + 0.1\, dz\, dx\, dy}$$

intersection w/ x-axis

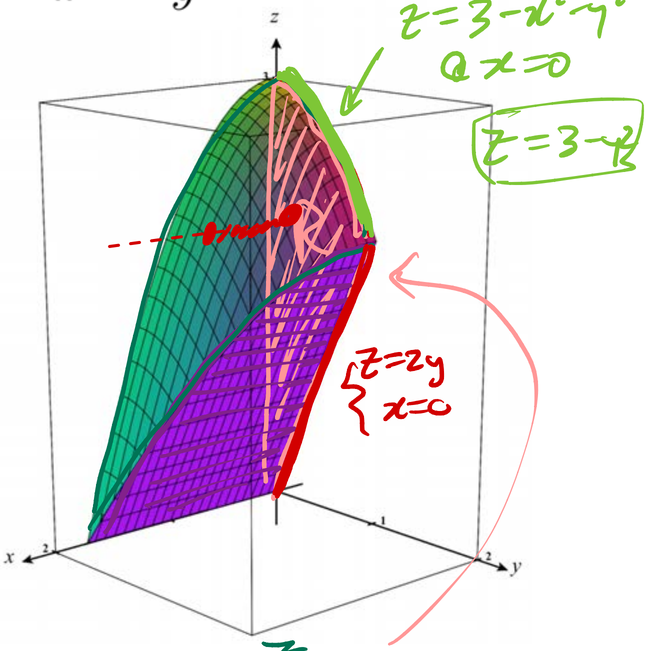
@ $y = 0$

$4 = x^2 + 1$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$

Example 104 (cont.) D: $2y \le z \le 3 - x^2 - y^2$



Case 3:

(b) x-simple

$$M = \iint_R \int_{g(y,z)}^{h(y,z)} \delta(x,y,z)\,dx\,dz\,dy$$

Top is $z=3-x^2-y^2$

Bot is 0

intersection

$z=2y$

$z=3-y^2$

(set $z=z$)

$2y=3-y^2$

$\Rightarrow y^2+2y-3=0$

$\Rightarrow (y+3)(y-1)=0$

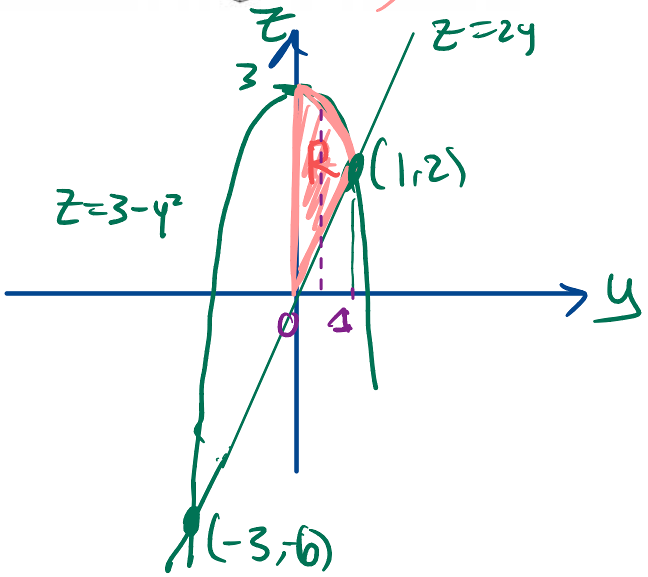
$\Rightarrow y=-3,1$

Separate solve for x

$z=3-x^2-y^2$

$\Rightarrow x^2=3-y^2-z$

$\Rightarrow x=\sqrt{3-y^2-z}$



$\sqrt{3-y^2-z}$

??

$$M=\int_0^1\int_{2y}^{3-y^2}\int_0^{\cancel{3-x^2-y^2}} x^2y+0.1\,dx\,dz\,dy$$

$y\in[0,1]$

$z\in[2y,3-y^2]$

$x\in[0,?]$

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers) (Starting w/ innermost)

Rule 1: Choose a variable appearing exactly twice for the next integral. Counting multiplicity!!

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

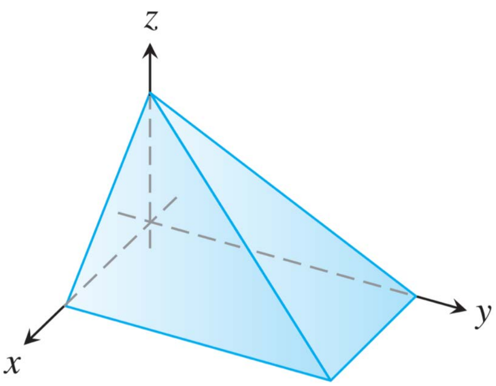
Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 105. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1,\ y + 2z = 2$.



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Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 105. *You try it!* Find the volume of the region in the first quadrant octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$. $, x=0, y=0$

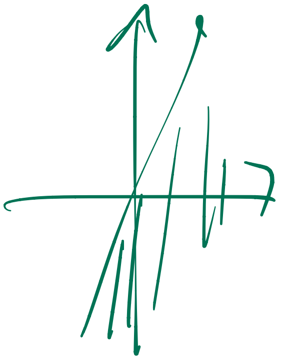
$z=0$

R1 z twice $z = 1 - x$ R6 Try $1 - x \le z \le 1 - \frac{1}{2}y$

$z = 1 - \frac{1}{2}y$ R2 cross out & R3 new

$1 - x \le 1 - \frac{1}{2}y$

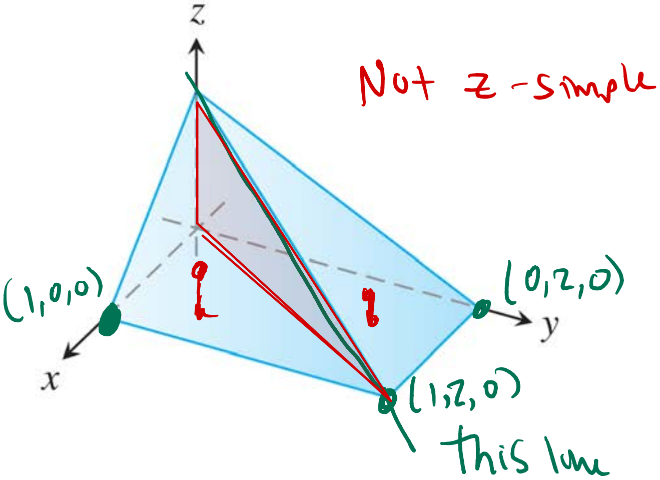
$\frac{1}{2}y \le x \Rightarrow y \le 2x$



??

