Practice Exam 1 solutions

1. Prove that

$$1 + 2 + 2^2 + \cdots 2^n = 2^{n+1} - 1.$$

Solution: We give a proof by induction. For $n = 0$ the left hand side equals 1, and the right hand side equals $2^1 - 1 = 1$, which proves the base case. Now suppose the equality holds for $k \geq 1$, and we show that it holds for $k + 1$. We have

$$\begin{aligned} 1 + 2 + 2^2 + \cdots 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1, \end{aligned}$$

which means that the equality holds when $n = k + 1$.

□

2. Prove or disprove:

$$2 + 4 + 6 + 8 + \cdots + 2n = (n-1)(n+2).$$

Solution: This is false, it should be $n(n+1)$ on the right hand side since the left hand side equals $2(1 + 2 + 3 + \cdots + n) = 2 \cdot \frac{n(n+1)}{2}$.

□

3. Valid? Prove or disprove.

$$(a) \quad \begin{array}{l} p \to q \\ q \vee r \\ \hline r \to (\neg q) \end{array} \qquad (b) \quad \begin{array}{c} p \to q \\ (\neg r) \vee (\neg q) \\ r \\ \hline (\neg p) \end{array}$$

Solution: *Part (a)* is invalid. When p, q, r are all true the assumptions are true but the conclusion is false. *Part (b)* is valid. We give a simple proof by contradiction. Suppose, seeking a contradiction, that the argument is invalid. Then, for some assignment the conclusion is false and the assumptions are all true. If the conclusion is false then p is true. By the first assumption and the fact that p is true we get that q is true. By the second assumption and the fact that q is true we get that r is false, but r is true by the third assumption, which is a contradiction.

□

4. Valid? Prove or disprove.

$$\begin{array}{r} \text{If I work hard, then I earn lots of money.} \\ \underline{\text{If I don't pay high taxes, then I don't work hard.}} \\ \text{If I work hard, then I pay high taxes.} \end{array}$$

Solution: The second assumption is the contrapositive of the conclusion, so the argument is clearly valid.

5. True or False questions.

(i) If $p \wedge q$ is true, then $p \vee q$ is true. TRUE

(ii) If $p \to q$ is true and $q \to p$ is true, then p is logically equivalent to q. TRUE

(iii) If $\mathcal{A}$ is a tautology and $\mathcal{B}$ is a contradiction, then $\mathcal{A} \wedge (\neg \mathcal{B})$ is a tautology. TRUE

(iv) If $\mathcal{A} \iff \mathcal{B}$ and $\mathcal{C}$ is any statement, then $(\mathcal{A} \to \mathcal{C}) \iff (\mathcal{B} \to \mathcal{C})$. TRUE

(v) If the premises of an argument are all contradictions, then the argument is valid. TRUE

(vi) The statement $(p \to q) \leftrightarrow (q \wedge (r \to s)$ evaluates to TRUE when all the atomic statements p, q, r, s are true. TRUE

6. In the math department there are 30 personal computers (PCs).

20	have A drives,
8	have 19-inch monitors,
25	are running Windows XP,
20	have at least two of these properties,
6	have all three properties.

(a) How many PCs have at least one property?

(b) How many have none of these properties?

(c) How many have exactly one?

SOLUTION: The number of PCs that have at least one property can be calculated as follows. Let A, B, C denote the sets of computers having A drives, 19-inch monitors, and those that are running Windows XP, respectively. Then the number PCs with at least one property is the number of elements in the set $A \cup B \cup C$. Note that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. We have

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + (|B| + |C| - |B \cap C|) - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|.
\end{aligned}$$

We already have $|A|, |B|, |C|$, and $|A \cap B \cap C|$. So, we just need to compute the number

$$|A \cap B| + |A \cap C| + |B \cap C|.$$

Note that this number is the number of elements in exactly 2 sets plus three times the number of elements in all 3 sets (draw a Venn diagram if you don't see this immediately, or read the simple proof below), which can also be said to be the number of elements in *at least* 2 sets plus two times the number of elements in all 3 sets. Hence, this number is $20 + 2 * 6 = 32$. So, the number of elements in at least one set is $20 + 8 + 25 - (32) + 6 = 27$.

Next, the number with none of the properties is the total number of computers minus the number that have at least one property, so $30 - 27 = 3$.

Finally the number with exactly one property is the number that have at least one property minus the number that have exactly two properties, so $27 - 20 = 7$. □

CLAIM: *For any finite sets A, B, C contained in some universal finite set U, the number $|A \cap B| + |A \cap C| + |B \cap C|$ equals the number of elements that are in at least 2 of the sets plus two times the number of elements that are in all three sets.* Proof: Denote by S the elements of U that are in at least 2 sets. We are asked to show that

$$|A \cap B| + |A \cap C| + |B \cap C| = |S| + 2|A \cap B \cap C|.$$

We prove the equality by examining how each element of U contributes to each side of the above equality.

Let x be an element of U. If x does not belong to S, then it will not contribute to either side of the equality. If x belongs to S, then either x is in exactly one of the sets $A \cap B, A \cap C, B \cap C$, or x is in all 3 of them. In the first case, x does not belong to all three sets and so contributes +1 to both the LHS and RHS. In the second case, x belongs to the triple-intersection and so contributes +3 to both the LHS and RHS. This finishes the proof of the claim. □

7. How many ways can you get a total of 6 when rolling two dice?

Solution: The possible ways to get 6 are if the die say $(1,5),(2,4),(3,3),(4,2),(5,1)$, so there are 5 possibilities.

8. How many three digit numbers contain the digits 2 and 5 but not 0, 3, or 7?

Solution: I will consider 3 digit numbers in the range 000-999. In this case, 2 of the digits must be 2 and 5, and the last digit can be one of 1,2,4,5,6,8, or 9. There are 7 choices for the unknown digit, and there are several ways to permute the three digits once all are chosen. If the chosen digit is 2 or 5, then there are 3 permutations, and if the chosen digit is 1,4,6,8, or 9, there are 6 permutations. So, there are a total of $2*3+5*6=36$ possible numbers.

□

9. In a group of 29 people, how many people must there be whose birthdays are in the same month?

Solution: There are 12 months, so in a group of 29 people, by the Strong Pigeon Hole Principle, there are $\lceil \frac{29}{12} \rceil = \lceil 2.4167 \rceil = 3$ people who have birthdays in the same month.

□