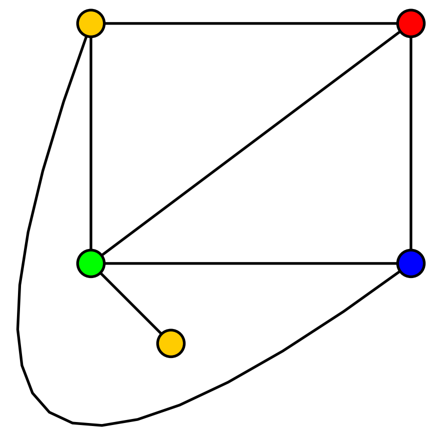
Practice Exam 3

1. Give an example of a planar graph with 5 vertices which has chromatic number 4, and draw a 4-colored planar model for your graph. Is it always true that no matter what graph you picked, it is not bipartite? Explain.



The graph whose model is shown. No graph with chromatic number 4 is bipartite, since a graph $\mathcal{G}$ is bipartite if and only if $\chi(\mathcal{G}) = 2$.

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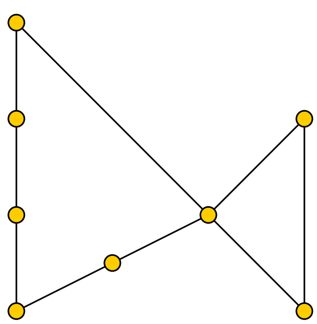
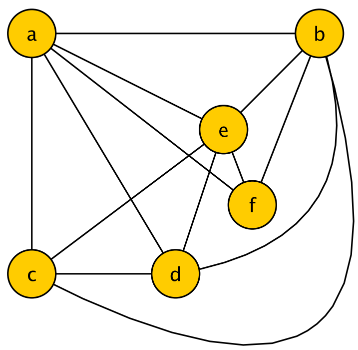
2. Give an example of a graph with 8 vertices that is Eulerian but not Hamiltonian.



3. Consider the following graph

$$\mathcal{G} = (\{a, b, c, d, e, f\}, \{ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, de, ef\}).$$

(a) Give a model for $\mathcal{G}$.



(b) Find a Hamiltonian path in $\mathcal{G}$. ***Solution:*** $abcdefa$ works.

(c) Find a subgraph of $\mathcal{G}$ that is isomorphic to K_4. ***Solution:*** $(\{a, b, e, f\}, \{ab, ae, af, be, bf, ef\})$.

(d) Find a subgraph of $\mathcal{G}$ that is homeomorphic to K_5. Is $\mathcal{G}$ planar? ***Solution:*** The subgraph obtained by deleting f and the three edges that contain it is K_5. Thus, $\mathcal{G}$ is not planar.

(e) Is $\mathcal{G}$ Eulerian? Justify your answer. ***Solution:*** No. Vertex f has odd degree.

(f) Find the degree sequence of $\mathcal{G}$. ***Solution:*** 5, 5, 5, 4, 4, 3.

(g) Find two non-isomorphic spanning trees of $\mathcal{G}$. ***Solution:***

$$\mathcal{T}_1 = (\{a, b, c, d, e, f\}, \{ab, bc, cd, de, ef\}),$$
$$\mathcal{T}_2 = (\{a, b, c, d, e, f\}, \{ab, ac, ad, ae, af\}).$$

4. Give an example of two graphs with 6 vertices that have the same degree sequence but are not isomorphic.

Solution: Both graphs below have degree sequence $3, 3, 3, 3, 3, 3$ but they are not isomorphic since one is connected and one is disconnected.

$$\mathcal{G}_1 = (\{a, b, c, d, e, f\}, \{ab, bc, cd, de, ef, fa\}),$$
$$\mathcal{G}_2 = (\{x, y, z, u, v, w\}, \{xy, yz, xz, uv, vw, uw\}).$$

5. Consider the following graphs.

$$\mathcal{G} = (\{a, b, c, d, e, f\}, \{ab, ac, ad, bd, cd, de, df\}),$$
$$\mathcal{H} = (\{x, y, z, w\}, \{xy, xz, xw\}).$$

(a) Enumerate (list and number) the subgraphs of $\mathcal{G}$ that are isomorphic to $\mathcal{H}$.