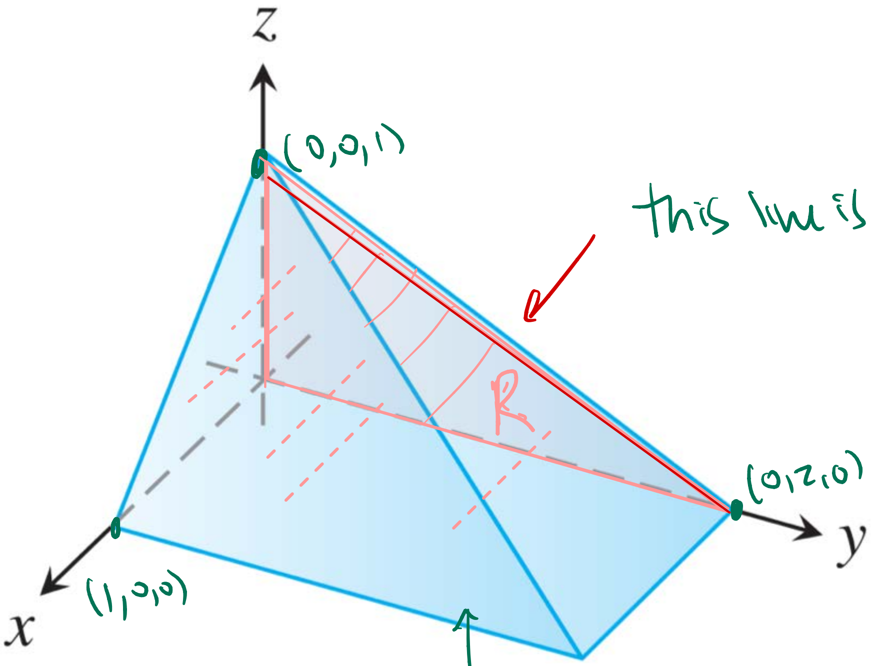
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$

quadrant

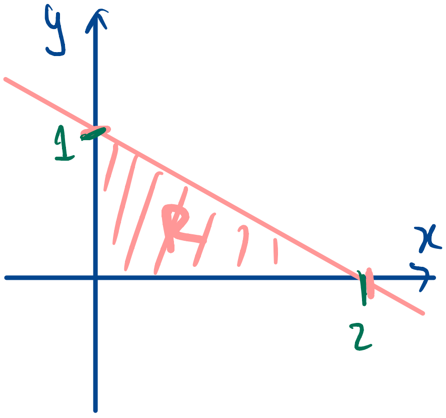


Not z-simple!

This line $\begin{cases} x+z=1 \\ y+2z=2 \end{cases}$ @ $z=0$ $x=1$ $y=2$ @ $x=y=0$ $z=1$



this line is $\begin{cases} x=0 \\ y+2z=2 \end{cases} \Rightarrow \begin{cases} x=0 \\ z=1-\frac{1}{2}y \end{cases}$



this plane is $x+z=1$

$\Leftrightarrow x=1-z$

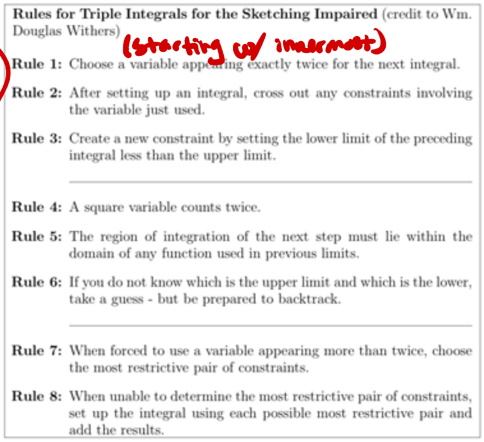
So

$$\boxed{\text{Vol} = \int_0^2 \int_0^{1-\frac{1}{2}y} \int_0^{1-z} 1 \, dx \, dz \, dy}$$

Example 106. Set up an integral for the volume of the region D defined by

$$\cancel{\overset{\textcircled{1}}{x}+y^2 \le 8}, \quad \cancel{y^2+2z^2 \le \overset{\textcircled{2}}{x}}, \quad \boxed{y \ge 0}$$

??



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Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Rule 1: pick x which shows up twice

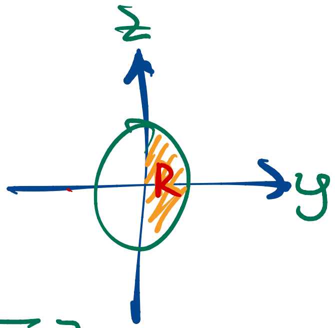
$$\text{bot} \rightarrow \overset{\textcircled{2}}{y^2+2z^2} \le x \le \overset{\textcircled{1}}{8-y^2} \leftarrow \text{top}$$

Rule 2: Cross out the constraints you just used.

Rule 3: Make new constraint bot ≤ top

$$R: \; y^2+2z^2 \le 8-y^2 \Rightarrow 2y^2+2z^2 \le 8$$

$$\Rightarrow y^2+z^2 \le 4$$



$$y \in [0, 2]$$

$$z \in \left[-\sqrt{4-y^2}, \sqrt{4-y^2}\right]$$

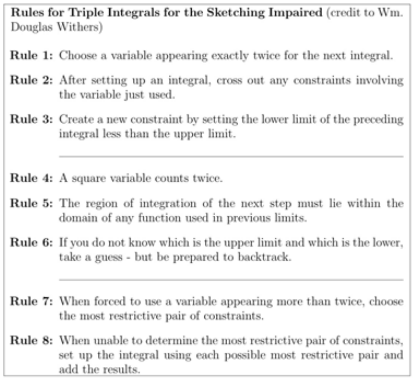
$$x \in \left[y^2+2z^2, 8-y^2\right]$$

$$\boxed{Vol = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{y^2+2z^2}^{8-y^2} 1 \, dx \, dz \, dy}$$

Example 107. Set up a triple iterated integral for the triple integral of $f(x, y, z) = x^3y$ over the region D bounded by

$\{f(x,y,z)\}$

$$x^2 + y^2 = 1, \quad ① \cancel{z = 0}, \quad ② \cancel{x + y + z = 2}.$$



Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

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Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Note x appears 3 times
y appears 3 times
z appears 2 times

Rule 1: Choose z for innermost integral.

