Practice Exam 2 (approximately 2x length of Exam 2)

1. Prove that
$$1 + 2 + 2^2 + \cdots 2^n = 2^{n+1} - 1.$$

Solution: Proof by induction. The base case when $n = 1$ reads $1 = 2^{1+1} - 1$ which is true. Next, we assume that the above formula holds for n, and note

$$1 + 2 + 2^2 + \cdots + 2^n + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1,$$

which is the RHS of the formula in the $n + 1$ case.

□

2. Prove or disprove:
$$2 + 4 + 6 + 8 + \cdots + 2n = (n - 1)(n + 2).$$

Solution: This seems false for all n? In particular, when $n = 1$ the equality $2 = (1-1)(1+2)$ is false. Also, when $n = 2$ the equality $2 + 4 = (2 - 1)(2 + 2)$ is false. It is worth noting that the inductive step actually works. It may be interesting to try to prove that for all n (perhaps except for finitely many lucky ones) that the above equality is *false*.

3. In the math department there are 30 personal computers (PCs).

20	have A drives,
8	have 19-inch monitors,
25	are running Windows XP,
20	have at least two of these properties,
6	have all three properties.

(a) How many PCs have at least one property? *Solution:*

(b) How many have none of these properties?

(c) How many have exactly one?

Solution: The number of PCs that have at least one property can be calculated as follows. Let A, B, C denote the sets of computers having A drives, 19-inch monitors, and those that are running Windows XP, respectively. Then the number PCs with at least one property is the number of elements in the set $A \cup B \cup C$. Note that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. We have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + (|B| + |C| - |B \cap C|) - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|. \end{aligned}$$

We already have $|A|, |B|, |C|$, and $|A \cap B \cap C|$. So, we just need to compute the number

$$|A \cap B| + |A \cap C| + |B \cap C|.$$

Note that this number is the number of elements in exactly 2 sets plus three times the number of elements in all 3 sets (draw a Venn diagram if you don't see this immediately, or read the simple proof below), which can also be said to be the number of elements in *at least* 2 sets plus two times the number of elements in all 3 sets. Hence, this number is $20 + 2 * 6 = 32$. So, the number of elements in at least one set is $20 + 8 + 25 - (32) + 6 = 27$.

Next, the number with none of the properties is the total number of computers minus the number that have at least one property, so $30 - 27 = 3$.

Finally the number with exactly one property is the number that have at least one property minus the number that have exactly two properties, so $27 - 20 = 7$. □

CLAIM: *For any finite sets A, B, C contained in some universal finite set U, the number $|A \cap B| + |A \cap C| + |B \cap C|$ equals the number of elements that are in at least 2 of the sets plus two times the number of elements that are in all three sets.* Proof: Denote by S the elements of U that are in at least 2 sets. We are asked to show that

$$|A \cap B| + |A \cap C| + |B \cap C| = |S| + 2|A \cap B \cap C|.$$

We prove the equality by examining how each element of U contributes to each side of the above equality.

Let x be an element of U. If x does not belong to S, then it will not contribute to either side of the equality. If x belongs to S, then either x is in exactly one of the sets $A \cap B, A \cap C, B \cap C$, or x is in all 3 of them. In the first case, x does not belong to all three sets and so contributes +1 to both the LHS and RHS. In the second case, x belongs to the triple-intersection and so contributes +3 to both the LHS and RHS. This finishes the proof of the claim. □

4. How many ways can you get a total of 6 when rolling two dice?

Solution: In my mind, the result of the "first" die being 4 and the result of the "second" die being 2 is different than the situation where the results are reversed. It perhaps would be a clearer distinction if I mentioned that one die were red and one were green, for example, but in my mind color is unnecessary attribute to distinguish different objects. Anyway, the answer I was looking for is 5 (the set of ordered results is $\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$. The other answer you may be thinking is 3 (with set of unordered results $\{\{1,5\},\{2,4\},\{3,3\}\}$.

□

5. How many three digit numbers contain the digits 2 and 5 but not 0, 3, or 7?

Solution: I will consider 3 digit numbers in the range 000-999. In this case, 2 of the digits must be 2 and 5, and the last digit can be one of 1,2,4,5,6,8, or 9. There are 7 choices for the unknown digit, and there are several ways to permute the three digits once all are chosen. If the chosen digit is 2 or 5, then there are 3 permutations, and if the chosen digit is 1,4,6,8, or 9, there are 6 permutations. So, there are a total of $2 * 3 + 5 * 6 = 36$ possible numbers.