Solution: There is one with middle vertex a and $\binom{5}{3} = 10$ with middle vertex d. They are the graphs

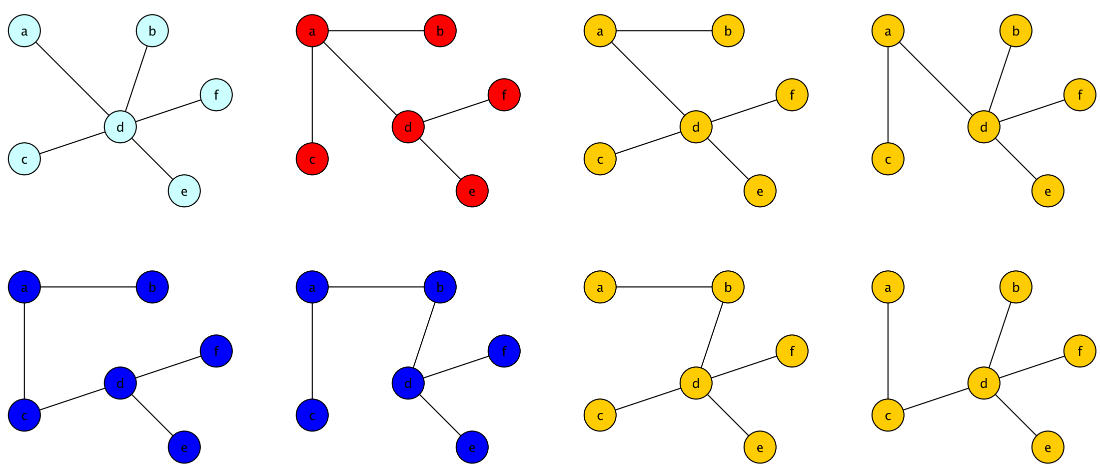
1. $(\{a,b,c,d\},\{ab,ac,ad\})$,
2. $(\{d,a,b,c\},\{da,db,dc\})$,
3. $(\{d,e,f,a\},\{de,df,da\})$,
4. $(\{d,e,f,b\},\{de,df,db\})$,
5. $(\{d,e,f,c\},\{de,df,dc\})$,
6. $(\{d,e,a,b\},\{de,da,db\})$,
7. $(\{d,e,a,c\},\{de,da,dc\})$,
8. $(\{d,e,b,c\},\{de,db,dc\})$,
9. $(\{d,f,a,b\},\{df,da,db\})$,
10. $(\{d,f,a,c\},\{df,da,dc\})$,
11. $(\{d,f,a,b\},\{df,da,db\})$.

(b) Enumerate the subgraphs of $\mathcal{G}$ that are isomorphic to K_3. ***Solution:*** There are two, with vertex sets $\{a,c,d\}$ and $\{a,b,d\}$.

(c) Find the subgraphs of $\mathcal{G}$ that have a connected component isomorphic to K_3.

Solution: There are a total of 16, found by adding any subset of 3 isolated vertices to one of the copies of K_3. For example $(\{a,c,d\} \cup S, \{ac,ad,cd\})$ where $S \subseteq \{b,e,f\}$.