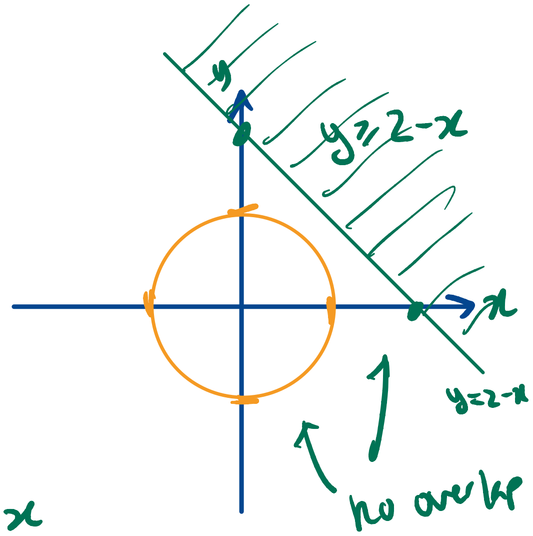
Next: $z = 2-x-y$ or $z = 0$ which on top?

⊗ Rule 6: Guess $2-x-y \leq z \leq 0$

Rule 2: Cross out constraints ① & ②

Rule 3: Make new constraint $2-x-y \leq 0$

$\Leftrightarrow y \geq 2-x$

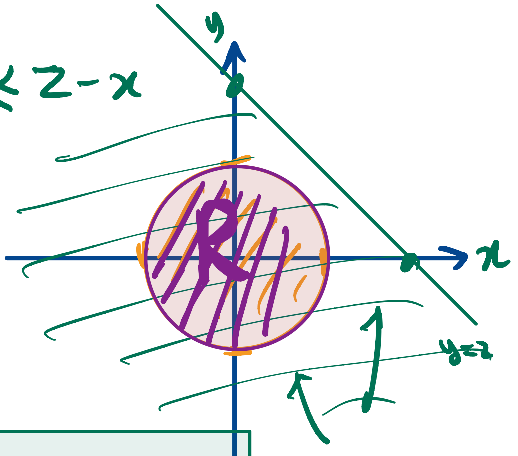


no overlap so Guess was wrong.

Rule 6: New guess $0 \leq z \leq 2-x-y$

Rule 2: cross out

Rule 3: Make new $0 \leq 2-x-y \Rightarrow y \leq 2-x$



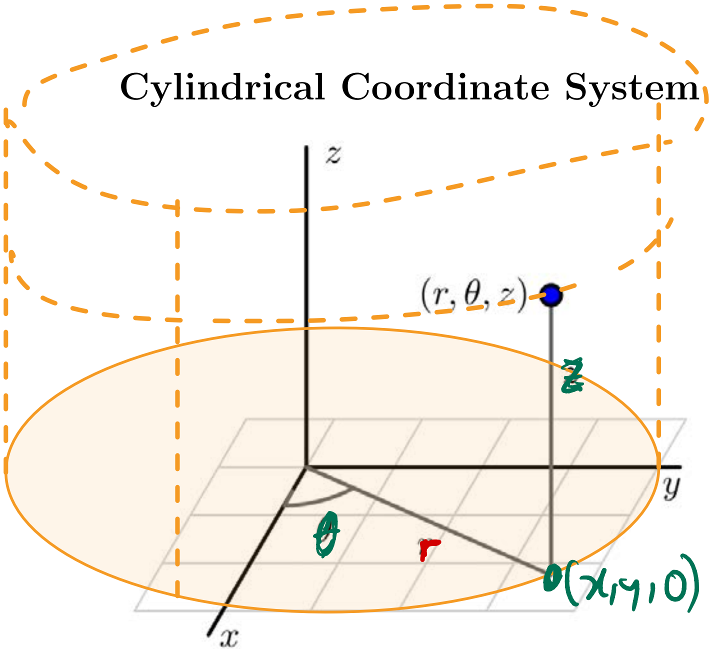
So $R: \begin{cases} x \in [-1, 1] \\ y \in [-\sqrt{1-x^2}, \sqrt{1-x^2}] \end{cases}$

and $z \in [0, 2-x-y]$

$$\boxed{M = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{2-x-y} x^3y \, dz \, dy \, dx}$$

§15.7 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinate System



Conventions: $r \geq 0$ (as before)

$\theta \in [0, 2\pi]$ or $[-\pi, \pi]$

Example 108. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

$x = -1$

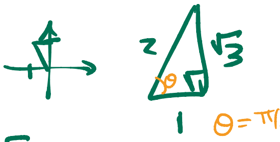
$y = \sqrt{3}$

$z = 3$

$\theta = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

$r = \sqrt{1+3} = \sqrt{4} = 2$

$z = 3$



$\theta = \pi/3$ in Quad II.

$$\boxed{(x,y,z)_{e} = \left(2, \frac{2\pi}{3}, 3\right)_{cy}}$$

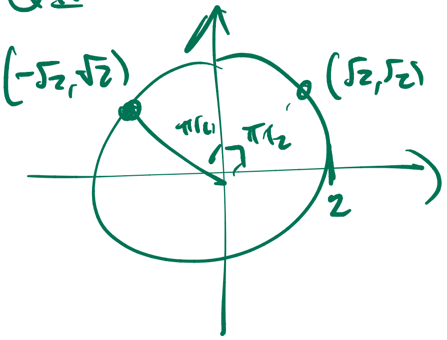
Cylindrical to Cartesian: $(x,y)_{e} = (r,\theta)_{p}$

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

QII



b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1) = (r, \theta, z)$

$x = 2 * \cos\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

$y = 2 * \sin\left(\frac{5\pi}{4}\right) = \sqrt{2}$

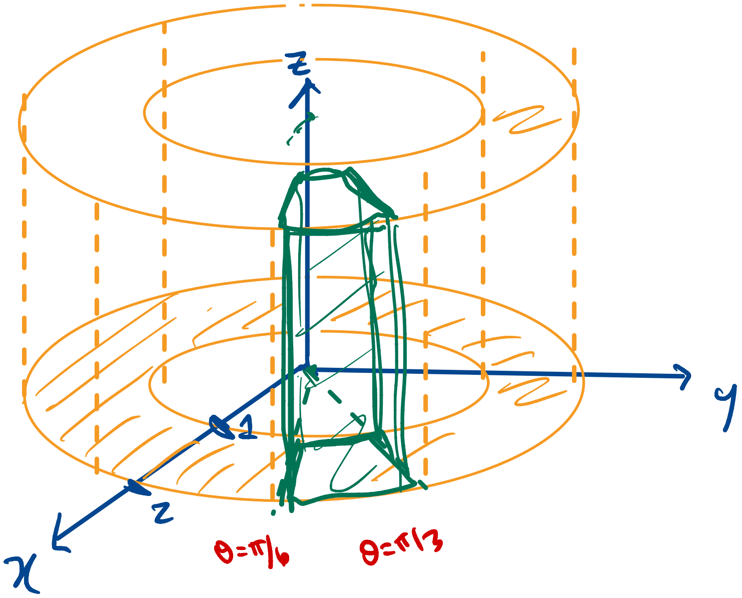
$z = 1.$

$$\boxed{(x,y,z) = (-\sqrt{2}, \sqrt{2}, 1)}$$

$dV = dx\,dy\,dz$ $\quad \nabla V = \nabla x \nabla y \nabla z$

Example 109. In xyz-space sketch the *cylindrical box* $\quad = \nabla r \nabla \theta \nabla z$??