6. In a group of 29 people, how many people must there be whose birthdays are in the same month?

Solution: There are 12 months, so in a group of 29 people, by the Strong Pigeon Hole Principle, there are $\lceil \frac{29}{12} \rceil = \lceil 2.4167 \rceil = 3$ people who have birthdays in the same month.

7. Consider the recursively defined sequence defined by $a_0 = 2$, $a_1 = -1$, and

$$a_n = -2a_{n-1} + 8a_{n-2}, \text{ for } n \geq 2.$$

(a) Find the first five terms of the sequence a_n.

Solution: We have

$$\begin{aligned} a_0 &= 2 \\ a_1 &= -1 \\ a_2 &= -2(-1) + 8(2) = 14 \\ a_3 &= -2(14) + 8(-1) = 36 \\ a_4 &= -2(36) + 8(14) = 40. \end{aligned}$$

(b) Solve the recurrence to find a closed form for a_n, $n \geq 0$.

Solution: The characteristic polynomial factors as

$$x^2 + 2x - 8 = (x+4)(x-2) = 0, \text{ if } x = -4, 2.$$

Thus, the general solution is $a_n = c_1(-4)^n + c_2 2^n$, with unknown constants that are solved for using the initial conditions $a_0 = 2$ and $a_1 = -1$, corresponding to the case $n = 0$ and $n = 1$, respectively. We have,

$$\begin{aligned} a_0 = c_1 + c_2 &= 2 \\ a_1 = -4c_1 + 2c_2 &= \text{-1} \end{aligned}$$

and solving for c_1, c_2 we get that $6c_2 = 7$ so $c_2 = 7/6$ and $c_1 = 5/6$. Thus, the closed formula is

$$a_n = \frac{5}{6}(-4)^n + \frac{7}{6}2^n.$$

(c) What is the tenth term of the sequence?

Solution: The tenth term corresponds to $n = 9$ since the indices start at zero. We plug into the closed formula to find $a_9 = 5(-4)^9/6 + 7(2)^9/6 = -217,856$.

8. Consider the following algorithm.

Input: $a_1, \ldots, a_n$ real numbers.

Procedure: Initialize: Set $s = 1$ and $t = a_1$.

Step 1: For $i = 2, \ldots, n$,

If $a_i < t$ set $s = 1$ and $t = a_i$,

If $a_i = t$ set s to $s + 1$.

Output: s, t.

(a) Describe the relationship between the output of the algorithm and the input values $a_1, \ldots, a_n$. That is, what are s and t?

Solution: s is the number of times the minimum t occurs.

(b) Find an accurate bound on the complexity of the algorithm in terms of the number of comparisons used. *Answer each question separately.*

Solution: The number of comparisons is equal to $2(n-1)$ in the worst case input.

9. Over a total of 11 years of teaching I have had 1,500 students. Find the largest integer n such that at least n of my previous students have the same initials. Recall, a person's initials are a string of the form $X.Y.$ where X is the first letter of their first name and Y is the first letter of their last name (for example, my initials are S.B.).

Solution: There are 26^2 different initials. By the pigeon hole principle there are $\lceil \frac{1500}{26^2} \rceil = 3$ students who have the same initials, at least.

10. The number of 5-letter words (not necessarily real English words) containing the string R-A-P, where the letters are consecutive and in order, is

Solution: $26^2 \cdot 3$.

11. A license plate is a string of 7 characters. How many license plates can be made if the license plate needs to contain 3 different letters and 4 different numbers?

Solution: $\binom{10}{4}\binom{26}{3}7!$.

12. How many license plates can be made if the license plate needs to contain 3 different letters and 4 different numbers and the letters must all be to the left of all the numbers, and furthermore the numbers must be increasing from left to right and the letters must be in alphabetical order?

Solution: $\binom{10}{4}\binom{26}{3}$.

13. True or false questions.

(a) $2 + 4 + 6 + \cdots + 2n = n(n + 1)$ for all integers $n \geq 1$. TRUE FALSE

(b) There exists an integer n such that for every integer m, if $m \geq n$ then $\lceil \frac{n}{m} \rceil = 1$. TRUE FALSE

(c) There are exactly 8 subsets of $\{a, b, c\}$, including the emptyset and the set itself. TRUE FALSE

Solution: True (Euler's trick times 2). True (any positive n will work). True (multiplication rule).