# §14.3: Partial Derivatives

**Goal:** Describe how a function of two (or three, later) variables is changing at a point $(a, b)$.

**Example 47.** Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point $(x, y)$. If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive $y$-direction), at what rate will our height change? What if we walk due east (the positive $x$-direction)?

Let's investigate graphically.

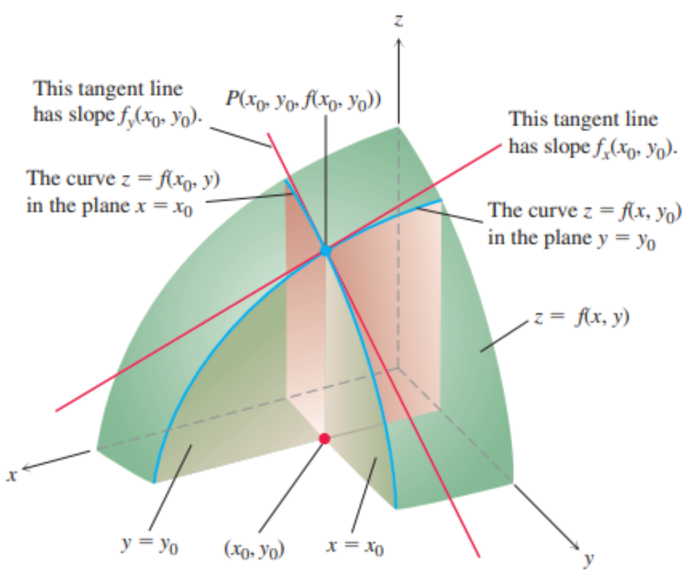
**Definition 48.** If $f$ is a function of two variables $x$ and $y$, its ____________________

are the functions $f_x$ and $f_y$ defined by

$$f_x(x, y) = \lim_{h \to 0} \underline{\hspace{6cm}} \qquad f_y(x, y) = \lim_{h \to 0} \underline{\hspace{6cm}}$$

Notations:

Interpretations:



**Example 49.** Find $f_x(1, 2)$ and $f_y(1, 2)$ of the functions below.

a) $f(x, y) = \sqrt{5x - y}$

b) $f(x, y) = \tan(xy)$

**Question:** How would you define the second partial derivatives?

**Example 50.** Find $f_{xx}$, $f_{xy}$, $f_{yx}$, and $f_{yy}$ of the function below.

a) $f(x, y) = \sqrt{5x - y}$

What do you notice about $f_{xy}$ and $f_{yx}$ in the previous example?

**Theorem 51** (Clairaut's Theorem). *Suppose $f$ is defined on a disk $D$ that contains the point $(a, b)$. If the functions $f, f_x, f_y, f_{xy}$, $f_{yx}$ are all continuous on $D$, then*

**Example 52.** *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

**Example 53.** How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted $Df$, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

**Example 54.** *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2 y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

**What does it mean?** In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \dots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

**Definition 55.** The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \dots, a_n)$ is

$$L(\mathbf{x}) =$$

**Example 56.** Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

**Question:** What do you notice about the equation of the linearization?

We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of $f$ near $\mathbf{a}$.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y) - (a,b)\|} = 0.$$

In particular, if $f$ is a function $f(x,y)$ of two variables, it is differentiable at $(a,b)$ its graph has a unique tangent plane at $(a,b,f(a,b))$.

**Example 57.** Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.

# §14.4 The Chain Rule

Recall the Chain Rule from single variable calculus:

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

**Example 58.** Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t + 1, 2 - t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

**Example 59.** Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where $F, u, v$ are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$

$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$

$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$

$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

Find $W_s(1,0)$ and $W_t(1,0)$.

**Application to Implicit Differentiation:** If $F(x, y, z) = c$ is used to *implicitly* define $z$ as a function of $x$ and $y$, then the chain rule says:

**Example 60.** Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.