# §15.1 Double Integrals, Iterated Integrals, Change of Order

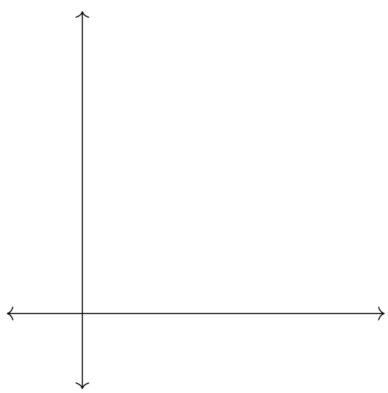
**Recall:** Riemann sum and the definite integral from single-variable calculus.

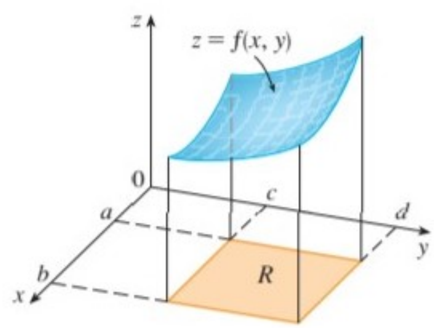
# Double Integrals

**Volumes and Double integrals** Let $R$ be the closed rectangle defined below:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x, y)$ be a function defined on $R$ such that $f(x, y) \geq 0$. Let $S$ be the solid that lies above $R$ and under the graph $f$.






**Question:** How can we estimate the volume of $S$?

**Definition 79.** The ____________________ of $f(x, y)$ over a rectangle $R$ is

$$\iint_R f(x,y)\ dA = \lim_{|P| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

if this limit exists.

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**Question:** How can we compute a double integral?

**Answer:**

Let $f(x, y) = 2xy$ and lets integrate over the rectangle $R = [1, 3] \times [0, 4]$.

We want to compute $\int_1^3 \int_0^4 f(x, y)\ dy\ dx$, but lets consider the slice at $x = 2$.

What does $\int_0^4 f(2, y)\ dy$ represent here?

In general, if $f(x,y)$ is integrable over $R = [a,b] \times [c,d]$, then $\int_c^d f(2,y)\,dy$ represents:

What about $\int_c^d f(x,y)dy$?

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$= \int_a^b A(x)dx =$$

This is called an ____________________.

**Example 80.** Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y\ dy\ dx.$

**Theorem 81** (Fubini's Theorem)**.** *If $f$ is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

*More generally, this is true if we assume that $f$ is bounded on $R$, $f$ is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.*

**Example 82.** *You try it!* Integrate:

a) $\displaystyle\int_0^2 \int_{-1}^1 x - y \; dy \; dx$ **easy**

b) $\displaystyle\int_0^1 \int_0^1 \frac{y}{1+xy} \; dx \; dy$ **medium**

c) $\displaystyle\int_1^4 \int_1^e \frac{\ln x}{xy} \; dx \; dy$ HARD!

**Example 83.** Compute $\iint_R xe^{e^{e^y}}\ dA$, where $R$ is the rectangle $[-1, 1] \times [0, 4]$.

*Hint: Fubini's Theorem.*

# §15.2 Double Integrals on General Regions

**Question:** What if the region $R$ we wish to integrate over is not a rectangle?

**Answer:** Repeat same procedure - it will work if the boundary of $R$ is smooth and $f$ is continuous.

**Example 84.** Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the $xy$-plane and whose top is $z = 2 - x - y$.

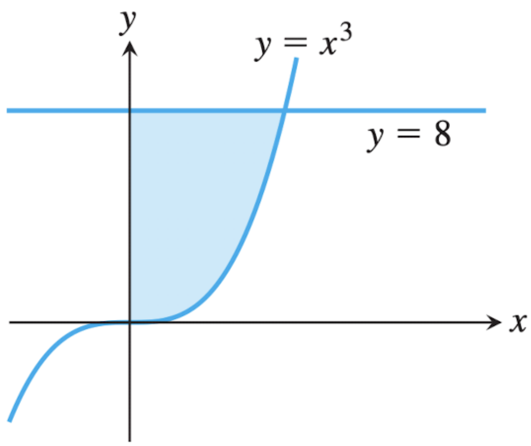
**Vertically simple:**

**Horizontally simple:**

**Example 85.** Write the two iterated integrals for $\iint_R 1\ dA$ for the region $R$ which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

**Example 86.** Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\ dA$, where $R$ is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.

**Example 87.** *You try it!* Write the two iterated integrals for $\iint_R 1\ dA$ for the region $R$ which is bounded by $x = 0, y = 8$, and $y = x^3$.



**Example 88.** Sketch the region of integration for the integral

$$\int_0^1 \int_{4x}^4 f(x, y)\ dy\ dx.$$

Then write an equivalent iterated integral in the order $dx\ dy$.

# §15.3 Area & Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

**Area:** If $R$ is a region bounded by smooth curves, then

$$\text{Area}(R) = \underline{\hspace{4cm}}$$

**Example 89.** Find the area of the region $R$ bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

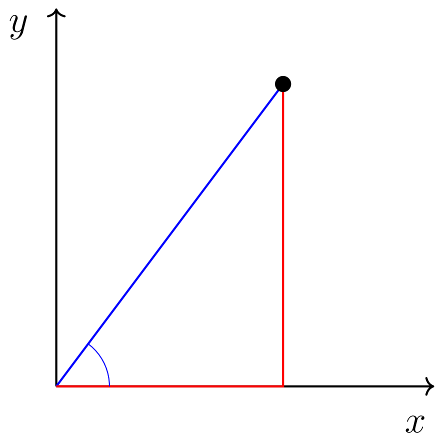
**Average Value:** The average value of $f(x, y)$ on a region $R$ contained in $\mathbb{R}^2$ is

$$f_{avg} = \underline{\hspace{5cm}}$$

**Example 90.** Find the average temperature on the region $R$ in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.

# §15.4 Double Integrals in Polar Coordinates

## Review of Polar Coordinates




**Cartesian coordinates:** Give the distances in ________ and ________ directions from ________

**Polar coordinates:**

- $r$ = distance from ________ to ________
- $\theta$ = angle between the ray ________ and the positive ____________

We can use trigonometry to go back and forth.

**Polar to Cartesian:**

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

**Cartesian to Polar:**

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

**Example 91.** a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

b) Graph the set of points $(x, y)$ that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the $xy$-plane.**

c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

**Goal:** Given a region $R$ in the $xy$-plane described in polar coordinates and a function $f(r, \theta)$ on $R$, compute $\iint_R f(r, \theta)\, dA$.

**Example 92.** Compute the area of the disk of radius 5 centered at $(0, 0)$.

**Remember:** In polar coordinates, the area form $dA =$____________________

**Example 93.** Compute $\iint_D e^{-(x^2+y^2)}\, dA$ on the washer-shaped region $1 \leq x^2+y^2 \leq 4$.

**Example 94.** Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

**Example 95.** *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.