§16.4 Divergence, Curl, Green's Theorem

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} =$

- $\nabla \times \mathbf{F} =$

How do we measure the change of a vector field?

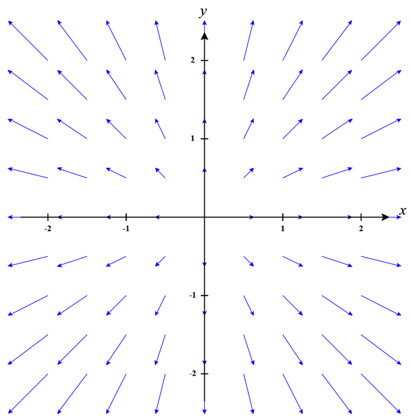
1. **Curl** (in $\mathbb{R}^3$)

   - Tells us ____________________
   - Measures ____
   - Is a ____________
   - Direction gives ____________________
   - Magnitude gives ____________________
   - $\operatorname{curl} \mathbf{F} =$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle$

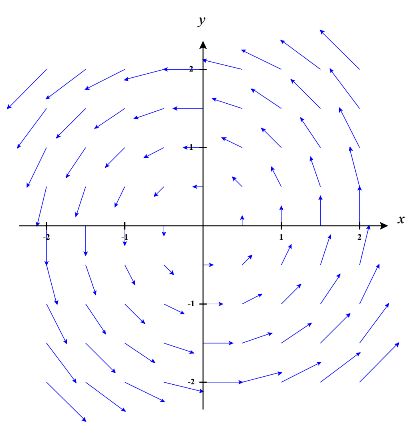
2. **Divergence** (in any $\mathbb{R}^n$)

   - Tells us ____________________
   - Measures ____
   - Is a ____________
   - $\operatorname{div} \mathbf{F} =$

Example 138. Let $\mathbf{F}(x, y) = \langle x, y \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Example 139. *You try it!* Let $\mathbf{F}(x, y) = \langle -y, x \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Question: How is this useful?

Answer: We can relate ______________________________ inside a region to the behavior of the vector field on the boundary of the region.

Theorem 140 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$*. If $\mathbf{F} = \langle P, Q\rangle$ has continuous partial derivatives around R, then*

a) Circulation form:

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C P\ dx + Q\ dy = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA = \iint_R Q_x - P_y\ dA$$

b) Flux form:

$$\int_C \mathbf{F}\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \iint_R (\nabla\cdot\mathbf{F})\ dA = \iint_R P_x + Q_y\ dA$$

Example 141. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0,$ and $y = 1$ oriented counterclockwise.

Example 142. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first octant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.

Example 143. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \leq t \leq \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y \rangle$.

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

§16.5, 16.6 Surfaces & Surface Integrals

Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$	$z = f(x, y)$
Implicit:	$F(x, y) = 0$	$F(x, y, z) = 0$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$	

Example 144. Give parameteric representations for the surfaces below.

a) $x = y^2 + \frac{1}{2}z^2 - 2$

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

c) $x^2 + y^2 + z^2 = 9$

d) $x^2 + y^2 = 25$

What can we do with this?

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is ____________________________

- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a rectangle of size ____________________ on the surface in $\mathbb{R}^3$.

- Thus, $\text{Area}(S) =$

Example 145. *You try it!* Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

Example 146. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

§16.6, 16.7 Flux Surface Integrals, Stokes' Theorem

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of ____________________

If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

Example 147. Find $\mathbf{r}_u \times \mathbf{r}_v$ and $\|\mathbf{r}_u \times \mathbf{r}_v\|$ when $z = f(x, y)$ so that S is the graph of a scalar function with domain in $\mathbb{R}^2$.

Example 148. Find $\mathbf{r}_u \times \mathbf{r}_v$ and $\|\mathbf{r}_u \times \mathbf{r}_v\|$ when S is a portion of a sphere of radius $\rho = a$, for some fixed constant a, using the standard spherical coordinates for your parametrization.

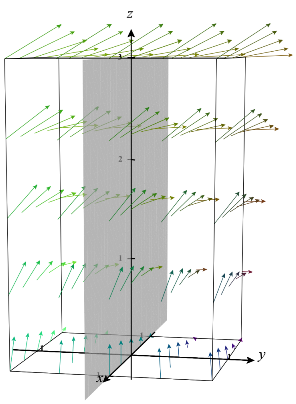
Example 149. Find the flux of $\mathbf{F} = \langle x, -y, z \rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.

Example 150. *You try it!* Compute $\iint_S G \cdot \mathbf{n}\, d\sigma$ the flux of G across the surface S.

$$G(x, y, z) = x^2, \quad S: \; x^2 + y^2 + z^2 = 1$$

Example 151. *You try it!* Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

Example 152. *You try it!* Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ positive, negative, or zero?



Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla\cdot\mathbf{F} = P_x + Q_y + R_z$

2. *curl:* $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

Example 153. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

Theorem 154 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

- If S is a region R in the xy-plane, then we get:

- An **oriented surface** is one where ______________________________

- S and C are oriented compatibly if:

Example 155. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by calculating the flux across the interior of C.

$$\mathbf{F} = \langle y, xz, x^2 \rangle$$

$$C : \text{boundary of } x + y + z + 1 \text{ in first octant,}$$

$$\text{oriented counter-clockwise from above.}$$

Example 156. *You try it!* Use Stokes' Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ the flux of $\mathbf{F}$ across S by calculating the circulation line integral around the boundary curve C of S.

$$\begin{aligned}
\mathbf{F} &= \langle 2z, 3x, 5y \rangle \\
S :& \mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, (4-r^2) \rangle \\
R :& r \in [0,2],\ \theta \in [0, 2\pi]
\end{aligned}$$