$$B = \{(r, \theta, z) \mid 1 \le r \le 2,\ \pi/6 \le \theta \le \pi/3,\ 0 \le z \le 2\}.$$



Triple Integrals in Cylindrical Coordinates

We have $dV =$ $r\,dr\,d\theta\,dz$

Example 110. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x+2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

rewrite $\quad x = r\cos\theta$

$z = x+2$

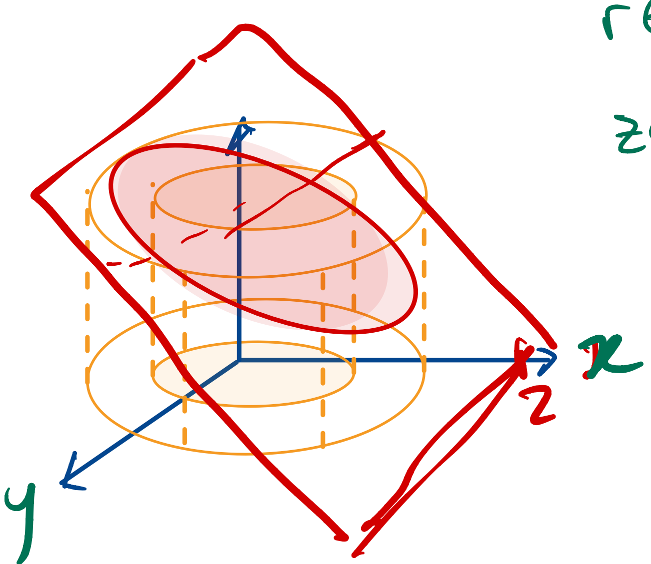
$z = r\cos\theta + 2$

$\theta \in [0, 2\pi]$

$r \in [1, 2]$

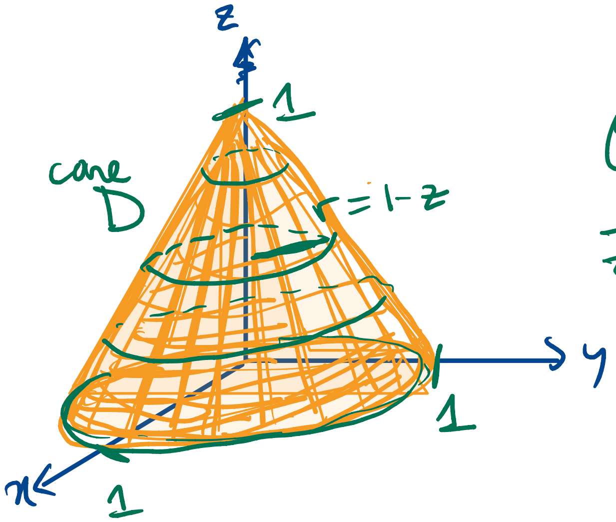
$z \in [0, x+2]$

$$\text{Vol} = \int_0^{2\pi} \int_1^2 \int_0^{\overset{???\ \boxed{r\cos\theta+2}}{\cancel{x+2}}} 1\, r\, dz\, dr\, d\theta$$



Example 111. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

Example 111. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.



$\theta \in [0, 2\pi]$

$z \in [0, 1]$

$r \in [0, 1-z]$

So $$M = \iiint_D \delta(r,\theta,z)\, r \, dr\, d\theta\, dz$$

$$M = \boxed{\int_0^{2\pi} \int_0^{1} \int_0^{1-z} z\, r \, dr\, dz\, d\theta}$$

other options also work

$$M = \int_0^{2\pi} \int_0^{1} \int_0^{1-r} z\, r \, dz\, dr\, d\theta$$

or even

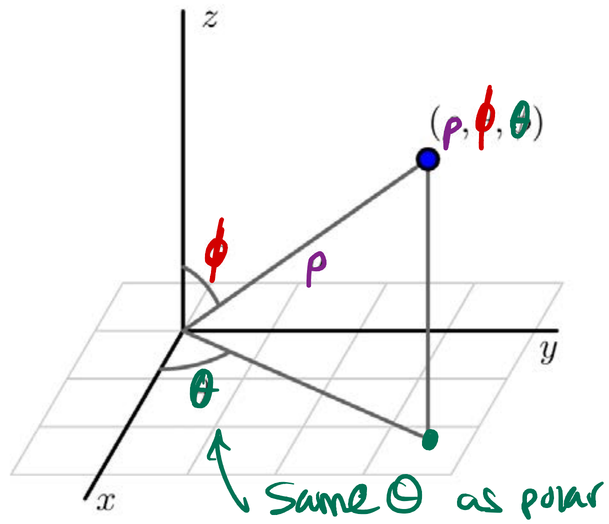
$$M = \int_0^{1} \int_0^{1-r} \int_0^{2\pi} z r \, d\theta\, dz\, dr$$ also fine!

ϕ "Fee" or "Fie" φ Scripty Phi

regular phi

Spherical Coordinate System

ρ rho "basically r"



Conventions: $\rho \geq 0$ (same as r)

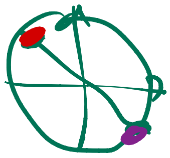
$\theta \in [0,2\pi)$ or $[-\pi,\pi)$

$\varphi \in [0,\pi]$??

Example 112. a) Find spherical coordinates for the point with Cartesian coordinates $(-2,2,\sqrt{8}) = (x,y,z)$ is on sphere of radius $\rho=4$.

$\rho = \sqrt{4+4+8} = \sqrt{16} = 4$

$\tan\theta = \frac{2}{-2} = -1 \quad \theta = \boxed{\frac{3\pi}{4},} \cancel{\frac{7\pi}{4}}$??



