## Practice Exam 1

**1.** Valid? Prove or disprove.

$$(a)\quad \begin{array}{c} p \to q \\ \underline{q \vee r} \\ r \to (\neg q) \end{array} \qquad (b)\quad \begin{array}{c} p \to q \\ (\neg r) \vee (\neg q) \\ \underline{r} \\ (\neg p) \end{array}$$

*Solution:* The first argument is invalid since there exists an assignment which makes the premises false and the conclusion true: for example, when $p, q, r$ are all true.

The second argument is valid and here is a proof.

Seeking a contradiction, suppose the argument is invalid and consider an assignment to the variables $p, q, r$ where the premises are true and the conclusion is false. If the conclusion is false, then $p$ is true. By the first premise and the fact that $p$ is true, $q$ must be true as well. Also, by the third premise $r$ is true. But the second premise asserts that either $r$ is false or $q$ is false, which is false based on what we have already deduced. Therefore, if the conclusion is false then at least one of the premises are false, so the argument is valid.

$\square$

**2.** Valid? Prove or disprove.

If I work hard, then I earn lots of money.
If I don't pay high taxes, then I don't work hard.
———
If I work hard, then I pay high taxes.

*Solution:* The second premise is the contrapositive of the conclusion. Hence, whenever the second premise is true, the conclusion is true (whether or not the first premise is true). So the argument is valid.

$\square$

**3.** True or False questions.

**(i)** If $p \wedge q$ is true, then $p \vee q$ is true. $\boxed{\textbf{True}}$

**(ii)** If $p \to q$ is true and $q \to p$ is true, then $p$ is logically equivalent to $q$. $\boxed{\textbf{True}}$

**(iii)** If $\mathcal{A}$ is a tautology and $\mathcal{B}$ is a contradiction, then $\mathcal{A} \wedge (\neg\mathcal{B})$ is a tautology. $\boxed{\textbf{True}}$

**(iv)** If $\mathcal{A} \iff \mathcal{B}$ and $\mathcal{C}$ is any statement, then $(\mathcal{A} \to \mathcal{C}) \iff (\mathcal{B} \to \mathcal{C})$. $\boxed{\textbf{True}}$

**(v)** If the premises of an argument are all contradictions, then the argument is valid. $\boxed{\textbf{True}}$

**(vi)** If there exists a premise of an argument that is a contradiction, then the argument is valid. $\boxed{\textbf{True}}$

**(vii)** If an argument is valid, then some premise of the argument is a contradiction. $\boxed{\textbf{False}}$

**(viii)** The statement $(p \to q) \leftrightarrow (q \wedge (r \to s)$ evaluates to TRUE when all the atomic statements $p, q, r, s$ are true. $\boxed{\textbf{True}}$

**(ix)** If $f : A \to B$ is a function and $|A| < |B|$ then $f$ is one-to-one. $\boxed{\textbf{False}}$

**4.** Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8\}$. Find $A \cap B$, $A \cup B$ and $(A \cap B) \times (A \cup B)$.

*Solution:*

$$\begin{aligned} A \cap B &= \{2\}, \\ A \cup B &= \{1, 2, 3, 4, 6, 8\}, \\ (A \cap B) \times (A \cup B) &= \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 6), (2, 8)\}. \end{aligned}$$

**5.** Let $A$ and $B$ be sets. Prove or give a counterexample: If $A \subseteq B$ and $A \subseteq B^c$, then $A = \emptyset$.

*Solution:* Suppose seeking a contradiction that $A \subseteq B$ and $A \subseteq B^c$ and $A \neq \emptyset$. Since $A$ is non-empty, it contains an element $x \in A$. Since $A \subseteq B$ and $x \in A$, the element $x$ belongs to $B$ as well. Since $A \subseteq B^c$ and $x \in A$, the element $x$ belongs to $B^c$ as well. Since $x$ belongs to both $B$ and $B^c$, we have found an element $x \in B \cap B^c$. But $B \cap B^c = \emptyset$, so it contains no elements. This is a contradiction, so it is impossible to have $A$ non-empty if $A \subseteq B$ and $A \subseteq B^c$.

□

**6.** Let $A$, $B$, and $C$ be sets. Prove or give a counterexample:

$$\text{If } C \subseteq (A \cup B), \text{ then } C \subseteq A \text{ or } C \subseteq B.$$

*Solution:* For example if $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$ then $C \subseteq (A \cup B)$ (in fact $C$ is equal to the union of $A$ and $B$, so in particular it is a subset), but $C$ is neither a subset of $A$ nor $B$.

$\square$

**7.** Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g$ is onto and $f$ is onto then $g \circ f$ is onto. Is the converse true? Is the statement still true if we relax the assumption to only insist that $g$ was onto? that $f$ was onto? *Solution:* Suppose $f, g$ are both onto. Let $z \in C$ be an arbitrary element and we need to show that it is possible to find $x \in A$ such that $g \circ f(x) = z$.

Since $z \in C$ and $g$ is onto, there exists $y \in B$ such that $g(y) = z$. Next, since $f$ is onto and $y \in B$, there exists $x \in A$ such that $f(x) = y$. I claim that this $x$ does what we want, that $g \circ f(x) = z$.

Indeed, $g \circ f(x) = g(f(x)) = g(y) = z$. Since we have shown that for any $z \in C$ there exists a $x \in A$ such that $g \circ f(x) = z$, we have shown that indeed $g \circ f$ is onto.

$\square$

For the remainder of the problem, it suffices to give examples where the following statements are false:

*If $g \circ f$ is one-to-one, then so is $f : A \to B$ and $g : B \to C$.*
Let $A = \{1, 2, 3\}, B = \{a, b, c\}$ and $C = \{1, 2\}$. Then $f, g$ defined below satisfy $g \circ f$ is onto, $g$ is onto, and $f$ is not onto.

$$\begin{array}{ll} f(1) = a & g(a) = 1 \\ f(2) = a & g(b) = 2 \\ f(3) = b & \end{array}$$

*If $g$ is one-to-one, then so is $g \circ f$.*
Let $A = \{1, 2\}, B = \{a, b, c\}$ and $C = \{1, 2, 3\}$. Then $f, g$ defined below satisfy $g$ is onto but $g \circ f$ is not onto.

$$\begin{array}{ll} f(1) = a & g(a) = 1 \\ f(2) = b & g(b) = 2 \\ & g(c) = 3 \end{array}$$

*If $f$ is one-to-one, then so is $g \circ f$.*
Let $A = \{1, 2\}, B = \{a, b, c\}$ and $C = \{1, 2, 3\}$. Then $f, g$ defined below satisfy $f$ is onto but $g \circ f$ is not onto.

$$\begin{array}{ll} f(1) = a & g(a) = 1 \\ f(2) = b & g(b) = 1 \\ f(3) = c & g(c) = 1 \end{array}$$