MATH 2551 GT-E w/ Dr. Sal Barone

- Dr. Barone, Prof. Sal, or just Sal, as you prefer

Daily Announcements & Reminders:

Goals for Today: Sections 12.1, 12.4, 12.5

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
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-

Big Idea: Extend differential & integral calculus.

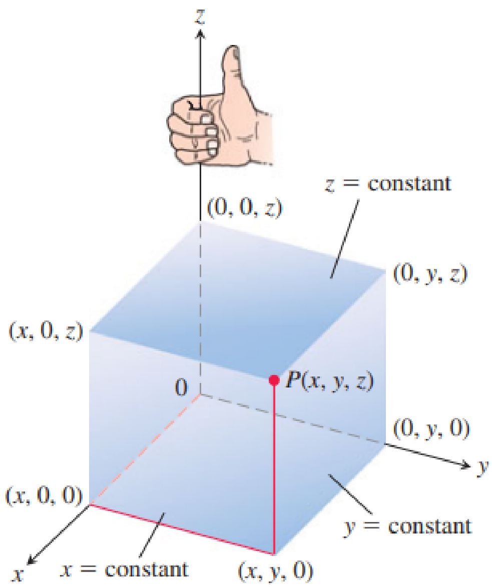
What are some key ideas from these two courses?

Differential Calculus | Integral Calculus

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

§12.1: Three-Dimensional Coordinate Systems



Question: What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x^2+y^2 = 1$?

§12.3, 12.4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = \underline{\hspace{8cm}}$$

This product tells us about ____.

In particular, two vectors are **orthogonal** if and only if their dot product is _______.

Example 2. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1.

2.

Definition 3. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ <u>in $\mathbb{R}^3$</u> is

Example 4. Find $\langle 1, 2, 0\rangle \times \langle 3, -1, 0\rangle$.

A Geometric Interpretation of $\mathbf{u} \times \mathbf{v}$

The cross product $\mathbf{u} \times \mathbf{v}$ is the vector

$$\boxed{\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n}}$$

where $\mathbf{n}$ is a unit vector which is normal to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

Since $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

$$\boxed{|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta}$$

Example 5. Find the area of the parallelogram determined by the points P, Q, and R.

$$P(1, 1, 1),\ Q(2, 1, 3),\ R(3, -1, 1)$$

§12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:

Example 6. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

Example 7. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.

§12.6 Quadric Surfaces

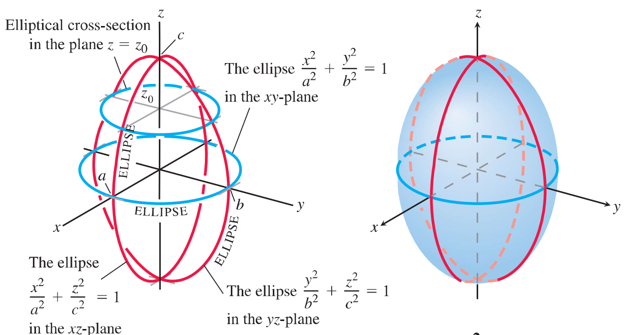
Definition 8. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

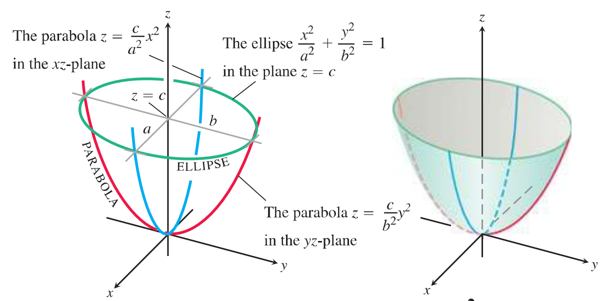
The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 9. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

