§15.5-15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

Taking the limit gives a

$$\underline{\hspace{6cm}} : \iiint_D f(x,y,z)\, dV$$

Important special case:

$$\iiint_D 1 \, dV = \underline{\hspace{5cm}}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\, dV, \qquad M_{xz} = \iiint_D y\,\delta\, dV, \qquad M_{xy} = \iiint_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta \, dA$ $\delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\,\delta\, dA, \qquad M_x = \iint_R y\,\delta\, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

Example 102. 1. **How to do the computation:**

Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$.

2. **What does it mean:** What shape is this the volume of?

3. **How to reorder the differentials:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\ dy\ dz$

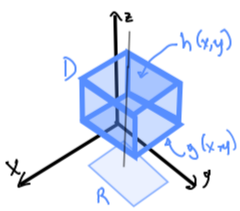
(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\ dx\ dy\ dz$

We will think about converting triple integrals to iterated integrals in terms of the

________________ of D on one of the coordinate planes.

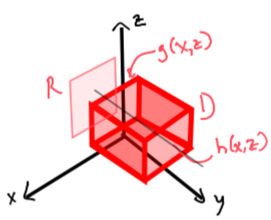
Case 1: z**-simple**) region. If R is the projection of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x,y)$ and $z = g(x,y)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz \right)\ dy\ dx$$



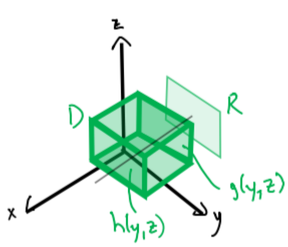
Case 2: y**-simple**) region. If R is the projection of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x,z)$ and $y = g(x,z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\ dy \right)\ dz\ dx$$



Case 3: x**-simple**) region. If R is the projection of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y,z)$ and $x = g(y,z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\ dx \right)\ dz\ dy$$



Example 104. Write an integral for the mass of the solid D in the first octant with $2y \leq z \leq 3 - x^2 - y^2$ with density $\delta(x, y, z) = x^2 y + 0.1$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

Example 104 (cont.)

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 105. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1,\ y + 2z = 2$.

Example 105. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$.

Example 106. Set up an integral for the volume of the region D defined by

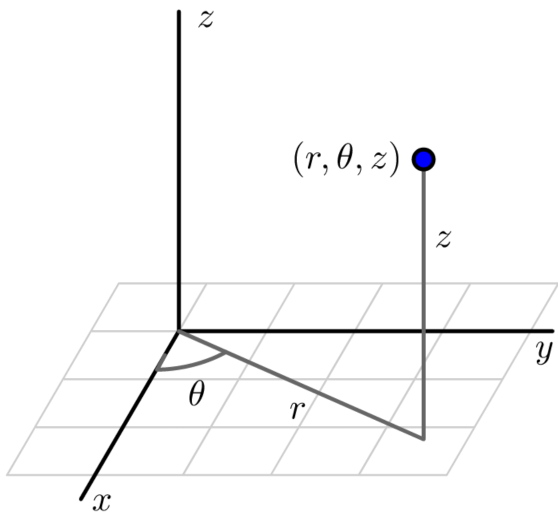
$$x + y^2 \leq 8, \quad y^2 + 2z^2 \leq x, \quad y \geq 0$$

Example 107. Set up a triple iterated integral for the triple integral of $f(x, y, z) = x^3y$ over the region D bounded by

$$x^2 + y^2 = 1, \quad z = 0, \quad x + y + z = 2.$$

§15.7 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinate System



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

Conventions:

Example 108. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

Example 109. In xyz-space sketch the *cylindrical box*

$$B = \{(r, \theta, z) \mid 1 \le r \le 2,\ \pi/6 \le \theta \le \pi/3,\ 0 \le z \le 2\}.$$

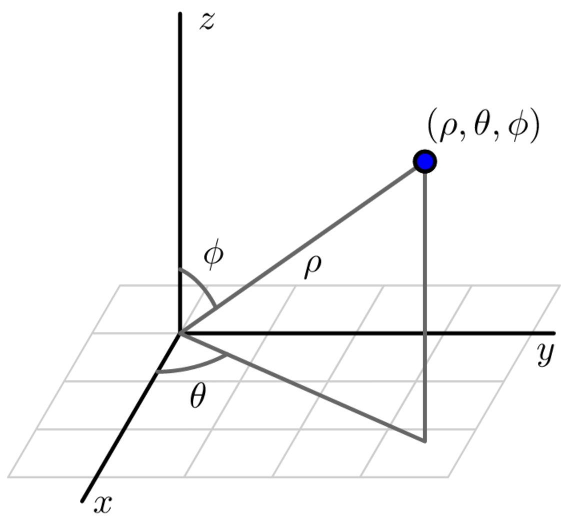
Triple Integrals in Cylindrical Coordinates

We have $dV =$ ____________________

Example 110. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x+2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 111. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

Spherical Coordinate System



Conventions:

Example 112. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

Spherical to Cartesian:

$$x = \rho \sin(\varphi)\cos(\theta)$$
$$y = \rho \sin(\varphi)\sin(\theta)$$
$$z = \rho \cos(\varphi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2 + y^2}}{z}$$

Example 113. In xyz-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid 1 \le \rho \le 2,\ 0 \le \varphi \le \pi/4,\ \pi/6 \le \theta \le \pi/3\}.$$

Triple Integrals in Spherical Coordinates

We have $dV =$ ______________________________

Example 114. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.

Example 115. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

§15.8 Change of Variables in Multiple Integrals

Thinking about single variable calculus: Compute $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$

Theorem 116 (Substitution Theorem)**.** *Suppose* $\mathbf{T}(u, v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

$$\iint_R f(x, y)\ dx\ dy = \iint_G f(\mathbf{T}(u, v))|\det(D\mathbf{T}(u, v))|\ du\ dv.$$

Example 117. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x - y}{2}\ dx\ dy$ via the transformation $x = u + v$, $y = 2v$.

1. **Find T:**

2. **Find G and sketch:**

3. **Find Jacobian:**

4. **Convert and use theorem:**

Example 118. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v, \; y = v.$$

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2} \, dx \, dy?$$

i) $u = x, v = y$

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

iii) $u = 2x - y, \; v = y^3$

iv) $u = y, \; v = 2x - y$

v) $u = 2x - y, \; v = y$

vi) $u = e^{(2x-y)^2}, \; v = y^3$

Theorem 119 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane and* $T(u_0, v_0) = (x_0, y_0)$*, then we have*

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

Example 120. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

1. **Find the rectangle G in the uv plane that is mapped to R**

2. **Evaluate $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ in terms of u and v without directly solving for $\mathbf{T}$ using the theorem above**

3. **Use the Substitution Theorem to compute the integral.**