Quad II

$\tan\varphi = \frac{\sqrt{4+4}}{\sqrt{8}} = \frac{\sqrt{8}}{\sqrt{8}} = 1 \quad \varphi = \frac{\pi}{4}$

$\boxed{\left(4, \frac{\pi}{4}, \frac{3\pi}{4}\right)}$

b) Find Cartesian coordinates for the point with spherical coordinates $(2,\pi/2,\pi/3)$. $= (\rho, \varphi, \theta)$

Spherical to Cartesian:

$$\boxed{\begin{aligned} x &= \rho\sin(\varphi)\cos(\theta) \\ y &= \rho\sin(\varphi)\sin(\theta) \\ z &= \rho\cos(\varphi) \end{aligned}}$$

Cartesian to Spherical:

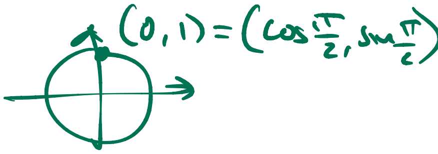
$$\rho^2 = x^2 + y^2 + z^2$$

$$\tan(\theta) = \frac{y}{x}$$

$$\tan(\varphi) = \frac{\sqrt{x^2+y^2}}{z}$$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



$x = \rho\sin(\varphi)\cos(\theta) = 2\sin(\pi/2)\cos(\pi/3) = 2(1) * \frac{1}{2} = 1$

$y = \rho\sin(\varphi)\sin(\theta) = 2\sin(\pi/2)\sin(\pi/3) = 2(1)\frac{\sqrt{3}}{2} = \sqrt{3}$

$z = \rho\cos(\varphi) = 2\cos(\pi/2) = 0$

So $\boxed{(x,y,z) = (1, \sqrt{3}, 0)}$

🌀 The two common lowercase phi symbols:

ϕ (curly phi) — sometimes called “script phi” or “open phi”

Looks like a curly or loopy “C” with a vertical line

Unicode: U+03D5

Often used in physics and engineering (e.g., magnetic flux)

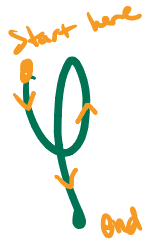
φ (straight phi) — often just “phi”

Looks like a circle with a vertical line through it

Unicode: U+03C6

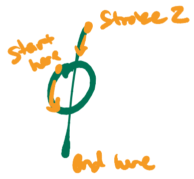
Often used in math, philosophy, and logic

How to draw them:



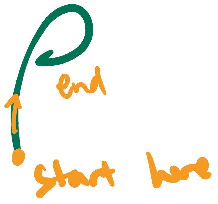
#strokes = 1

Scripty phi



#strokes = 2

phi

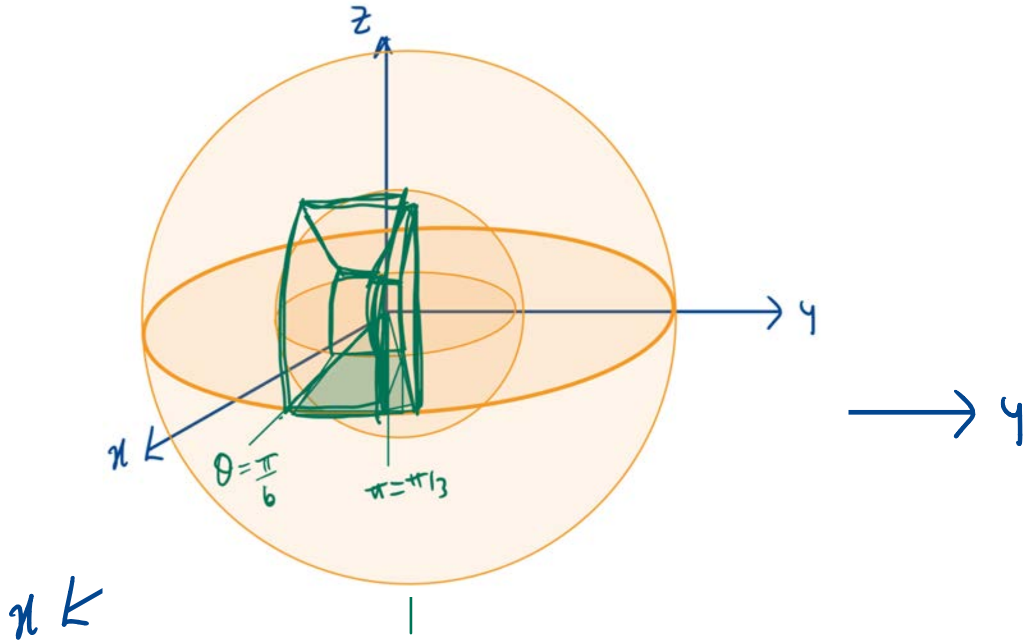


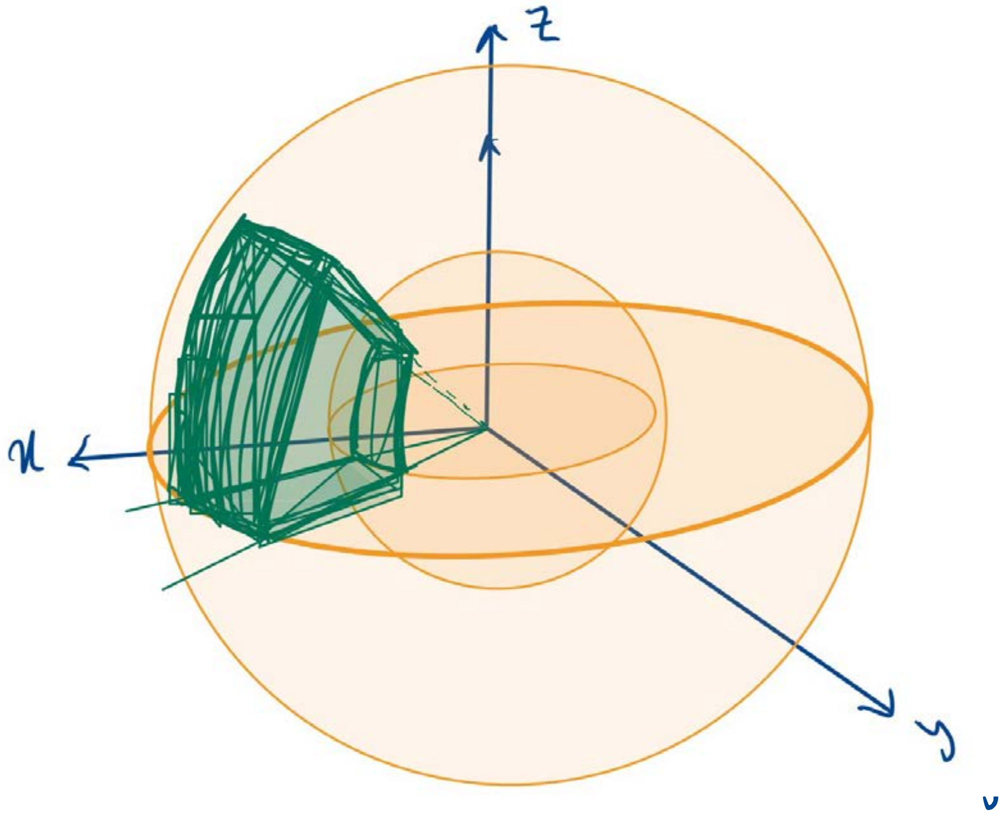
#strokes = 1.