(d) Enumerate the spanning trees of $\mathcal{G}$. Are all spanning trees of $\mathcal{G}$ isomorphic? ***Solution:*** There are a total of 8, which I found by deleting edges from cycles. Isomorphic graphs have the same color:



12. Prove that any subgraph of a bipartite graph is bipartite.

Proof. Let $\mathcal{G}$ be a bipartite graph and $\mathcal{H}$ be a subgraph of $\mathcal{G}$. Since $\mathcal{G}$ is bipartite there exists a bipartition V_1, V_2 of $V(\mathcal{G})$, that is, V_1, V_2 is a partition of $\mathcal{G}$ such that if $v, w \in V_i$, $i = 1, 2$, then $(v, w) \notin E(\mathcal{G})$. In particular, since $V(\mathcal{H}) \subseteq V(\mathcal{G})$ we define $\tilde{V}_1 = V_1 \cap V(\mathcal{H})$ and $\tilde{V}_2 = V_2 \cap V(\mathcal{H})$. Note that $\tilde{V}_1, \tilde{V}_2$ is a partition of $V(\mathcal{H})$, and furthermore it is a bipartition since if $v, w \in \tilde{V}_i$, $i = 1, 2$, then $v, w \in V_i$ as well (since $\tilde{V}_i \subseteq V_i$), and so $v, w \notin E(\mathcal{H})$ since $v, w \notin E(\mathcal{G})$ and $E(\mathcal{H}) \subseteq E(\mathcal{G})$.

□

13. Let $\mathcal{G} = (V, E)$ be a *(non-empty)* graph. Prove that if m is the minimum degree of a vertex of $\mathcal{G}$ and M is the maximum degree of a vertex of $\mathcal{G}$, then

$$m \leq \frac{2|E|}{|V|} \leq M.$$

Proof. We freely assume that

$$\sum_{v \in V} \deg v = 2|E|.$$

In particular, if

$$m = \min_{v \in V} \deg v, \text{ and}$$
$$M = \max_{v \in V} \deg v,$$

then we have

$$\sum_{v \in V} m \leq \sum_{v \in V} \deg v \leq \sum_{v \in V} M$$

and hence since $\sum_{v \in V} c = |V| \cdot c$ for any constant c and using the first equality above we have

$$m|V| \leq 2|E| \leq M|V|$$

from which it follows that

$$m \leq \frac{2|E|}{|V|} \leq M.$$

□

14. Prove that the two graphs below are isomorphic by exhibiting an explicit isomorphism between them (be sure to justify that your map is an isomorphism).

$$\mathcal{G} = (\{a, b, c, d\}, \{ab, ac, ad, bc, cd\}),$$
$$\mathcal{H} = (\{x, y, z, w\}, \{xy, xw, xz, yw, yz\}).$$

Solution: Define the map f as follows

$$f : \{a, b, c, d\} \to \{x, y, z, w\}$$
$$a \mapsto x$$
$$b \mapsto z$$
$$c \mapsto y$$
$$d \mapsto w$$

This map is clearly bijective (it is onto and the domain and co-domain have the same number of elements, for example). We check that it is edge preserving:

$$ab \mapsto xz, \text{ and } xz \overset{?}{\in} E(\mathcal{H}) \quad \checkmark$$
$$ac \mapsto xy, \text{ and } xy \overset{?}{\in} E(\mathcal{H}) \quad \checkmark$$
$$ad \mapsto xw, \text{ and } xw \overset{?}{\in} E(\mathcal{H}) \quad \checkmark$$
$$bc \mapsto zy, \text{ and } zy \overset{?}{\in} E(\mathcal{H}) \quad \checkmark$$
$$cd \mapsto yw, \text{ and } yw \overset{?}{\in} E(\mathcal{H}) \quad \checkmark$$

In particular, every edge of $\mathcal{G}$ is mapped to an edge of $\mathcal{H}$ and no edge is mapped to more than once (or equivalently when they have the same number of edges, every edge of $\mathcal{H}$ is mapped to by an edge of $\mathcal{G}$).

15. TRUE OR FALSE Any two graphs with exactly 4 vertices and 5 edges are isomorphic.
(The answer is NOT *what you might expect!)*

That's true. In general you may not have isomorphic graphs when the number of edges and vertices are equal, but for small number of vertices (or small number of edges) this is actually true. In particular, every 4 vertex graph with 5 edges is isomorphic to a subgraph of K_4 with one edge removed.