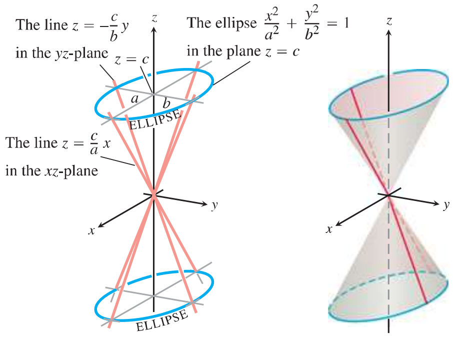
TABLE 12.1 Graphs of Quadric Surfaces



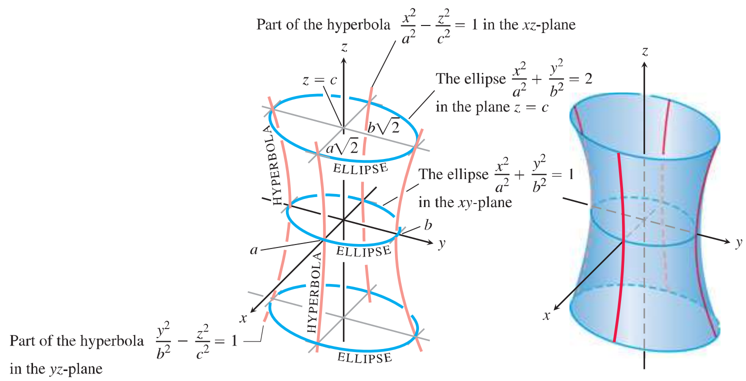
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



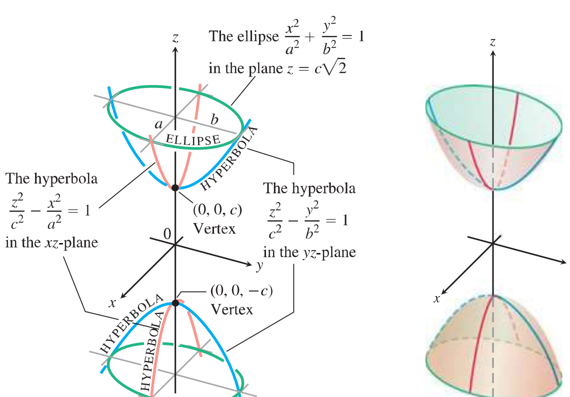
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



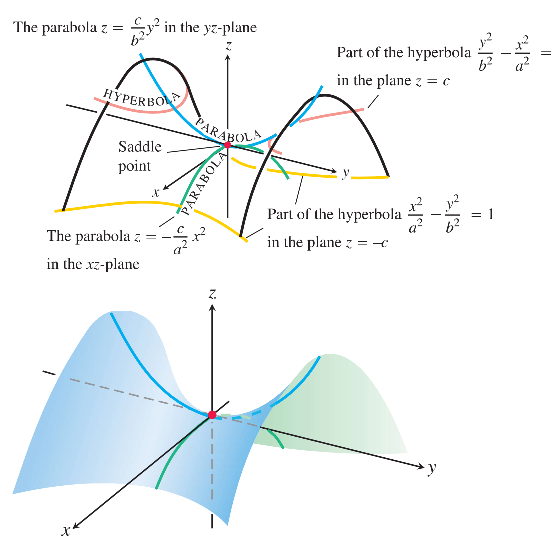
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

§13.1 Curves in Space & Their Tangents

The description we gave of a line last week generalizes to produce other one-dimensional graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well. We said that a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ produces a straight line when graphed.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called ________________ the curve C, and $\mathbf{r}$ is called a ________________ of C.

Example 10. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Check your intuition

§13.2: Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 11. Compute $\lim_{t \to e} \langle t^2, 2, \ln(t) \rangle$.

And with continuity:

Example 12. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \frac{1}{t^2 - 4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

And with derivatives:

Example 13. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives ______________________________
- $|\mathbf{r}'(t)|$ gives ______________________________
- $\mathbf{r}''(t)$ gives ______________________________

Let's see this graphically

Example 14. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

And with integrals:

Example 15. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t) \rangle \, dt$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 16. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \rangle \ m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, use calculus to reconstruct his flight path.