rho

Example 113. In *xyz*-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid 1 \leq \rho \leq 2, \overset{\frac{\pi}{4}}{\cancel{0}} \leq \varphi \leq \underset{2}{\cancel{\pi/4}}, \ \pi/6 \leq \theta \leq \pi/3\}.$$

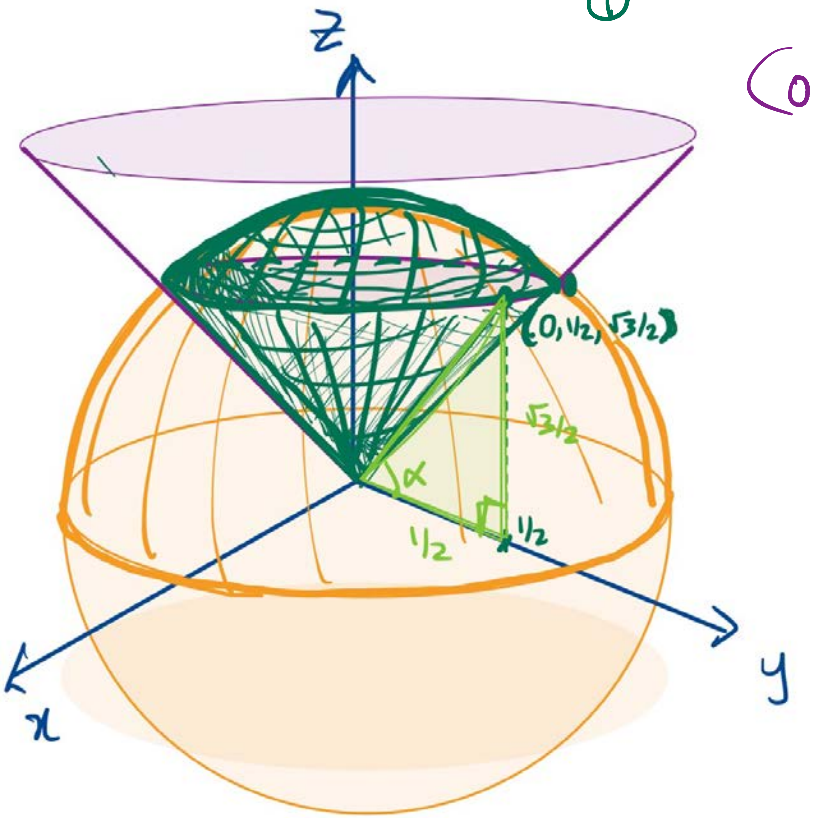




Triple Integrals in Spherical Coordinates

We have $dV = \rho^2 \sin\varphi \quad d\varphi\, d\theta\, d\rho$

Example 114. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $\boxed{x^2+y^2+z^2 = 1}$ ① and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.



(option 1) algebraic

$\rho \in [0,1)$

$\rho = \sqrt{x^2+y^2+z^2} = 1$ (eqn. of sphere)

$\boxed{\rho = 1.}$

For cone

① $\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \end{cases}$

$z = \sqrt{3}\sqrt{x^2+y^2}$

② $z = \rho \cos(\varphi)$

$z \overset{①}{=} \sqrt{3}\sqrt{\rho^2 \sin^2\varphi \underbrace{\cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta}_{1}}$

$z = \sqrt{3}\sqrt{\rho^2 \sin^2\varphi} = \sqrt{3}\rho \sin\varphi.$

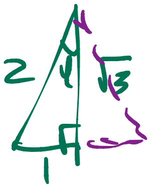
$\cancel{\rho}\cos\varphi \overset{②}{=} \sqrt{3}\cancel{\rho}\sin\varphi \quad (\rho \neq 0)$

$\Rightarrow \frac{1}{\sqrt{3}} = \tan\varphi \Rightarrow \boxed{\varphi = \pi/6}$

eqn of the cone.

$\varphi \in [0, \pi/6]$?? ✓

~~$[\pi/6, \pi/2]$??~~



$\tan\varphi = 1/\sqrt{3}$

$\Rightarrow \varphi = \pi/6$

$\varphi \in [0, \pi/6]$

$\theta \in [0, 2\pi]$

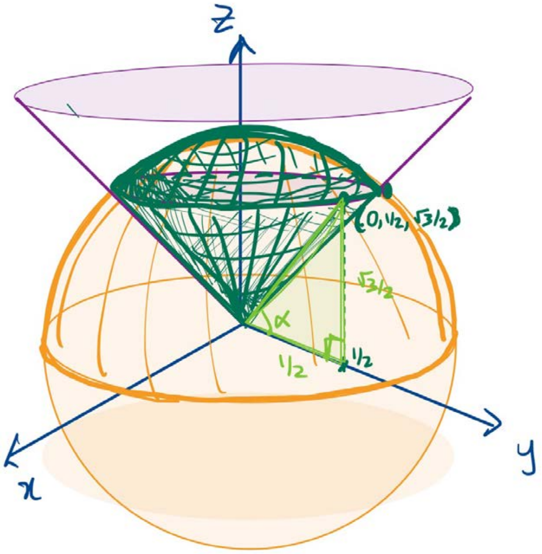
$\rho \in [0, 1]$

$$\boxed{V = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^2 \sin\varphi \, d\varphi\, d\theta\, d\rho}$$

