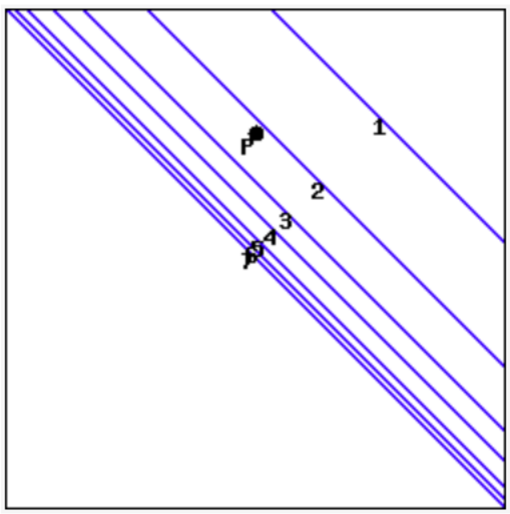
§14.5 Directional Derivatives & Gradient Vectors

Example 61. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?



Let's go back to our hill example again, $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2,1)$ if we move in the direction $\langle -1, 1 \rangle$?

Definition 62. The ____________________________ of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) =$$

if this limit exists.

E.g. for our hill example above we have:

Note that $D_{\mathbf{i}} f =$ $D_{\mathbf{j}} f =$ $D_{\mathbf{k}} f =$

Definition 63. If $f : \mathbb{R}^n \to \mathbb{R}$, then the ________________ of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ____ (or ________) defined by

$$\nabla f(\mathbf{p}) =$$

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

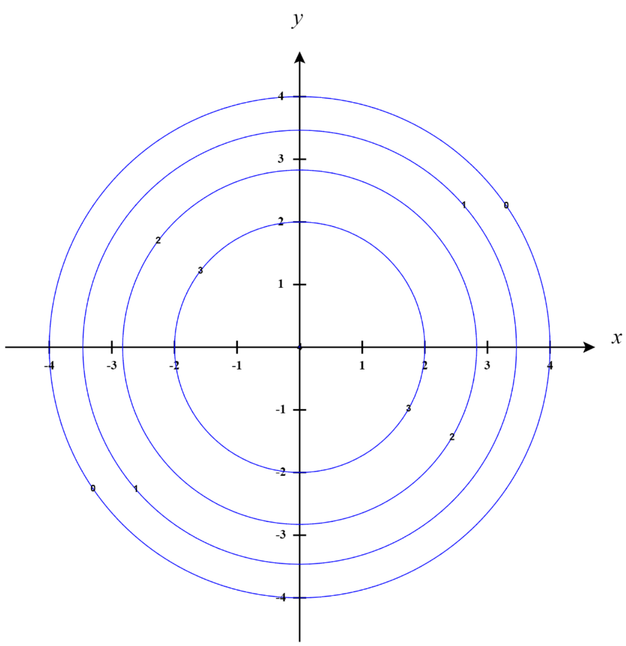
$$D_{\mathbf{u}} f(\mathbf{p}) =$$

Example 64. *You try it!* Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x, y) = \ln(x^2 + y^2), \mathbf{p} = (-1, 1), \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1, 2, 1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

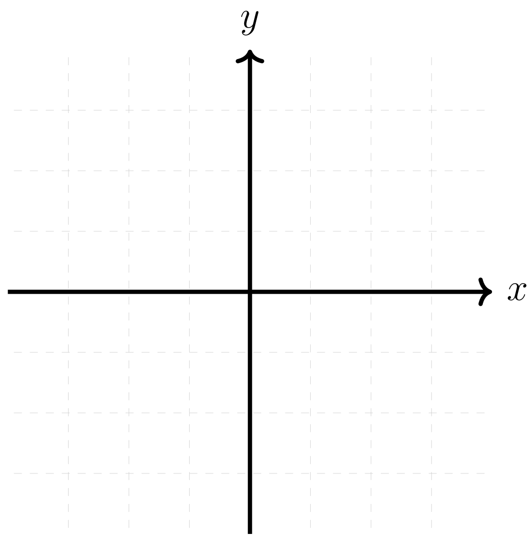
Example 65. If $h(x,y) = 4 - \dfrac{1}{4}x^2 - \dfrac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.



Note that the gradient vector ∇f is ________________ to the level curves of the function ____________.

Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is ____

Example 66. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f \mid_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.



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what's the latex symbol \nabla

The LaTeX symbol `\nabla` renders as:

$$\nabla$$

It is called **"nabla"** or the **del operator**, and it is used primarily in vector calculus. It represents the vector differential operator:

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right]$$

Common Uses:

- **Gradient** of a scalar function f:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\right]$$

- **Divergence** of a vector field $\vec{F}$:

$$\nabla \cdot \vec{F}$$

- **Curl** of a vector field $\vec{F}$:

$$\nabla \times \vec{F}$$

- **Laplacian** of a scalar field f:

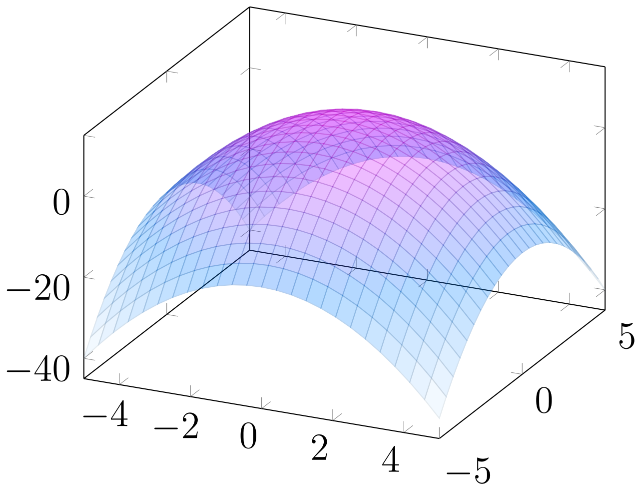
$$\nabla^2 f = \nabla \cdot \nabla f$$

So in summary, `\nabla` is a compact and powerful symbol in multivariable calculus, especially when working with fields and differential operators.

+ Tools

§14.6 Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$\boxed{x^2 + y^2 + z = 10, P = (-1, 3, 0)}$

Example 67. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

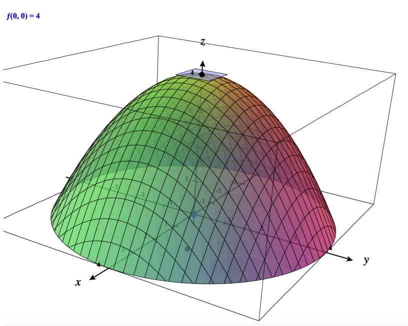
This should look familiar: it's ______________________

§14.7 Optimization: Local & Global

Gradient: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$.

What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?



Definition 68. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a ____________________ value of f if $f(a, b)$ ____ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a ____________________ value of f if $f(a, b)$ ____ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $\boxed{z = x^2 - y^2}$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a ____________________

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

Definition 69. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$________________ or where $Df(a, b)$ ____________________ is called a ____________________ of f.

Example 70. Find the critical points of the function

$$f(x, y) = x^3 + y^3 - 3xy.$$

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0, 0)$.

a) $f(x, y) = 3x + y^3 + 2y^2$

b) $g(x, y) = \cos(x) + \sin(x)$

c) $h(x, y) = \dfrac{4}{x^2 + y^2}$

d) $k(x, y) = x^2 + y^2$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x,y)$ at (a,b) is

$$Hf(a,b) =$$

Theorem 72 (2nd Derivative Test)**.** *Suppose (a,b) is a critical point of $f(x,y)$ and f has continuous second partial derivatives. Then we have:*

- *If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) > 0$, $f(a,b)$ is a local minimum*
- *If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) < 0$, $f(a,b)$ is a local maximum*
- *If $\det(Hf(a,b)) < 0$, f has a saddle point at (a,b)*
- *If $\det(Hf(a,b)) = 0$, the test is inconclusive.*

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 73. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 70.

Two Local Maxima, No Local Minimum: The function $g(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 74. *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

Closed:

Bounded:

Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 75. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

§14.8 Optimization: Global & Constrained, Lagrange Multipliers

Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.

2. Find all critical points of f on the boundary of R

3. Evaluate f at each critical point as well as at any endpoints on the boundary.

4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 76. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Constrained Optimization

Goal: Maximize or minimize $f(x, y)$ or $f(x, y, z)$ subject to a *constraint*, $g(x, y) = c$.

Example 77. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

Objective function:

Constraint equation:

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.