§16.1 Line Integrals of Scalar Functions

Chapter 16: Vector Calculus

Goals:

- Extend ____________ integrals to ____________ objects living in higher-dimensional space
- Extend the ________________________________ in new ways

We will use tools from everything we have covered so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Example 121. Suppose we build a wall whose base is the straight line from $(0,0)$ to $(1,1)$ in the xy-plane and whose height at each point is given by $h(x,y) = 2x + y^2$ meters. What is the area of this wall?

Definition 122. The **line integral** of a scalar function $f(x, y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x, y)\ ds =$$

What things can we compute with this?

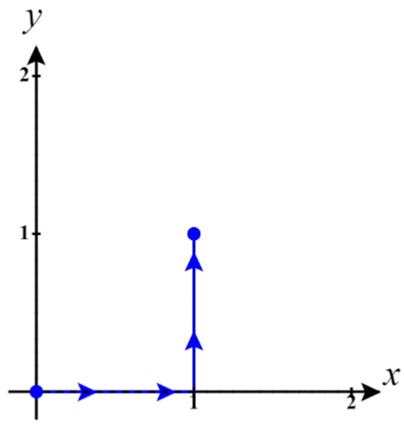
- If $f = 1$:
- If $f = \delta$ is a density function:
- If f is a height:

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \leq t \leq b$
2. Compute $ds = \|\mathbf{r}'(t)\|\ dt$
3. Substitute: $\int_C f(x, y, z)\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt$
4. Integrate

Example 123. *You try it!* Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \leq t \leq \frac{1}{10}$.

Example 124. Compute $\int_C 2x + y^2 \; ds$ along the curve C pictured below.



Example 125. *You try it!* Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \leq t \leq b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \leq t \leq b$ is also a parameterization of C with the same orientation

b) $\mathbf{r}(2t)$ for $a/2 \leq t \leq b/2$ is also a parameterization of C with the same orientation

c) $\mathbf{r}(-t)$ for $a \leq t \leq b$ is also a parameterization of C with the opposite orientation

d) $\mathbf{r}(-t)$ for $-b \leq t \leq -a$ is also a parameterization of C with the opposite orientation

e) $\mathbf{r}(b-t)$ for $0 \leq t \leq b-a$ is also a parameterization of C with the opposite orientation

Example 126. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2 + 1$ from $(2, 5)$ to $(-1, 2)$ and use it to write the integral $\int_C x^2 + y^2 \, ds$ as an integral with respect to your parameter.

§16.2 Vector Fields & Vector Line Integrals

Vector Fields:

Definition 127. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain.

Examples:

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-
-
-
-

Graphically: For each point (a, b) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b)$ with its base at (a, b).

Tools: CalcPlot3d
Field Play

Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The ______________________________ by a field **F** on an object moving along a curve C is given by

Example 128. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x, y) = \langle x, y\rangle$ N. Find the work done by **F** on a moving object from $(0, 3)$ to $(3, 0)$ in a straight line, where x, y are measured in meters.

1. The ______________________ along a curve C of a velocity field $\mathbf{F}$ for a fluid in motion is given by

When C is ___________, this is called __________________. C is called ______________ if it does not intersect itself.

Example 129. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x \rangle$ cm/s around the unit circle, parameterized counterclockwise.

Example 130. *You try it!* What is the circulation of $\mathbf{F}(x, y) = \langle x, y \rangle$ around the unit circle, parameterized counterclockwise?

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$
4. Integrate

Idea: ____________ across a plane curve of a 2D-vector field measures the flow of the field across that curve (instead of along it).

We compute this with the integral

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

The sign of the flux integral tells us whether the net flow of the field across the curve is in the direction of ________ or in the opposite direction.

We can choose $\mathbf{n}$ to be either of

Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \pm \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle\, dt$ (sign based on choice of normal)
4. Integrate

Example 131. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0\rangle$ cm/s across the quarter of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ in the first quadrant, oriented away from the origin.

§16.3 Conservative Vector Fields & Fundamental Theorem

Definition 132. A vector field $\mathbf{F}$ is **path independent** on an open region D if ______________________________ for all paths C in the region that have the same endpoints.

When $\mathbf{F}$ is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F} \cdot d\mathbf{r}.$$

Example 133. If C is any closed path and $\mathbf{F}$ is path independent on a region containing C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} =$$

Question: Given $\mathbf{F}$, how do we tell if it is path independent on a particular region?

For example, is $\mathbf{F}(x, y) = \langle x, y \rangle$ a path independent vector field on its domain?

Example 134. *You try it!* Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x \rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

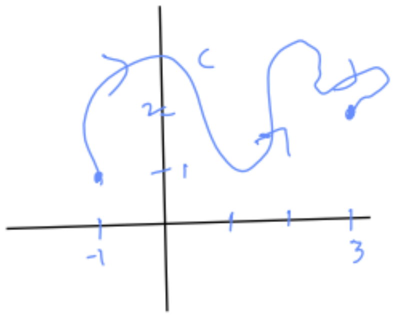
A different idea: Suppose $\mathbf{F}$ is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a ____________________ for $\mathbf{F}$. In this case we also say that $\mathbf{F}$ is **conservative.**

Is $\mathbf{F}(x, y) = \langle x, y \rangle$ conservative?

Theorem 135 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

$$\int_C \nabla f \cdot \mathbf{T}\, ds = f(B) - f(A)$$

Example 136. Compute $\int_C \langle x, y\rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.



It follows that **every conservative field is path independent.**

In fact, by carefully constructing a potential function, we can show the converse is also true: ________________________________

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0,0,0\rangle$, then $\mathbf{F}$ is conservative.

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} =$
- This is also called the **mixed-partials test**, because

Example 137. Evaluate $\int_C (10x^4 - 2xy^3)\, dx - 3x^2y^2\, dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$.