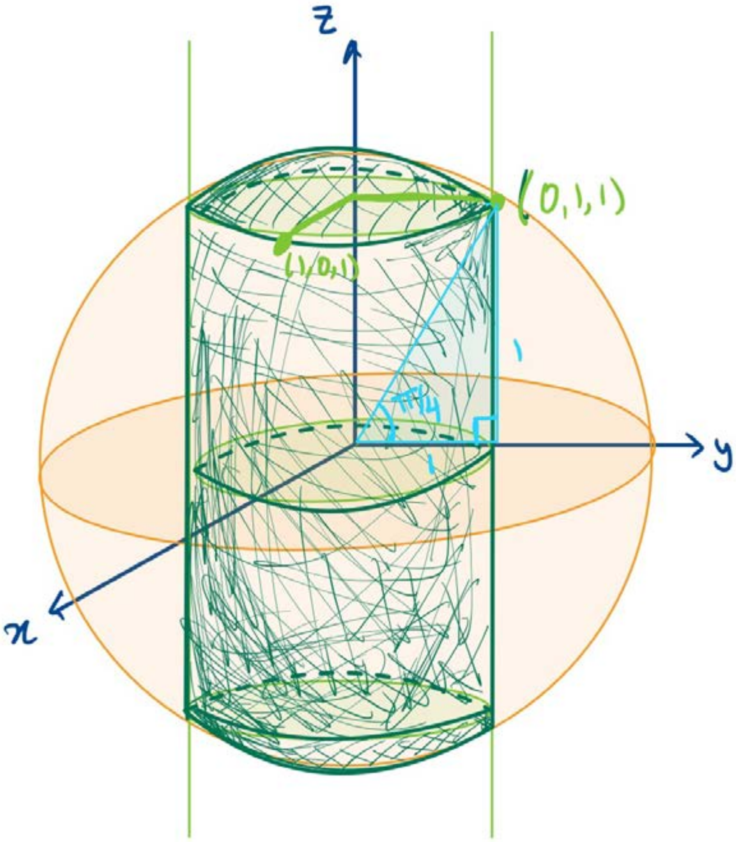
Triple Integrals in Spherical Coordinates

We have $dV =$ ______________________________

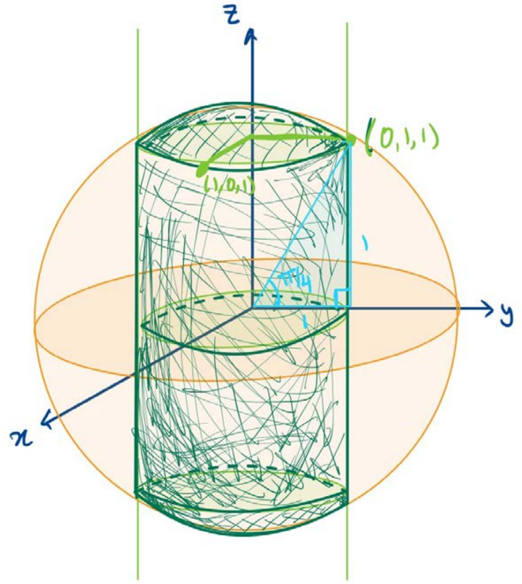
Example 114. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.



Example 115. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.



Example 115. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.



Sphere:

$x^2+y^2+z^2=2 \iff \rho=\sqrt{2}$

intersection

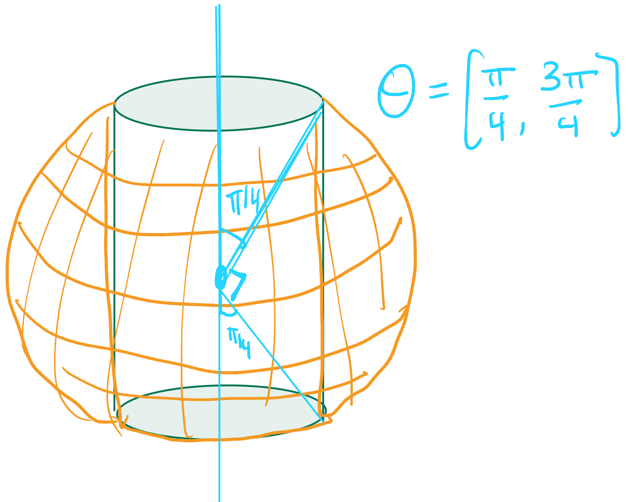
$x^2+y^2+z^2=2$ & $x^2+y^2=1$

$\Rightarrow 1+z^2=2$

$\Rightarrow z^2=1 \Rightarrow z=\pm 1$

$\underline{\text{one point}}$ in intersection

is $(0,1,1)_{\mathcal{C}} = (1, \frac{\pi}{2}, \pi/4)_S$



Same idea as last example

$x^2+y^2 = \rho^2 \sin^2\varphi$ so

cylinder is $\rho^2\sin^2\varphi = 1$

$\Rightarrow \rho\sin\varphi = 1$

$\Rightarrow \rho = \csc\varphi$

we want $\underline{\text{outer}}$ part outside cylinder & inside sphere so

$\varphi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$\rho \in [\csc\varphi, \sqrt{2}]$

and $\theta \in [0, 2\pi)$

$$\boxed{\text{Vol} = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_{\csc\varphi}^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta}$$

§15.8 Change of Variables in Multiple Integrals

Thinking about single variable calculus: Compute $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

rectangular

$dA = dx\,dy \qquad dV = dx\,dy\,dz$

polar coord.

$dA = r\,dr\,d\theta$

cyclindrical

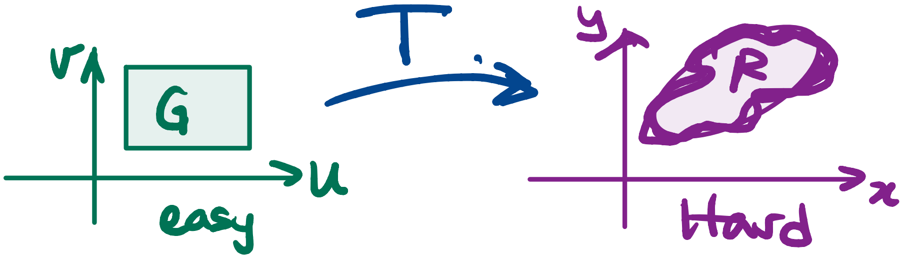
$dV = r\,dr\,d\theta\,dz$

Spherical coord.

$dV = \rho^2 \sin\varphi\,d\varphi\,d\rho\,d\theta$

Theorem 116 (Substitution Theorem)**.** *Suppose* **T**(u, v) *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(\mathbf{T}(u, v))\boxed{\left|\det(D\mathbf{T}(u, v))\right|\,du}\,dv.$$



The "Jacobian determinant" measures how the ΔxΔy needs to get converted to ΔuΔv

e.g. If $|\det DT| = 2$ Then $2 *$ Area G = Area R.

Example 117. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x - y}{2}\,dx\,dy$ via the transformation $x = u + v$, $y = 2v$.

① $x = u + v$

② $y = 2v$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

1. **Find T:** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u+v \\ 2v \end{bmatrix} = T(u, v)$

$y \in [0, 4]$

$x \in \left[\frac{y}{2}, \frac{y}{2} + 1\right]$

@ $y = 0 \Rightarrow 0 = y = 2v \Rightarrow v = 0$

@ $y = 4 \Rightarrow 4 = y = 2v \Rightarrow v = 2$ $\Big\}\ v \in (0, 2)$

$u \in (?, ?)$

$v \in (?, ?)$

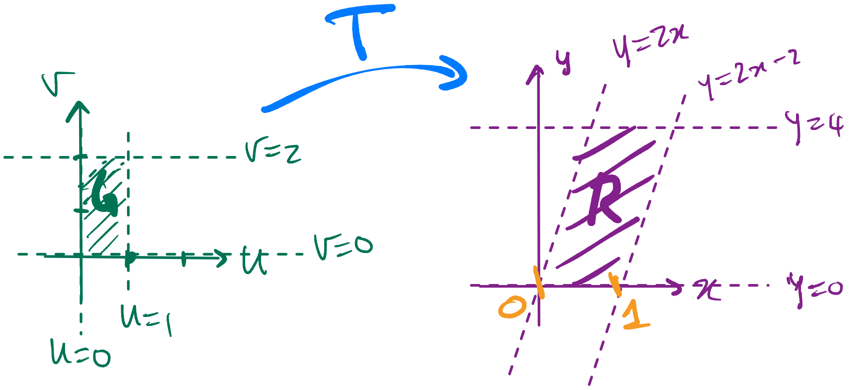
@ $x = \frac{y}{2} \Rightarrow \frac{y}{2} = \overset{①}{x} = u + v \Rightarrow u = \frac{y}{2} - v \overset{②}{=} \frac{2v}{2} - v \Rightarrow u = 0$

@ $x = \frac{y}{2} + 1 \Rightarrow \frac{y}{2} + 1 = u + v \Rightarrow \frac{2v}{2} + 1 = u + v \Rightarrow u = 1$ $\Big\}$

$u \in [0, 1]$

2. **Find G and sketch:** $G = [0,1]\times[0,2] \ni (u,v)$

IF $y\in[0,4]$ then $x\in\left(\frac{y}{2}, \frac{y}{2}+1\right]$



$$T\left(\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}u+v\\2v\end{bmatrix}$$

3. **Find Jacobian:**

$$DT = \begin{bmatrix} x_u & x_v\\ y_u & y_v\end{bmatrix} = \begin{bmatrix}1&1\\0&2\end{bmatrix}$$

$x = u+v$

$y = 2v$

$|\det DT| = 2$

Note Area $G = 2$

Area $R = 4$

In alg

Area $T(S) =$ $|\det A_T| *$ Area S

4. **Convert and use theorem:**

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|\,du\,dv.$$

$$\int_0^4\int_{y/2}^{y/2+1}\frac{2x-y}{2}\,dx\,dy \overset{!}{=} \iint_R f(x,y)\,dx\,dy$$

$$= \iint_G f(u+v,2v) * 2\,du\,dv = \int_0^2\int_0^1 \frac{2(u+v)-2v}{\cancel{2}} * \cancel{2}\,du\,dv$$

$$= \int_0^2\int_0^1 2u\,du\,dv = \int_0^2 u^2\Big|_0^1\,dv = \int_0^2 1\,dv = v\Big|_0^2 = \boxed{2}$$

Example 118. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v, \;\; y = v.$$

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\, dx\, dy?$$

i) $u = x, v = y$

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

iii) $u = 2x - y,\ v = y^3$

iv) $u = y,\ v = 2x - y$

v) $u = 2x - y,\ v = y$

vi) $u = e^{(2x-y)^2},\ v = y^3$

Example 118. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v, \;\; y = v.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u + \frac{1}{2}v \\ v \end{bmatrix} = T(u,v)$$

The Jacobian MATRIX

So $DT = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

and $|DT| = 1$

The Jacobian determinant

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\, dx\, dy?$$

i) $u = x, v = y$ does nothing 😑

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$?

iii) $u = 2x - y,\; v = y^3$

iv) $u = y,\; v = 2x - y$

v) $u = 2x - y,\; v = y$

vi) $u = e^{(2x-y)^2},\; v = y^3$

Try (iii) $\iint_G v\,\underbrace{u\,e^{u^2}}_{\text{Can do } w\text{-sub } \checkmark} * |DT|\,du\,dv$ Seems helpful.

Try (iv) $\iint_G u^3\,\underbrace{v\,e^{v^2}}_{\text{Can do } w\text{-sub } \checkmark} * |DT|\,du\,dv$ Seems helpful.

try (v) $\iint_G v^3\,u\,e^{u^2} * |DT|\,du\,dv$ " —— " ✓

might be easier to figure out bounds for these two?

try (vi) $\iint_G v\,(?!?)\,u\,du\,dv$ how to isolate 2x-y??

Theorem 119 (Derivative of Inverse Coordinate Transformation). *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane and* $T(u_0, v_0) = (x_0, y_0)$*, then we have*

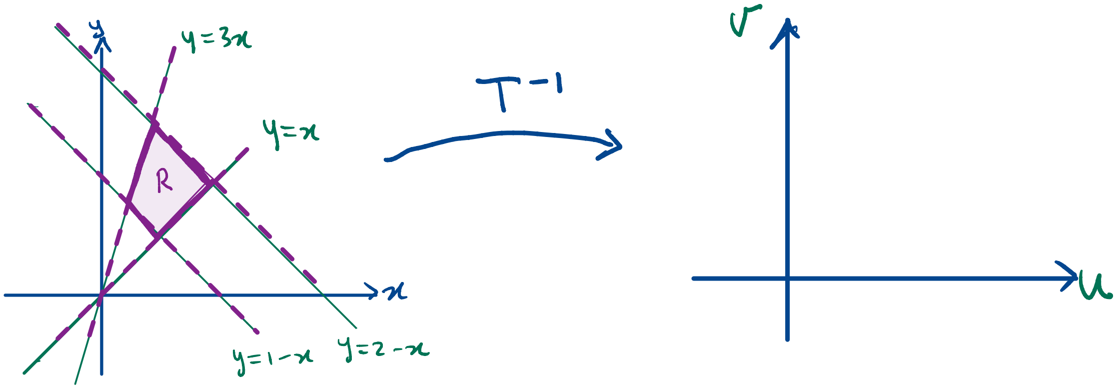
$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

General fact about invertible matrices

$$\det A = \frac{1}{\det A^{-1}} \quad \checkmark$$

Example 120. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y=x, y=3x, y=1-x$, and $y=2-x$. Consider the coordinate transformation $u=x+y, v=y/x$.

1. **Find the rectangle G in the uv plane that is mapped to R**



top-left boundary

top-right boundary

bot-left boundary

bot-right boundary

2. **Evaluate $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ in terms of u and v without directly solving for T using the theorem above**

3. **Use the Substitution Theorem to compute the integral.**